Dynamic Event-triggered Control under Packet Losses: The Case with Acknowledgements

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***Abstract*—In this paper, a dynamic ETC strategy for nonlinear state-feedback systems is proposed that results in guarantees for a finite $\mathcal{L}_p$-gain from disturbance input to performance output and a strictly positive lower bound on the inter-event times despite the presence of packet losses. The proposed dynamic ETC strategy has several advantages with respect to the commonly studied static ETC strategy including significantly larger average inter-event times. The proposed design methodology results in tradeoffs between the maximum allowable number of successive packet dropouts, (minimum and average) inter-event times and $\mathcal{L}_p$-gains, which will be illustrated by means of a numerical example.**

I. Introduction

In contrast to traditional control setups, networked control systems (NCSs) do not rely on dedicated point-to-point connections but employ shared communication media to transmit the sensor and actuation data. For this reason, NCSs are less expensive to install, more flexible and easier to maintain. Moreover, in case the communication is wireless, physical limitations of wired links can be overcome. However, (packet-based) networked communication suffers from inevitable imperfections such as variable transmission intervals, communication constraints and packet dropouts. These networked-induced imperfections lead to many technical challenges which need to be resolved to enable successful implementation of NCSs.

One of these challenges is to minimize the number of transmissions over time while ensuring desired stability and control performance properties despite the presence of packet losses. This is desired as in many application the communication resources are limited and possibly shared with other tasks. In most digital control setups, the transmission instants are generated in a *time-triggered* fashion and often according to a fixed sampling rate. In general, this enhances the predictability and the ease of implementation. However, as a *time-triggered* control scheme transmits information irrespective of the status of the plant, it often results in redundant utilization of the communication medium. To deal with the scarcity of communication resources, it seems more natural to determine the transmission instants on the basis of state measurements. This results in more resource-aware control methodologies of which *event-triggered* control and *self-triggered* control are prominent examples, see [11] for a recent overview.

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Event-triggered control strategies aim to reduce the utilization of communication resources by letting the transmission times depend on, *e.g.*, state measurements of the system. Although many ETC strategies were proposed before, only a few dealt with the possible occurrence of packet loss. Examples include [1], [14]–[16] in which stochastic optimal control approaches are used to minimize a cost function consisting of a quadratic control cost and a communication cost. Another approach is to combine time-triggered and event-triggered solutions in the sense that in case a packet loss is detected, the ETC scheme is interrupted and transmissions are scheduled according to time-based specifications until the controller successfully receives the plant measurements, see [9], [13], [25]. In [24] it was shown that the design of a triggering rule of the form as in [21] can be related to a maximum allowable number of successive packet drops (MANSD). Unfortunately, in case of disturbances, no positive minimal inter-event time can be guaranteed with this approach. For this reason, in [18] a periodic event-triggered control (PETC) scheme is considered in the sense that the triggering condition is only evaluated at equidistant instances in time. As such, a lower-bound on the inter-event times is enforced despite the presence of disturbances.

In this paper, we propose a new framework that allows the design of *dynamic* event-triggering strategies [4], [6], [19], [20] that can deal with the presence of packet loss. In contrast to static event-trigger mechanisms (ETMs) that are commonly adopted (also in the papers mentioned before dealing with packet loss in the context of ETC), in dynamic ETMs, the inter-event times do not converge to the enforced minimal inter-event time when the system's state approaches the desired equilibrium, see, *e.g.*, Example 3 in [2] and the numerical example in [4]. The numerical example in this paper will also illustrate this important issue. Moreover, dynamic ETMs result in considerably larger average inter-event times than static ETMs. Given these benefits of dynamic ETMs and the presence of packet losses in many practical networked control applications, it is important to provide such a design framework of dynamic ETMs being robust to such losses. Note that in the existing contributions on dynamic ETMs [4], [6], [19], [20] did not incorporate packet losses in their analysis and design.

The remainder of this paper is organized as follows. First, we present the necessary preliminaries and notational conventions in Section II, followed by the problem statement in Section III. The introduction of a mathematical model of the NCS and a more mathematically rigorous problem formulation is presented in Section IV. In Section V we present design conditions for the proposed *dynamic* event-triggering strategy. Finally, we demonstrate how the presented theory leads to tradeoffs between the maximum allowable number of successive packet dropouts (MANSD), (minimum and average) inter-event times and $\mathcal{L}_p$-gains by means of a numerical example in Section VI. We provide concluding remarks in Section VII.

II. Definitions and Preliminaries

$\mathbb{N}$ denotes the set of all non-negative integers, $\mathbb{N}_{>0}$ the set of positive integers, $\mathbb{R}$ the field of real numbers and $\mathbb{R}_{\geq 0}$ the set of all non-negative reals. For N vectors $x_i \in \mathbb{R}^{n_i}$, $i \in \bar{N}$, we denote the vector obtained by stacking all vectors in one (column) vector $\bar{x} \in \mathbb{R}^n$ with $n = \sum_{i=1}^N n_i$ by $(x_1, x_2, \ldots, x_N)$, *i.e.*, $(x_1, x_2, \ldots, x_N) = \begin{bmatrix} x_1^\top & x_2^\top & \cdots & x_N^\top \end{bmatrix}^\top$. By $|\cdot|$ and $\langle \cdot, \cdot \rangle$ we denote the Euclidean norm and the usual inner product of real vectors, respectively. I denotes the identity matrix of appropriate dimensions. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$, and in addition, it is unbounded. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be locally Lipschitz continuous if for each $x_0 \in \mathbb{R}^n$ there exist constants $\delta > 0$ and $L > 0$ such that $|x - x_0| \leqslant \delta \Rightarrow |f(x) - f(x_0)| \leqslant L|x - x_0|$.

We recall now some definitions given in [8] that will be used for describing an NCS in terms of a hybrid model later. A *compact hybrid time domain* is a set $\mathcal{D} = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ with $J \in \mathbb{N}_{>0}$ and $0 = t_0 \leq t_1 \ldots \leq t_J$. A *hybrid time domain* is a set $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that $\mathcal{D} \cap ([0, T] \times \{0, \ldots, J\})$ is a compact hybrid time domain for each $(T, J) \in \mathcal{D}$. A *hybrid trajectory* is a pair $(\operatorname{dom}\xi, \xi)$ consisting of a hybrid time domain $\operatorname{dom}\xi$ and a function ξ defined on $\operatorname{dom}\xi$ that is absolutely continuous in t on $(\operatorname{dom}\xi) \cap (\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{N}$. For the hybrid system $\mathcal{H}$ given by the state space $\mathbb{R}^n$, the input space $\mathbb{R}^{n_w}$ and the data (F, G, C, D), where $F : \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ is continuous, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map, and C and D are subsets of $\mathbb{R}^n$, a hybrid trajectory $(\operatorname{dom}\xi, \xi)$ with $\xi : \operatorname{dom}\xi \to \mathbb{R}^n$ is a *solution to* $\mathcal{H}$ for a locally integrable input function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_w}$ if

1) For all $j \in \mathbb{N}$ and for almost all t such that $(t, j) \in \operatorname{dom}\xi$, we have $\xi(t, j) \in C$ and $\dot{\xi}(t, j) = F(\xi(t, j), w(t))$.

2) For all $(t, j) \in \operatorname{dom}\xi$ such that $(t, j+1) \in \operatorname{dom}\xi$, we have $\xi(t, j) \in D$ and $\xi(t, j+1) = G(\xi(t, j))$.

Hence, the hybrid system $\mathcal{H}$ is of the form

$$\dot{\xi} = F(\xi, w), \quad \xi \in C \tag{1a}$$

$$\xi^+ \in G(\xi), \quad \xi \in D, \tag{1b}$$

where we denoted $\xi(t_{j+1}, j+1)$ as in item 2) above as ξ^+.

In addition, for $p \in [1, \infty)$, we introduce the $\mathcal{L}_p$-norm of a function ξ defined on a hybrid time domain $\operatorname{dom}\xi = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\}$ with J possibly ∞ and/or $t_J = \infty$ by

$$\|\xi\|_p = \Big(\sum_{j=0}^{J-1} \int_{t_j}^{t_{j+1}} |\xi(t, j)|^p dt \Big)^{1/p} \tag{2}$$

provided the right-hand side is well-defined and finite. In case $\|\xi\|_p$ is finite, we say that $\xi \in \mathcal{L}_p$. Notice that this definition is essentially identical to the usual $\mathcal{L}_p$-norm in case a function is defined on a subset of $\mathbb{R}_{\geq 0}$.

Lemma 1. *Consider $a, b \in \mathbb{R}$ and some constant $\varepsilon > 0$, then it holds that $2ab \leqslant (1/\varepsilon)\, a^2 + \varepsilon b^2$.*

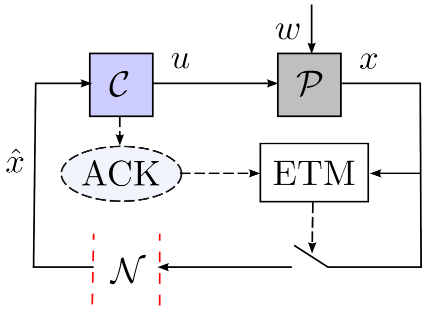

Fig. 1: Schematic representation of the event-triggered control configuration of an NCS discussed in this paper.

III. NCS Model and Problem Statement

In this section, we introduce an event-triggered NCS with communication constraints and network-induced imperfections such as time-varying transmission intervals and packet losses. Based on this description, we also provide the problem statement considered in this paper.

A. Networked Control Configuration

In this paper, we consider the static state-feedback control configuration as depicted in Fig. 1. The continuous-time plant $\mathcal{P}$ is given by

$$\mathcal{P} : \begin{cases} \dot{x} = f_p(x, u, w) \\ z = q(x, w), \end{cases} \tag{3}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state vector of the system, $w \in \mathbb{R}^{n_w}$ is a disturbance input, $u \in \mathbb{R}^{n_u}$ is the control input and z is the performance output of the system. The function $f_p : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_x}$ is assumed to be continuously differentiable. The controller $\mathcal{C}$ is given by

$$u = g_c(\hat{x}), \tag{4}$$

where $\hat{x} \in \mathbb{R}^{n_x}$ represents the most recently received state measurement by the controller $\mathcal{C}$. The function $g_c : \mathbb{R}^{n_x} \to \mathbb{R}^{n_u}$ is assumed to be continuous. Observe that in case of ideal (continuous) communication, $\hat{x}(t) = x(t)$ for all $t \in \mathbb{R}$.

In most NCSs, data is sent across the network in the form of discrete packages. For this reason, sensor measurements of the state x are sampled and transmitted to the controller at times t_j, $j \in \mathbb{N}$. If no packet loss has occurred, the corresponding elements of the most recent state information $\hat{x}(t)$ available at the controller is updated, *i.e.*, $\hat{x}(t_j^+) = x(t_j)$. In case of a packet loss, the sensor information available at the controller will not be updated, *i.e.*, $\hat{x}(t_j^+) = \hat{x}(t_j)$. To cope with these packet losses, we adopt the following assumption, which has been used in several works before, see, *e.g.*, [9], [12], [13], [18], [24].

Standing Assumption 1. *The number of successive packet dropouts $\delta \in \mathbb{N}$ occurred since the last successful transmission is bounded by δ_{max}, where δ_{max} denotes the maximum allowable number of successive dropouts.*

In between updates, $\hat{x}$ evolves according to

$$\dot{\hat{x}}(t) = \hat{f}(\hat{x}, w), \tag{5}$$

Observe that in case a zero-order hold device (ZOH) is employed, $\hat{f} = 0$, meaning that $\hat{x}$ is kept constant between transmissions.

B. Event-based Communication

In time-based NCSs, the transmission intervals are determined by means of time-based specifications regardless of the actual state of the system. As such, the transmission intervals typically have to satisfy $\epsilon \le t_{j+1} - t_j \le \tau_{mati}$, for all $j \in \mathbb{N}$, where $\epsilon \in (0, \tau_{mati}]$ can be chosen arbitrarily small and τ_{mati} denotes the *maximum allowable transmission interval* (MATI) as used in [3], [12], [17], [23]. This MATI bound is based on the worst-case situation of the system as no state information is used. However, in case the communication resources are scarce (especially when the communication medium is shared with other devices or users and/or if the communication bandwidth is limited), a resource-aware control strategy diverting from the worst-case time bounds is a necessity. By letting the transmission instants depend on state measurements, this resource-aware control perspective aims to communicate only when needed to achieve desired stability and performance criteria. In this way, transmission intervals can be prolonged for states not corresponding to the worst-case situation and the average transmission intervals can be significantly be enlarged beyond the worst-case value.

The resource-aware control approach that we adopt in this paper is an event-triggered control scheme in which, in contrast to time-based NCSs, transmissions are determined by a state-dependent triggering condition. The design of such an event-trigger mechanism (ETM) is often such that stability or other properties of the closed-loop system can be guaranteed. Moreover, it is important to guarantee a lower-bound on the transmission intervals in order to exclude Zeno-behaviour and to enable practical implementation. This lower bound is also referred to as the *minimal inter-event time* (MIET). In this work, we consider a *dynamic* triggering condition [4], [6], [20] of the form

$$t_0 = 0,\, t_{j+1} := \inf\left\{t > t_j + \tau_{miet} \mid \eta(t) < 0\right\}, \tag{6}$$

where η is the solution to the hybrid system [7]

$$\begin{cases} \dot{\eta} = \Psi(o) \\ \eta^+ = \eta_0(o), \text{ when } \eta < 0 \end{cases} \tag{7}$$

where o represents the information available at the ETM such as the state vector $x \in \mathbb{R}^{n_x}$ and transmission error $e := \hat{x} - x \in \mathbb{R}^{n_x}$ and where Ψ and η_0 are well-designed functions which will be specified in Section V. Observe that condition (6) enforces a robust positive MIET $\tau_{miet} > 0$ as the next event can only take place after at least τ_{miet} time units. Moreover, observe that we assume the usage of a transmission control protocol (TCP) meaning that an acknowledgement is sent whenever a package has arrived at the controller as illustrated in Figure 1. As such, the number of successive packet dropouts and thus the transmission error e are available at the ETM. Since the triggering condition is based on a dynamic variable η instead of a static expression depending on state x and transmission error e, the event-trigger condition given by (6) and (7), is referred to as a *dynamic* event triggering condition, see [4], [6], [20]. The main reason for employing *dynamic* ETMs is that *static* ETMs generally reduce to approximately time-triggered periodic communication in presence of disturbances when the state is close to the origin, as observed in [2], [4].

C. Problem Formulation

A time-triggered control setup can incorporate packet losses in a straightforward, although conservative, manner by regarding a packet loss as a prolongation of the transmission interval. Hence, the stability guarantees derived for the case where packet losses are absent remain satisfied when MATI is chosen as

$$\tau'_{mati} := \frac{\tau_{mati}}{\delta_{max} + 1}, \tag{8}$$

where τ_{mati} is the obtained transmission interval bound for the packet loss-free case using tools as in, *e.g.*, [3] and given by

$$\tau_{mati} = \begin{cases} \frac{1}{Lr}\operatorname{arctan}\left(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}-1\right)+1+\lambda}\right), & \gamma > L \\ \frac{1}{L}\frac{1-\lambda}{1+\lambda}, & \gamma = L \\ \frac{1}{Lr}\operatorname{arctanh}\left(\frac{r(1-\lambda)}{2\frac{\lambda}{\lambda+1}\left(\frac{\gamma}{L}-1\right)+1+\lambda}\right), & \gamma < L, \end{cases} \tag{9}$$

and $r = \sqrt{|(\gamma/L)^2 - 1|}$ where γ, L and λ are some positive constants which will be specified in Section V. This bound has the following property.

Lemma 2. *[3] If τ_{mati} is chosen as given in (9), then the solution to $\tilde{\phi}(0) = \lambda^{-1}$ and $\dot{\tilde{\phi}} = -2L\tilde{\phi} - \gamma(\tilde{\phi}^2 + 1)$ satisfies $\tilde{\phi}(t) \in [\lambda, \lambda^{-1}]$ for all $t \in [0, \tau_{mati}]$ with $\tilde{\phi}(\tau_{mati}) = \lambda$.*

For ETC systems, such an extension is less trivial as the next transmission depends on state-values and is therefore not exactly known in time. In this work, the problem addressed is formulated as follows.

Problem 1. *Propose a design procedure for τ_{miet}, Ψ and η_0 which results in guarantees for a finite $\mathcal{L}_p$-gain ($p \in [1, \infty)$) despite the occurrence of packet losses and a reduction in communication compared to a time-triggered solution.*

Because of the occurrence of packet losses in many practical situations and the technical difficulty as indicated before the problem formulation, solving this problem can be seen as an important extension of [4] in which packet loss was excluded.

IV. Hybrid Model of the ETC scheme

The hybrid system framework as developed in [8] allows to effectively analyze $\mathcal{L}_p$-stability for time-triggered and event-triggered NCSs as shown [3], [4], [12], [17], [19], [20]. In this section, we introduce an event-triggered NCS model incorporating packet losses.

To capture packet dropouts, we introduce the auxiliary variables $\tau \in \mathbb{R}_{\ge}0$, $q \in \{0, 1\}$ and $\delta \in \Delta$, where $\Delta := \{0, 1, \ldots, \delta_{max}\}$. The scalar variable τ is an internal clock variable. The Boolean q determines whether or not the packet has dropped at a transmission event. To be more precise, when $q(t_j) = 0$, $j \in \mathbb{N}$, the transmission at time t_j is successful resulting in an update of the state information available at the controller is updated according to $\hat{x}(t_j^+) = x(t_j)$. When

$q(t_j) = 1$, the packet with measurement data is lost at time t_j which implies that $\hat{x}$ is not be updated and thus $\hat{x}(t_j^+) = \hat{x}(t_j)$. Hence, the update of $\hat{x}(t_j^+)$ can be written as follows

$$\hat{x}^+ = (1-q)x + q\hat{x}. \tag{10}$$

The integer variable δ is used to keep track of the number of successive packet losses. Using the auxiliary variables q and δ, we can now write the closed-loop system (3),(4),(6) and (7) as the following hybrid system $\mathcal{H}$,

$$\mathcal{H} := \begin{cases} \left.\begin{array}{l} \dot{x} = f(x,e,w) \\ \dot{e} = g(x,e,w) \\ \dot{q} = 0 \\ \dot{\delta} = 0 \\ \dot{\tau} = 1 \\ \dot{\phi} = f_\phi(\phi,\tau) \\ \dot{\eta} = \Psi(x,e,\phi) \end{array}\right\}, \text{ when } \xi \in C \\ \left.\begin{array}{l} x^+ = x \\ e^+ = qe \\ q^+ \in \bar{q}(\delta) \\ \delta^+ = q(\delta+1) \\ \tau^+ = 0 \\ \phi^+ = q\phi + (1-q)\lambda^{-1} \\ \eta^+ = q\eta + (1-q)\eta_0(e,\phi) \end{array}\right\}, \text{ when } \xi \in D, \end{cases} \tag{11}$$

where $\tau \in \mathbb{R}_{\geq 0}$, $\phi \in \mathbb{R}_{\geq 0}$, $\eta \in \mathbb{R}_{\geq 0}$ and $\xi := (x,e,q,\delta,\tau,\phi,\eta) \in \mathbb{X} := \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \{0,1\} \times \Delta \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. Noting that $e = \hat{x} - x$ with (3), (4), (5) and (11), we obtain that

$$f(x,e,w) = f_p(x, g_c(x+e), w) \tag{12}$$

$$g(x,e,w) = \hat{f}(e+x,w) - f(x,e,w). \tag{13}$$

Moreover, the set-valued map $\bar{q} : \Delta \rightrightarrows \{0,1\}$ is given by

$$\bar{q}(\delta) := \begin{cases} \{0,1\}, & \text{when } \delta < \delta_{max} \\ \{0\}, & \text{when } \delta = \delta_{max}. \end{cases} \tag{14}$$

Observe that this map ensures that Standing Assumption 1 is satisfied. The function f_ϕ and the constant λ will be specified in the next section. The flow set C and jump set D are given by

$$\begin{aligned} C &:= \{\xi \in \mathbb{X} \mid 0 \leqslant \tau \leqslant \tau_{miet} \text{ or } \eta \geq 0\} \\ D &:= \{\xi \in \mathbb{X} \mid \tau > \tau_{miet} \text{ and } \eta < 0\}. \end{aligned} \tag{15}$$

Notice that these sets reflect the ETC condition given in (6). Furthermore, observe that if $\tau_{miet} \in \mathbb{R}_{>0}$, Zeno-behaviour is excluded from the system $\mathcal{H}$ since the next transmission event can only occur after τ_{miet} time units have elapsed, regardlessly whether the most recent transmission attempt was successful or not.

In this work, the performance of hybrid system $\mathcal{H}$ is defined as the attenuation of the output z, in terms of an induced $\mathcal{L}_p$-gain with $p \in [1,\infty)$.

Definition 1. *The hybrid system $\mathcal{H}$ is said to be $\mathcal{L}_p$-stable with an $\mathcal{L}_p$-gain less than or equal to θ from input w to output z, if there exists a $\mathcal{K}_\infty$-function β such that for any exogenous input $w \in \mathcal{L}_p$, and any initial condition $\xi(0,0) \in \mathbb{X}$, each corresponding solution to $\mathcal{H}$ satisfies*

$$\|z\|_{\mathcal{L}_p} \leq \beta(|(x(0,0), e(0,0), \eta(0,0))|) + \theta\|w\|_{\mathcal{L}_p}. \tag{16}$$

The problem that we consider in this paper is formulated in a more formal way as follows.

Problem 2. *Given an event-triggered NCS described by hybrid system $\mathcal{H}$, a desired $\mathcal{L}_p$-gain $\theta \in \mathbb{R}_{\geq 0}$ and a maximum number of successive packet losses δ_{max}, determine conditions for the value of τ_{miet} and for the functions Ψ, f_ϕ and η_0 as in (7), such that the system $\mathcal{H}$ is $\mathcal{L}_p$-stable with an $\mathcal{L}_p$-gain less than or equal to θ, while rendering τ_{miet} and the inter-event times $t_{j+1} - t_j$, $j \in \mathbb{N}$, large (on average).*

V. ETM DESIGN

A sufficient condition for $\mathcal{L}_p$-stability with $\mathcal{L}_p$-gain less than or equal to θ is the existence of a *storage function* S, which is a positive definite, radially unbounded function that satisfies the dissipation inequality $\dot{S} \leq \theta^p|w|^p - |q(x,w)|^p$ during flow, where $\theta^p|w|^p - |q(x,w)|^p$ is the supply rate, and satisfies $S^+ \leqslant S$ during jumps, see, *e.g.*, [12], [22]. In this section, we first provide conditions which lead to such a storage function. By means of these conditions, we can construct ETMs as given in (6) and (7) such that the system $\mathcal{H}$ is $\mathcal{L}_p$-stable with an $\mathcal{L}_p$-gain less than or equal to θ in presence of packet losses.

A. Preliminaries

As starting point to construct a suitable storage function, we consider the following well-known conditions as presented in [3], [17].

Condition 1. *There exist a locally Lipschitz function $W : \mathbb{N} \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$, a continuous function $H : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}_{\geq 0}$ and constants $L \geqslant 0$, $\underline{\alpha}_W$, $\bar{\alpha}_W$, and $0 < \lambda < 1$ such that*

- *for all $e \in \mathbb{R}^{n_e}$, $W(e)$ satisfies*

$$\underline{\alpha}_W|e| \leqslant W(e) \leqslant \bar{\alpha}_W|e|, \tag{17}$$

- *for all $x \in \mathbb{R}^{n_x}$, $w \in \mathbb{R}^{n_w}$ and almost all $e \in \mathbb{R}^{n_e}$ it holds that*

$$\left\langle \frac{\partial W(e)}{\partial e}, g(x,e,w) \right\rangle \leqslant LW(e) + H(x,w). \tag{18}$$

In addition, there exist a locally Lipschitz function $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geqslant 0}$, $\mathcal{K}_\infty$-functions $\underline{\alpha}_V$ and $\bar{\alpha}_V$, a continuous function $\varrho : \mathbb{R}^{n_x} \to \mathbb{R}_{\geqslant 0}$, and a constant $\gamma > 0$, such that

- *for all $x \in \mathbb{R}^{n_x}$*

$$\underline{\alpha}_V(|x|) \leqslant V(x) \leqslant \bar{\alpha}_V(|x|). \tag{19}$$

- *for all $x \in \mathbb{R}^{n_x}$, $w \in \mathbb{R}^{n_w}$ and almost all $e \in \mathbb{R}^{n_e}$*

$$\begin{aligned} \langle \nabla V(x), f(x,e,w) \rangle \leqslant& -\varrho(x) - H^2(x,w) \\ &+ \gamma^2 W^2(e) + \mu(\theta^p|w|^p - |q(x,w)|^p), \end{aligned} \tag{20}$$

for some $\mu > 0$ and $\theta \geqslant 0$.

Let us remark that for linear systems it is possible to systematically find functions V and W that satisfy Condition 1 by means of a linear matrix inequality (LMI), see [4], [12] for more details. In Section VI, a non-linear numerical example is provided.

Observe that at a successful transmission event, *i.e.*, when $\xi \in \mathcal{D}$ and $q = 0$, W satisfies

$$W(e^+) \leqslant \lambda W(e), \tag{21}$$

for all $\lambda \in (0, 1)$ since $e^+ = 0$. In case of a packet loss, *i.e.*, when $\xi \in \mathcal{D}$ and $q = 1$, W satisfies

$$W(e^+) = W(e). \tag{22}$$

Consider now the function $f_\phi : \mathbb{R} \to \mathbb{R}$ which defined as

$$f_\phi(\phi, \tau) = \begin{cases} -2L\phi - \gamma(\phi^2 + 1), & \text{for } \tau \in [0, \tau_{miet}] \\ 0, & \text{for } \tau > \tau_{miet} \end{cases} \tag{23}$$

where λ, $L > 0$ and $\gamma > 0$ are the constants as given in Condition 1. Observe that ϕ only decreases at most δ_{max} phases of length $[0, \tau_{miet}]$ before it is being reset to λ^{-1} as when $\tau \geqslant \tau_{miet}$, ϕ is being held constant. So $\dot{\phi} = -2L\phi - \gamma(\phi^2+1)$ is at most active over a time span of $[0, (\delta_{max} + 1)\tau_{miet}]$.

B. Design Conditions

In this section we discuss present design conditions for the ETM as given in (6).

Theorem 1. *Suppose that Condition 1 holds and that there exists a function* $\Psi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ *that satisfies*

$$\Psi(x, e, \phi) \leqslant \begin{cases} M_1(x, e, \phi, w), & \text{for } 0 \leq \tau \leq \tau_{miet} \\ M_2(x, e, \phi, w), & \text{for } \tau > \tau_{miet} \end{cases} \tag{24}$$

and

$$\Psi(x, e, \phi) \geqslant 0, \text{ for } 0 \leq \tau \leq \tau_{miet}, \tag{25}$$

with $\tau_{miet} = \frac{\tau_{mati}}{\delta_{max}+1}$ *where* τ_{mati} *as in (9) and where*

$$M_1(x, e, \phi, w) := \varrho(x) + \big(H(x, w) - \gamma\phi W(e)\big)^2 \tag{26}$$

$$M_2(x, e, \phi, w) := \varrho(x) + H^2(x, w) - 2\gamma\phi W(e)H(x, w) - \left(\gamma^2 + 2\gamma\phi L\right) W^2(e), \tag{27}$$

and suppose that $\eta_0(e, \phi)$ *is given by*

$$\eta_0(e, \phi) = \gamma\phi W^2(e). \tag{28}$$

Then, the ETM described by (6) and (7) guarantees that the system $\mathcal{H}$ *given in (11) is* $\mathcal{L}_p$*-stable with an* $\mathcal{L}_p$*-gain less than or equal to* θ *and with a MIET* τ_{miet} *equal to* τ'_{mati}.

The proof is omitted for brevity and can be found in [5]. Observe that since (25) assures that $\eta(t) \geqslant 0$ for $0 \leq \tau \leq \tau_{miet}$, the dynamic triggering condition (6) and (7) can be modified to a *static* triggering condition

$$t_0 = 0, t_{j+1} := \inf\left\{t > t_j + \tau_{miet} \mid \Psi(x, e, \phi) < 0\right\}, \tag{29}$$

which would correspond to the usual design in most works on ETC that adopt static ETMs. Observe that the static ETM given in (29) triggers before the ETM given by (6) and (7). As consequence, the ETM given in (29) also results in an $\mathcal{L}_p$-gain less than or equal to θ for the closed-loop system $\mathcal{H}$ as can be seen from the proof in the appendix.

Remark 1. *The semi-positive definite function* $\varrho : \mathbb{R}^{n_x} \to \mathbb{R}_{\geqslant 0}$ *in (20) and the constant* $\lambda \in (0, 1)$ *in (21) can be chosen arbitrarily. Observe from (24)-(27) that if* ϱ *is chosen positive definite, the upper bound on* $\Psi(x, e, \phi)$ *increases which is beneficial for the growth of* η *for* $0 \leqslant \tau \leqslant \tau_{miet}$*. As consequence, larger inter-event times can be expected. Furthermore,* λ *affects the size of* $\eta_0(e, \phi)$ *which might cause* η *to jump to a larger value which is beneficial for the inter-event times as well. However, both* ϱ *and* λ *affect* τ_{miet}*. Hence, the choice for* ϱ *and* λ *relies on a tradeoff between the minimum inter-event time* τ_{miet} *and the expected average inter-event time*

Remark 2. *In general, the disturbance* w *is not available at the ETM. For this reason, one might need to introduce some conservatism in the choice of* $H(x, w)$ *as in (18) in order to construct a function* Ψ *which satisfies (24) and does not depend on* w*. Due to this conservatism,* γ *as in (20) increases which will lead to a smaller* τ_{miet} *as can be seen from (9).*

In Section V of [4], a systematic procedure is presented to find a function Ψ that satisfies (24) and (25) for linear systems.

VI. Numerical Example

In this section, we illustrate the use of the derived ETM design conditions by means of a numerical example and we show that these conditions lead to a design tradeoff between performance (in terms of an induced $\mathcal{L}_2$-gain), robustness (in terms of MANSD) and the utilization of communication resources (in terms of the MIET and average inter-event times).

A. Model description

Consider the non-linear system

$$\mathcal{P} : \begin{cases} \dot{x} = -x \sin x + u + w \\ z = x, \end{cases} \tag{30}$$

where $x \in \mathbb{R}$ and $u \in \mathbb{R}$. We consider the control law $u = -k(x + e)$ implemented in a ZOH fashion. This leads to

$$\dot{x} = -x \sin x - kx - ke + w \tag{31a}$$

$$\dot{e} = x \sin x + kx + ke - w, \tag{31b}$$

as in (11).

B. ETM design

In order to construct a suitable ETM by means of the design conditions presented above, we first need to obtain the functions H and W and constants L and γ satisfying Condition 1. To find $H(x, w)$ and L as in (18), we first derive from (31b) that

$$|\dot{e}| \leqslant |x \sin x| + |kx - w| + k|e|. \tag{32}$$

Given that $|x \sin x| \leqslant |x|$, we obtain

$$|\dot{e}| \leqslant |x| + |kx - w| + k|e|. \tag{33}$$

By taking $W(e) = |e|$, we find that (18) is satisfied with

$$L = k \tag{34}$$

$$H(x, w) = |x| + |kx - w|. \tag{35}$$

Now consider the candidate storage function

$$V(x) = 2kx^2. \tag{36}$$

Observe that this function only constitutes a valid candidate storage function, in the sense that one can find $\mathcal{K}_\infty$-functions

$\underline{\alpha}_V$ and $\bar{\alpha}_V$ such that (19) holds, for $k > 0$. Given (30), we have that

$$\langle \nabla V(x), f(x,e,w) \rangle = -4kx^2 \sin x - 4k^2x^2 - 4k^2xe + 4kxw. \quad (37)$$

Since $-x^2 \sin x \leqslant x^2$, we obtain

$$\langle \nabla V(x), f(x,e,w) \rangle \leqslant (4k - 4k^2)x^2 - 4k^2xe + 4kxw. \quad (38)$$

To see whether we can find constants γ and θ such that (20) holds, consider the following

$$\langle \nabla V(x), f(x,e,w) \rangle + H^2(x,w) + \varrho(x) - \gamma^2 W^2(e) \leqslant \varrho(x) + (4k - 4k^2)x^2 - 4k^2xe + 4kxw + H^2(x,w) - \gamma^2 e^2. \quad (39)$$

By using Lemma 1, we obtain that

$$H^2(x,w) \leqslant 2x^2 + 2(kx - w)^2 \quad (40a)$$
$$-4k^2xe \leqslant 2x^2 + 2k^4e^2. \quad (40b)$$

Substitution of (40) in (39) yields,

$$\langle \nabla V(x), f(x,e,w) \rangle + H^2(x,w) + \varrho(x) - \gamma^2 W^2(e) \leqslant \varrho(x) + (4 + 4k - 2k^2)x^2 + 2w^2 + (2k^4 - \gamma^2)e^2. \quad (41)$$

By choosing $\varrho(x) = \rho x^2$, we can conclude from the latter inequality that (20) is satisfied for $k > 1+\sqrt{3+\frac{\varrho}{2}}$, $\gamma = \sqrt{2}k^2$, $p = 2$, $\mu = 2(k^2 - 2k - (2 + \frac{\varrho}{2}))$ and $\theta = \sqrt{\frac{1}{k^2 - 2k - (2+\frac{\varrho}{2})}}$.

At last, we need to construct the function $\Psi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}$ satisfying (24) and (25). By means of Lemma 1, we have that

$$M_1 \geqslant \varrho + (1 - \varepsilon_1)H^2 + (1 - \varepsilon_1^{-1})\gamma^2 W^2 \phi^2 \quad (42a)$$
$$M_2 \geqslant \varrho + (1 - \varepsilon_2)H^2 - \gamma(2\phi L + \gamma(1 + \phi^2 \varepsilon_2^{-1}))W^2, \quad (42b)$$

where the arguments of W, H and ϱ are omitted. As H is a function of the disturbance w, consider the following lower-bound that is independent of w

$$H^2(x,w) \geqslant x^2. \quad (43)$$

On the basis of (42) and (43), we can now define the function Ψ that satisfies (24) and (25)

$$\Psi(x,e,\tau) = \begin{cases} \Psi_1(x,e,\tau), & \text{for } 0 \leq \tau \leq \tau_{miet} \\ \Psi_2(x,e,\tau), & \text{for } \tau > \tau_{miet}, \end{cases} \quad (44)$$

where

$$\Psi_1 = \rho x^2 + \max(0, (1 - \varepsilon_1)x^2 + (1 - \varepsilon_1^{-1})\gamma^2 \phi^2 e^2) \quad (45a)$$
$$\Psi_2 = (\rho + (1 - \varepsilon_2))x^2 - \gamma(2\phi L + \gamma(1 + \phi^2 \varepsilon_2^{-1}))e^2. \quad (45b)$$

For the simulation results which are presented next, we choose $k = 4$ and $\rho = 2$ resulting in an induced $\mathcal{L}_2$-gain of $\theta = \sqrt{\frac{1}{5}}$ and a minimal inter-event time $\tau_{miet} = \frac{\tau_{mati}}{\delta_{max}+1}$ with, based on (9), $\tau_{mati} = 5.715 \cdot 10^{-2}$. Observe that the $\mathcal{L}_2$-gain can be reduced by increasing the control gain k. However, this also affects γ resulting in smaller minimal and average inter-event times. Hence, better performance in terms of a lower induced $\mathcal{L}_2$-gain comes at the cost of network utilization.

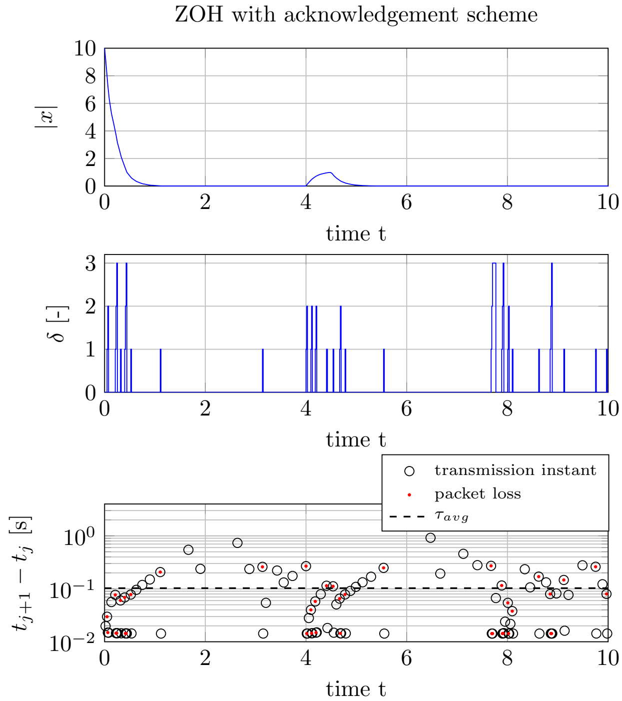

Fig. 2: The evolution of $|x(t)|$, the number of successive packet dropouts $\delta(t)$ and the inter-event times $t_{j+1} - t_j$ for the dynamic ETM given in (6) corresponding to the case that $k = 4$, $\rho = 2$ and $\delta_{max} = 3$ resulting in $\theta = \sqrt{\frac{1}{5}}$.

C. Simulation results

The average inter-event times τ_{avg} presented below are obtained by taking the average over 100 simulations of the system on the time interval $[0, 10]$ and with w a zero-mean random signal with $|w(t)| \leqslant 0.05$ for $t \in [0, 4] \cup (4.5, 10]$, and w(t) = 5 for the time interval $(4, 4.5]$, initial condition $x(0) = 10$ and $\lambda = 0.2$.

Figure 2 shows the evolution of $|x(t)|$, the number of successive packet dropouts $\delta(t)$ and the inter-event times $t_{j+1} - t_j$ for the dynamic ETM given in (6) with $\delta_{max} = 3$. Observe that despite the presence of packet losses and the disturbance w, the average inter-event time is significantly larger than the enforced lower-bound.

Figure 3 illustrates how the maximum allowable number of successive packet losses δ_{max} affects the average inter-event time τ_{avg} relative to the minimal inter-event time τ_{miet} for both the dynamic ETM as given in (6) and the static ETM as given in (29). Observe that, due to the noise, the inter-event times of the static ETMs are close to the MIETs, *i.e.*, $\tau_{avg}/\tau_{miet} \approx 1$ as observed in [2]. In contrast to the static ETMs, the dynamic ETMs show a reduction in network utilization of at least 80% with respect to time-triggered communication based on τ_{miet} despite the occurrence of packet losses.

VII. Conclusion

In this work, we proposed a *dynamic* event-triggered control scheme for a class of non-linear systems which guarantees

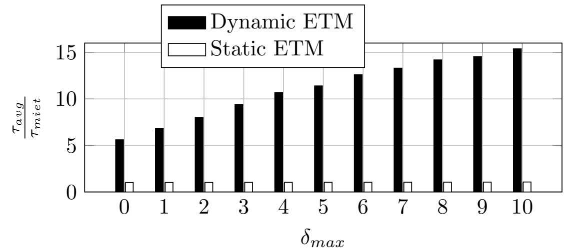

Fig. 3: Tradeoff between the maximum allowable number of successive packet losses and the relative average inter-event time τ_{avg}/τ_{miet} for both the dynamic ETM as given in (6) and the static ETM as given in (29) for the case $k = 4$ and $\rho = 2$ resulting in $\theta = \sqrt{\frac{1}{5}}$.

simultaneously a finite $\mathcal{L}_p$-gain and a robust positive MIET despite of the presence of packet losses. To be more concrete, the proposed ETC scheme allows a maximum allowable number of successive packet losses without jeopardizing the desired performance properties. This forms an important extension to ETC scheme developed in [4] in which packet loss was excluded.

In a numerical example we illustrated that the dynamic ETC algorithm shows a significant reduction of communication with respect to the corresponding time-triggered implementation in contrast to static ETC strategies which approximate the time-triggered solution. Moreover, the numerical results showed that the proposed theory allows a clear tradeoff between performance, in terms of an $\mathcal{L}_p$ gain, robustness, in terms of a maximum number of allowable successive packet losses, and network utilization, in terms of average transmission intervals.

Building upon [10], [12], the proposed ETC methodology allows to include variable delays and output-based and/or decentralized triggering as possible extension.

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