Observer Design for a Class of Piecewise Affine Hybrid Systems

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ABSTRACT

A methodology for the design of observers is proposed for a special class of hybrid dynamical systems, which are motivated by traffic and manufacturing applications. The class of hybrid systems is characterized as switched system models with constant drift and constant output, rendering all subsystems unobservable by themselves. However, an observer can still be derived due to the fixed switching pattern, even though the switching times may be unknown. A main step in the observer design methodology is the usage of a discrete-time linear observer based on the discretized hybrid dynamics at the event times that are visible. Using this step, a continuous-time observer is built that incorporates additional modes compared to the original hybrid system. This continuous-time observer is shown to asymptotically reconstruct the state of the original system under suitable assumptions. Manufacturing and traffic applications are used to illustrate the proposed observer design methodology.

Categories and Subject Descriptors

D.4.8 [**Performance**]: Modeling and prediction

Keywords

Observer design, Piecewise affine hybrid systems, Manufacturing systems, Traffic applications

1. INTRODUCTION

Nowadays, logic decision making and control actions are combined with continuous (physical) processes in many technological (cyber-physical) systems. Such systems are labeled *hybrid* as they have interacting continuous and discrete dynamics. Hybrid models are not only important in such man-made systems, but also in describing the behavior of many mechanical, biological, electrical and economical systems. Therefore, in the past decades, the structural properties of hybrid systems have been investigated by many researchers. This led to various techniques for controller synthesis, see e.g. [14, 18] for recent overviews. However, many of these controller synthesis methods are based on the assumption that the full state variable of the hybrid system is available, which is hardly ever the case in practice. This renders the *design of observers* for hybrid systems, providing good estimates of both continuous and discrete states, of crucial importance. Despite this high practical relevance, there are only a few results on hybrid observer design, see, e.g., [2, 5, 7, 12, 17, 21, 24] and the references therein. Theoretical results regarding the fundamental question of the existence of an observer are related to notions such as final state observability, reconstructability and final state determinability on which also some work in the area of hybrid systems appeared, see, for instance, [4, 6, 8, 9, 11, 23].

In this paper we are interested in designing observers for a special class of piecewise affine hybrid systems ($PWAHS$) [11], motivated by switching servers in manufacturing systems serving multiple products consecutively, and traffic applications such as signalized intersections. The considered hybrid system is autonomous with the mode dynamics consisting of constant drift and the output within a mode being constant. Only at some times the output reveals information about the currently active mode. In particular, this implies that all subsystems are unobservable, eliminating many currently available solutions for synthesizing hybrid observers proposed in the literature. Furthermore, the mode transitions are state-dependent, which is a property that will turn out to be useful, and are therefore a priori unknown. In addition, during a mode transition some specific state variables might exhibit jumps. The order in which the modes are traversed is fixed and periodic.

For this class of PWAHS, of which the full details are specified later, we propose a methodology for designing continuous-time observers. This methodology consists of a few main steps. First, the system is sampled (with varying sampling periods) at so-called visible event times, i.e., times at which the output changes during a mode transition, resulting in a linear time-varying periodic system. Based on the resulting sampled system a periodic discrete-time observer is derived with the guarantee that the observer's state converges asymptotically to the (original) system's state. Next, this observer is used as a stepping stone for designing an observer

HSCC'13, April 8–11, 2013, Philadelphia, Pennsylvania, USA.

in continuous time. This requires the inclusion of additional modes in the observer structure and additional reset laws at visible event times to ensure the asymptotic recovery of the system's state. A formal proof of the asymptotic recovery of the system's state is provided. Via an example of a traffic application we demonstrate the effectiveness of the proposed observer.

The remainder of this paper is organized as follows. Section 2 introduces the considered class of PWAHS and presents a two-buffer switching server as an introductory example. Section 3 presents the sampled system at the event times. In Section 4 the method for the observer design is presented. First, a discrete-time observer is presented for the sampled system. Next, the continuous-time observer is presented. In Section 5, a signalized traffic intersection with three flows is presented for which an observer is derived. Conclusions are provided in Section 6.

Nomenclature

In this paper we use $\{e_1, e_2, \ldots, e_M\}$ as the standard orthonormal basis in $\mathbb{R}^M$ in which e_i is the vector which contains a 1 at the i-th entry, and zeros elsewhere. By $\mathbb{R}_+$ we denote the set of non-negative reals, i.e., $\mathbb{R}_+ := [0, \infty)$. Furthermore, the product of matrices is considered as a left multiplication, i.e., $\prod_{i=1}^{3} A_i = A_3 A_2 A_1$.

2. CLASS OF PIECEWISE AFFINE HYBRID SYSTEMS

In this section, we present the dynamics of the class of piecewise affine hybrid systems (PWAHS) studied in this paper. Before doing so, we first present an illustrative example of a manufacturing system to motivate the structure of the class. In fact, this manufacturing system is used as a running example throughout the paper.

2.1 Illustrative example

Consider a single server that serves two different job types denoted by $n = 1, 2$, see Figure 1. Each job type n has a separate buffer in which x_n jobs are stored. Jobs arrive at buffer 1 with a constant arrival rate denoted by $\lambda_1 > 0$. After service in buffer 1 the jobs move to buffer 2. The server can only serve one job type at a time and operates based on a clearing policy, i.e., it completely empties the buffer of one job type before it switches to the next job type. The processing speed of job type n is denoted by $\mu_n > 0$. Switching to job type n requires a setup time with duration $\gamma_n \geq 0$ and at least one setup time is non-zero, i.e., $\gamma_1 + \gamma_2 > 0$. The only (measurement) information we get from the server is when the server is processing job type 2.

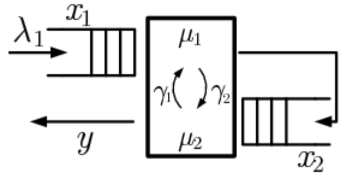

Figure 1: Two-product switching server.

To model this server system, we use a continuous state that consists next to the buffer contents x_n, $n = 1, 2$, also of the remaining setup time at the server, denoted by x_0. Therefore, $x = \begin{bmatrix} x_0 & x_1 & x_2 \end{bmatrix}^\top \in \mathbb{R}_+^{N+1}$ with $N = 2$ being the number of job types and $\mathbb{R}_+ = [0, \infty)$. For each job type n the system has two modes, one for setting up to serve the job type and the other for serving the job type. Hence, this results in $Q = 4$ modes (discrete states) in this case and the modes are denoted by $q \in \mathcal{Q} := \{1, 2, \ldots, Q\}$. The modes 1, 2, 3, and 4 represent setting up the server to serve job type 1, serving job type 1, setting up the server to serve job type 2, and serving job type 2, respectively. Note that the order in which the modes are traversed is fixed. The system evolves from mode 1 via modes 2, 3, and mode 4 back to mode 1 after which the cycle is repeated. In each mode $q \in \mathcal{Q}$ the continuous state x has a constant drift vector, denoted by f_q, i.e., $\dot{x} = f_q$. For the example system, these drift vectors are given by

$$f_1 = f_3 = \begin{bmatrix} -1 \\ \lambda_1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ \lambda_1 - \mu_1 \\ \mu_1 \end{bmatrix}, \quad f_4 = \begin{bmatrix} 0 \\ \lambda_1 \\ -\mu_2 \end{bmatrix}.$$

Furthermore, a transition occurs in modes 1 and 3 to the next modes 2 and 4, respectively, when $x_0 = 0$ indicating that the setup time has elapsed. A transition from mode 2 to mode 3 occurs when $x_1 = 0$ (buffer of job type 1 is empty) and from mode 4 to mode 1 when $x_2 = 0$ (buffer of job type 2 is empty).

Due to the cyclic behavior in the way the nodes are traversed, it holds at the k-th event time t_k, $k \in \mathbb{N}_{\geq 1}$, that the system switches from mode $q = ((k-1) \bmod Q) + 1$ to the next mode $\sigma(q)$ where $\sigma : \mathcal{Q} \to \mathcal{Q}$ is given by

$$\sigma(q) := 1 + (q \bmod Q). \tag{1}$$

In addition, at the event time t_k the setup time x_0 instantaneously increases with constant $\alpha_q(t_k^-) \in \mathbb{R}^+$, $q \in \mathcal{Q}$, given by

$$\alpha_1 = \alpha_3 = 0, \quad \alpha_2 = \gamma_2, \quad \alpha_4 = \gamma_1,$$

i.e., we have a reset of the continuous state variable given by

$$x(t_k^+) = \begin{bmatrix} x_0(t_k^+) & x_1(t_k^+) & x_2(t_k^+) \end{bmatrix}^\top = \begin{bmatrix} \alpha_{q(t_k^-)} & x_1(t_k^-) & x_2(t_k^-) \end{bmatrix}^\top = x(t_k^-) + \alpha_q e_1 \tag{2}$$

(as $x_0(t_k^-) = 0$, see also Lemma 2.7 below), where t_k^- denotes the time just before the k-th event. Note that e_1 is the unit vector with a 1 at the first entry and zeros elsewhere.

This reset law shows that discontinuities only appear in x_0, while x_n, $n = 1 \ldots, N$, evolve continuously in time. Combining the above, leads to the following overall dynamics that can be compactly written as

$$\left.\begin{aligned} \dot{x} &= f_q \\ \dot{q} &= 0 \\ y &= h_q \end{aligned}\right\} \qquad \text{if } e_{k_q}^\top x \geq 0, \tag{3a}$$

$$\left.\begin{aligned} x^+ &= x + \alpha_q e_1 \\ q^+ &= \sigma(q) \end{aligned}\right\} \qquad \text{if } e_{k_q}^\top x = 0 \tag{3b}$$

with $k_1 = k_3 = 1$, $k_2 = 2$ and $k_4 = 3$ selecting the flow and jump sets (recall that e_{k_q} is the k_q-th unit vector in

$\{e_1, e_2, \ldots, e_{N+1}\}$). Note that we expressed the example in terms of the modeling framework of jump-flow systems advocated in [13]. In fact, solutions/executions of the system under study can be interpreted in the sense of [13]. Initial conditions for this system are given by $x(0) \in \mathbb{R}_+^{N+1}$ with $x_0(0) = \alpha_Q$, and $q(0) = 1$. Besides we use the convention that $t_0 = 0$.

Measurement information regarding the knowledge of when the server is processing job type 2 is included via the output y, with

$$h_1 = h_2 = h_3 = 0, \qquad h_4 = 4.$$

Hence, as long as the system is in mode 4, this is directly seen in the output y. When the system is in one of the other modes $q \in \mathcal{Q} \setminus \{4\}$, the output y is equal to 0 and no information is available from the server.

2.2 General dynamics

In this section we provide the general description of the class of PWAHS under study, which includes the single server system discussed in the previous section as a particular case. Essentially, the general dynamics of the class of systems is given by (3) with continuous state

$$x = \begin{bmatrix} x_0 & x_1 & \ldots & x_N \end{bmatrix}^\top \in \mathbb{R}_+^{N+1}$$

with $N \in \mathbb{N}_{\geq 1}$, discrete state $q \in \mathcal{Q} = \{1, 2, \ldots, Q\}$ with $Q \in \mathbb{N}_{\geq 1}$ and output $y \in \mathcal{Q}_0 := \mathcal{Q} \cup \{0\}$. The *data* of the system are given by the drift vectors $f_q \in \mathbb{R}^{N+1}$, the outputs $h_q \in \mathcal{Q}_0$, and $k_q \in \{1, 2, \ldots, N+1\}$ for each mode $q \in \mathcal{Q}$ together with the reset parameters α_q, $q \in \mathcal{Q}$. In addition, we have for outputs h_q, $q \in \mathcal{Q}$ that $h_q \in \{0, q\}$ for all $q \in \mathcal{Q}$. As in the single server system example, the general dynamics exhibits a *cycle*, i.e., a sequence of Q consecutive modes being repeated over time. Some other special characteristics in the data being inherited from server-like systems in manufacturing and traffic applications are summarized below.

Assumption 2.1. *For all $q \in \mathcal{Q}$ it holds that $e_{k_q}^\top f_q < 0$ and $e_i^\top f_q \geq 0$ when $i \in \{1, \ldots, N+1\} \setminus \{k_q\}$.*

This assumption guarantees that only one continuous state component decreases (being the one that also triggers the mode transition).

Assumption 2.2. *For all $q \in \mathcal{Q}$*

$$e_1^\top f_q = -1, \qquad \text{if } k_q = 1, \tag{4a}$$

$$e_1^\top f_q = 0, \qquad \text{if } k_q \neq 1. \tag{4b}$$

This assumption expresses that x_0 is indeed a timer-related variable with only 0 and -1 as slopes. In case x_0 acts as a timer that triggers the next event (i.e., $k_q = 1$) then $e_1^\top f_q = -1$, otherwise it is 0.

Assumption 2.3. *For all $q \in \mathcal{Q}$ it holds that*

$$\alpha_q > 0 \Leftrightarrow k_{\sigma(q)} = 1. \tag{5}$$

This assumption states that if a mode transition in mode q governs a jump in the state x_0, this state x_0 decreases in mode $\sigma(q)$ and triggers the next mode transition (and vice versa).

Assumption 2.4.

$$\sum_{q=1}^{Q} \alpha_q > 0. \tag{6}$$

If translated in terms of the server system example, this assumption states that during a cycle of modes at least one setup of non-zero duration is present.

Assumption 2.5. *For all $q \in \mathcal{Q}$ it holds that*

$$k_q = 1 \Rightarrow k_{\sigma(q)} \neq 1. \tag{7}$$

This assumption expresses that a setup mode is followed by an operational mode ($k_q \neq 1$).

Assumption 2.6. *There is at least one $q \in \mathcal{Q}$ such that $h_q = q$.*

This assumption states that we get at least some information from the system.

Throughout the paper we assume that all the mentioned assumptions are true (without further reference).

2.3 Basic results

In this section we derive some basic results for the class of PWAHS under study. To do so, let us denote, as before, by t_k the k-th time occurrence of a transition and $t_0 = 0$. Then we can prove the following lemma.

Lemma 2.7. *For any trajectory of the PWAHS (3) it holds that $x_0(t_k^-) = 0$ for all $k \in \mathbb{N}_{\geq 1}$.*

Proof. We prove this statement using induction. Suppose the statement holds for some $k \in \mathbb{N}_{\geq 1}$, i.e., $x_0(t_k^-) = 0$. At event time t_k, when going from mode q to mode $\sigma(q)$, two situations can occur, being $\alpha_q > 0$ or $\alpha_q = 0$. In the first case, it holds that $x_0(t_k^+) = x_0(t_k^-) + \alpha_q$. Then due to (3), (4a) and (5) it follows that $\dot{x}_0 = -1$ for $t \in [t_k, t_{k+1})$ until $x_0 = 0$. Hence $x_0(t_{k+1}^-) = 0$. In the second case, i.e., $\alpha_q = 0$, no jump occurs and $x_0(t_{k+1}^-) = x_0(t_k^-)$ and $\dot{x} = 0$ for $t \in [t_k, t_{k+1})$ due to (4b) and (5). Therefore, $x_0(t_{k+1}^-) = x_0(t_k^-) = 0$. To complete the proof we need $x_0(t_1^-) = 0$. If $\alpha_Q > 0$, $x_0(t_1^-) = 0$ follows by the same reasoning as in the first case above. If $\alpha_Q = 0$, we immediately have $x_0(t_0) = 0$ and can use the reasoning in the second case to conclude $x_0(t_1^-) = 0$, thereby completing the proof. □

Lemma 2.8. *Consider $q \in \mathcal{Q}$ such that $\alpha_q > 0$. The dwell time in mode $\sigma(q)$ is equal to α_q.*

Proof. From Assumption 2.2 and Assumption 2.3 we know that if $\alpha_q > 0$ we have $e_1^\top f_{\sigma(q)} = -1$ and $k_{\sigma(q)} = 1$. Due to (3b) and Lemma 2.7 it holds that $x_0(t_k^+) = \alpha_q$ for the event at time t_k where the system switches from mode q to mode $\sigma(q)$. In addition, in mode $\sigma(q)$ state x_0 decreases according to $\dot{x}_0 = -1$ until $x_0(t_{k+1}^-) = 0$. Hence, the dwell time in mode $\sigma(q)$ is therefore given by

$$\frac{\alpha_q}{-e_1^\top f_{\sigma(q)}} = \alpha_q.$$

□

For the class of PWAHS the following statements can be made regarding Zeno behavior and fixed points:

Lemma 2.9. *Zeno behavior is not present in system (3).*

PROOF. Assumption 2.3 and Assumption 2.4 imply that there exists at least one mode in the cycle with $\alpha_q > 0$, $q \in \mathcal{Q}$. Lemma 2.8 shows that the dwell time in mode $\sigma(q)$ is α_q. Therefore, the dwell time of a cycle is bounded away from zero given the cyclic behavior, and no Zeno behavior occurs. □

LEMMA 2.10. *System* (3) *does not contain any fixed points.*

PROOF. The condition $e_{k_q}^\top f_q < 0$ in Assumption 2.1 guarantees that $e_{k_q}^\top x$ decreases at constant rate, until it reaches the jump criterion $e_{k_q}^\top x = 0$. Therefore, every mode is left in finite time. Since Zeno behavior is excluded, the system has no fixed points. □

LEMMA 2.11. *The system* (3) *is a positive system in the sense that if* $x(0) \in \mathbb{R}_+^{N+1}$ *with* $x_0(0) = \alpha_Q$*, and* $q(0) = 1$*, then* $x(t) \in \mathbb{R}_+^{N+1}$ *for all* $t \in \mathbb{R}_+$.

PROOF. The proof follows similar reasoning as the proof of Lemma 2.7 using Assumptions 2.1 and 2.2. □

2.4 Visible and invisible modes and event times

In the remainder of this paper we use the following notations in which we make a distinction between visible and invisible modes, and visible and invisible events.

A mode $q \in \mathcal{Q}$ is called *visible* if $h(q) = q$ and otherwise it is called *invisible*. The set of visible modes is denoted by $\mathcal{Q}_v$, i.e,

$$\mathcal{Q}_v = \{q \in \mathcal{Q} \mid h_q = q\}. \tag{8}$$

Transitions to and from visible modes are called *visible events* and transitions from invisible modes to invisible modes are called *invisible events*. To describe the visible events we define the set

$$\mathcal{V} = \mathcal{Q}_v \cup \{q \in \mathcal{Q} \mid \sigma(q) \in \mathcal{Q}_v\},$$

which is enumerated as $\{v_1, v_2, \ldots, v_V\} \subseteq \mathcal{Q}$ with $1 \le v_1 < v_2 < \ldots < v_V \le Q$. Using the above notation, if at time t_k the k-th event occurs jumping from mode $q(t_k^-)$ to mode $\sigma(q(t_k^-))$, this event is visible if and only if $q(t_k^-) \in \mathcal{V}$. Hence, loosely speaking, we know when the system enters or leaves visible modes and we know when the corresponding events occur.

In the remainder we will use j as the visible event counter and denote visible event times by $t_j^v = t_{k(j)}$, $j \in \mathbb{N}_{\ge 1}$. where $k(j)$ translates the visible event j into the corresponding ordinary event k, i.e.,

$$k(j) = v_{1+(j-1 \bmod V} + \lfloor (j-1)/V \rfloor \cdot Q, \tag{9}$$

where $\lfloor r \rfloor$ denotes the largest integer smaller than $r \in \mathbb{R}$. Due to the cyclic behavior in the way the (visible) nodes are traversed, it holds that visible event $v_{1+(j-1 \bmod V)}$ has successive visible event $\sigma_v(v_{1+(j-1 \bmod V)})$ where $\sigma_v : \mathcal{V} \to \mathcal{V}$, given by

$$\sigma_v(v_l) := v_{1+(l \bmod V)}, \qquad l = 1, 2, ..., V. \tag{10}$$

3. SAMPLING THE HYBRID SYSTEM

One of the main ideas in the observer design methodology is to derive the desired continuous-time observer for system (3) from a discrete-time observer. To that end, we sample the system at the visible event times t_j^v, $j \in \mathbb{N}_{\ge 1}$. To easily derive the sampled data, we split its computation into three parts. First the system is sampled at all events t_k, $k \in \mathbb{N}_{\ge 1}$. Second, the state dimension is reduced by removing the timer variable x_0. In the third step the sampled system description is limited to only the visible events.

3.1 Sampling at all event times

Let $x(t_k^-)$ denote the state at time t_k just before the jump from mode $q(t_k^-)$ to mode $\sigma(q(t_k^-))$. Sampling at all event times t_k, $k \in \mathbb{N}_{\ge 1}$, results in the system

$$x(t_{k+1}^-) = \tilde{A}_{q(t_k^-)} x(t_k^-) + \tilde{a}_{q(t_k^-)}, \tag{11a}$$

$$t_{k+1} = t_k + \tilde{C}_{q(t_k^-)} x(t_k^-) + \tilde{c}_{q(t_k^-)} \tag{11b}$$

in which

$$\tilde{A}_{q(t_k^-)} = I + \frac{f_{\sigma(q(t_k^-))} e_{k_{\sigma(q(k))}}^\top}{-e_{k_{\sigma(q(t_k^-))}}^\top f_{\sigma(q(t_k^-))}}, \quad \tilde{a}_{q(t_k^-)} = \alpha_{q(t_k^-)} A_{q(t_k^-)} e_1, \tag{12a}$$

$$\tilde{C}_{q(t_k^-)} = \frac{e_{k_{\sigma(q(t_k^-))}}^\top}{-e_{k_{\sigma(q(t_k^-))}}^\top f_{\sigma(q(t_k^-))}}, \qquad \tilde{c}_{q(t_k^-)} = \alpha_{q(t_k^-)} C_{q(t_k^-)} e_1 \tag{12b}$$

as can be derived from system (3). Note that the system (11) is periodic with period Q in the sense that $\tilde{A}_{k+Q} = \tilde{A}_k$ for $k \in \mathbb{N}_{\ge 1}$, and similar expressions hold for $\tilde{a}_k$, $\tilde{C}_k$ and $\tilde{c}_k$.

REMARK 3.1. *3 Notice that from Assumptions 2.1 and 2.2 we obtain that* $\tilde{A}_k$, $\tilde{a}_k$, $\tilde{C}_k$, *and* $\tilde{c}_k$ *only contain non-negative elements, resulting in a positive system, which is to be expected given Lemma 2.11.*

The sampled system (11) can be used to write the system as a timed automaton [3] with a single clock, due to the linear dynamics and cyclic behavior. Then, the continuous state can be derived via a linear combination of the clock and the sampled state. However, observability and observer design for timed automata considers the discrete state reconstruction, see e.g., [20], which is straightforward for the class of systems under consideration. Furthermore, we are also interested in reconstructing the continuous state.

Example

For the illustrative system in Section 2.1, we obtain a 4-periodic system for which the matrices in (12) are given as

$$\tilde{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\mu_1}{\mu 1 - \lambda 1} & 1 \end{bmatrix}, \qquad \tilde{a}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\tilde{C}_1 = \begin{bmatrix} 0 & \frac{1}{\mu_1 - \lambda_1} & 0 \end{bmatrix}, \qquad \tilde{c}_1 = 0,$$

$$\tilde{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ \lambda_1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \tilde{a}_2 = \begin{bmatrix} 0 \\ \gamma_2 \lambda_1 \\ 0 \end{bmatrix},$$

$$\tilde{C}_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad \tilde{c}_2 = \gamma_2,$$

$$\tilde{A}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\lambda_1}{\mu_2} \\ 0 & 0 & 0 \end{bmatrix}, \qquad \tilde{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\tilde{C}_3 = \begin{bmatrix} 0 & 0 & \frac{1}{\mu_2} \end{bmatrix}, \qquad \tilde{c}_3 = 0,$$

$$\tilde{A}_4 = \begin{bmatrix} 0 & 0 & 0 \\ \lambda_1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \tilde{a}_4 = \begin{bmatrix} 0 \\ \gamma_1\lambda_1 \\ 0 \end{bmatrix},$$

$$\tilde{C}_4 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad \tilde{c}_4 = \gamma_1.$$

3.2 State reduction

From Lemma 2.7 we have $x_0(t_k^-) = 0$ for all $k \in \mathbb{N}_{\geq 1}$. Therefore, we can consider the reduced state $\bar{x} = \begin{bmatrix} x_1 & \dots & x_N \end{bmatrix}^\top = \begin{bmatrix} 0 & I \end{bmatrix} x \in \mathbb{R}_+^N$, for which the dynamics at the event times becomes

$$\bar{x}(t_{k+1}^-) = \bar{A}_{q(t_k^-)}\bar{x}(t_k^-) + \bar{a}_{q(t_k^-)}, \tag{13a}$$

$$t_{k+1} = t_k + \bar{C}_{q(t_k^-)}\bar{x}(t_k^-) + \bar{c}_{q(t_k^-)}. \tag{13b}$$

where

$$\bar{A}_{q(t_k^-)} = \begin{bmatrix} 0 & I \end{bmatrix} \tilde{A}_{q(t_k^-)} \begin{bmatrix} 0 & I \end{bmatrix}^\top, \quad \bar{a}_{q(t_k^-)} = \begin{bmatrix} 0 & I \end{bmatrix} \tilde{a}_{q(t_k^-)},$$

$$\bar{C}_{q(t_k^-)} = \tilde{C}_{q(t_k^-)} \begin{bmatrix} 0 & I \end{bmatrix}^\top, \qquad \bar{c}_{q(t_k^-)} = \tilde{c}_{q(t_k^-)}.$$

Note that due to this reduction no discontinuities in $\bar{x}(t)$ occur, as they only appear in x_0 (2). Hence, $\bar{x}(t_k^-) = \bar{x}(t_k^+) = \bar{x}(t_k)$.

Example
For the illustrative system in Section 2.1, these reduced matrices are as follows:

$$\bar{A}_1 = \begin{bmatrix} 0 & 0 \\ \frac{\mu_1}{\mu_1-\lambda_1} & 1 \end{bmatrix}, \ \bar{a}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \bar{C}_1 = \begin{bmatrix} \frac{1}{\mu_1-\lambda_1} & 0 \end{bmatrix}, \ \bar{c}_1 = 0,$$

$$\bar{A}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{a}_2 = \begin{bmatrix} \gamma_2\lambda_1 \\ 0 \end{bmatrix}, \ \bar{C}_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \bar{c}_2 = \gamma_2,$$

$$\bar{A}_3 = \begin{bmatrix} 1 & \frac{\lambda_1}{\mu_2} \\ 0 & 0 \end{bmatrix}, \quad \bar{a}_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \bar{C}_3 = \begin{bmatrix} 0 & \frac{1}{\mu_2} \end{bmatrix}, \quad \bar{c}_3 = 0,$$

$$\bar{A}_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{a}_4 = \begin{bmatrix} \gamma_1\lambda_1 \\ 0 \end{bmatrix}, \ \bar{C}_4 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \bar{c}_4 = \gamma_1.$$

3.3 Sampling at visible event times

Let $\bar{x}(t_j^v)$ denote the reduced state vector at t_j^v, the j^{th} visible event. From (13) it is clear that sampling at the visible events results in the system

$$\bar{x}(t_{j+1}^v) = A_{q(t_j^{v-})}\bar{x}(t_j^v) + a_{q(t_j^{v-})}, \tag{14a}$$

$$t_{j+1}^v = t_j^v + C_{q(t_j^{v-})}\bar{x}(t_j^v) + c_{q(t_j^{v-})}. \tag{14b}$$

Since we have V different visible events, the system (14) is a periodic linear system with period V. The system matrices in (14) for all $j \in \mathbb{N}_{\geq 1}$ are presented below. For $j \neq lV$, $l \in \mathbb{N}_{\geq 1}$, we have

$$A_j = \prod_{q=v_j}^{\sigma_v(v_j)-1} \bar{A}_q, \tag{15a}$$

$$a_j = \sum_{q=v_j}^{\sigma_v(v_j)-1} \left(\prod_{r=q+1}^{\sigma_v(v_j)-1} \bar{A}_r \right) \bar{a}_q, \tag{15b}$$

$$C_j = \sum_{q=v_j}^{\sigma_v(v_j)-1} \bar{C}_q \prod_{r=v_j}^{q-1} \bar{A}_r, \tag{15c}$$

$$c_j = \sum_{q=v_j}^{\sigma_v(v_j)-1} \bar{c}_q + \sum_{q=v_j+1}^{\sigma_v(v_j)-1} \bar{C}_q \sum_{r=v_j}^{q-1} \left(\prod_{s=r+1}^{q-1} \bar{A}_s \right) \bar{a}_r \tag{15d}$$

For $j = lV$, $l \in \mathbb{N}_{\geq 1}$, we have

$$A_j = \prod_{q=1}^{\sigma_v(v_j)-1} \bar{A}_q \prod_{q=v_j}^{Q} \bar{A}_q, \tag{15e}$$

$$a_j = \sum_{q=v_j}^{Q+\sigma_v(v_j)-1} \left(\prod_{r=q+1}^{Q+\sigma_v(v_j)-1} \bar{A}_r \right) \bar{a}_q, \tag{15f}$$

$$C_j = \sum_{q=v_j}^{Q+\sigma_v(v_j)-1} \bar{C}_q \prod_{r=v_j}^{q-1} \bar{A}_r, \tag{15g}$$

$$c_j = \sum_{q=v_j}^{Q+\sigma_v(v_j)-1} \bar{c}_q + \sum_{q=v_j+1}^{Q+\sigma_v(v_j)-1} \bar{C}_q \sum_{r=v_j}^{q-1} \left(\prod_{s=r+1}^{q-1} \bar{A}_s \right) \bar{a}_r \tag{15h}$$

with $A_{j+V} = A_j$, $j, \mathbb{N}_{\geq 1}$, and similar expressions for a_j, C_j, c_j and v_j, $j \in \mathbb{N}_{\geq 1}$.

Example
For the illustrative system in Section 2.1, we obtain a 2-periodic system (14) for which the matrices above are given by

$$A_1 = \begin{bmatrix} 1 & \frac{\lambda_1}{\mu_2} \\ 0 & 0 \end{bmatrix}, \qquad a_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & \frac{1}{\mu_2} \end{bmatrix}, \qquad c_1 = 0,$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ \frac{\mu_1}{\mu_1-\lambda_1} & 1 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} \gamma_2\lambda_1 \\ \frac{\lambda_1\mu_1\gamma_1}{\mu_1-\lambda_1} \end{bmatrix},$$

$$C_2 = \begin{bmatrix} \frac{1}{\mu_1-\lambda_1} & 0 \end{bmatrix}, \qquad c_2 = \frac{\gamma_1\mu_1}{\mu_1-\lambda_1} + \gamma_2.$$

4. OBSERVER DESIGN

Notice that we only receive information about the system's state when visible events occur. Therefore, from the occurrence of visible events we need to reconstruct the system's state.

We therefore build an observer in two steps. First, starting from the dynamics (14) we build a linear time-varying (periodic) discrete-time observer to reconstruct the system's state at visible event times. Next, we use the dynamics (3) to make an open-loop prediction of the system's state, which is corrected (if necessary) at the next visible event time.

4.1 Discrete-time observer at visible event times

Our first goal is to build an observer which reconstructs the state at visible event times t_j^v, as described by the dynamics (14). To that end, we can use a Luenberger observer

$$\hat{\bar{x}}(t_{j+1}^v) = A_{q(t_j^{v-})}\hat{\bar{x}}(t_j^v) + a_{q(t_j^{v-})} + L_{q(t_j^{v-})}(t_{j+1}^v - \hat{t}_{j+1}^v), \tag{16a}$$

$$\hat{t}_{j+1}^v = t_j^v + C_{q(t_j^{v-})}\hat{\bar{x}}(t_j^v) + c_{q(t_j^{v-})}, \tag{16b}$$

where $L_1, L_2, ..., L_V$ are the observer gains. Evolution of the observation error $\bar{e}(t_j^v) = \bar{x}(t_j^v) - \hat{\bar{x}}(t_j^v)$ is given by

$$\bar{e}(t_{j+1}^v) = \left[A_{q(t_j^{v-})} - L_{q(t_j^{v-})}C_{q(t_j^{v-})}\right]\bar{e}(t_j^v).$$

ASSUMPTION 4.1. *There exist $L_1, L_2, ..., L_V$ such that the observer* (16) *leads to*

$$\lim_{j\to\infty} \|\bar{e}(t_j^v)\| = 0.$$

Finding observer gains for this periodic system satisfying (4.1) is a known observer design problem, cf. [16, 19, 22]. For the example presented in the next section, we exploit the periodicity of this system and use a simple sequential algorithm presented in [15] for determining the time-varying observer gains to guarantee deadbeat convergence to zero at visible event times.

4.2 Continuous-time observer

Starting from the *reduced* state estimates at visible event times, as generated by the observer (16), we will now provide a state estimate $\hat{x} \in \mathbb{R}^{N+1}$ of the full state x of (3) (i.e., including x_0). In fact, we will guarantee that the estimated states $\hat{x}(t_j^{v+})$ just after visible events satisfy

$$[0\ I]\hat{x}(t_j^{v+}) = \hat{\bar{x}}(t_j^v), \tag{17}$$

$j \in \mathbb{N}_{\geq 1}$, indicating that just after the visible events the estimated states (without timer x_0) and the estimates $\hat{\bar{x}}(t_j^v)$ of the discrete-time observer (16) coincide. In addition, we use the dynamics (3) to make an open-loop prediction of the system's state after visible events. This open-loop prediction is updated using the observer (16) as soon as a new visible event happens.

However, notice that due to estimation errors, the predicted occurrence of the next visible event, $\hat{t}_{j+1}^v$ based on (16b), can be either sooner or later than the actual occurrence of the next visible event t_{j+1}^v. In the latter case ($t_{j+1}^v \leq \hat{t}_{j+1}^v$) we can simply update the observer state according to (16), but in the former case we cannot (yet) use (16), as the duration to the next visible event $t_{j+1}^v - \hat{t}_{j+1}^v$ is not known yet. Hence in this case ($t_{j+1}^v > \hat{t}_{j+1}^v$), we have to determine the continuous-time observer dynamics for the period from predicted to actual occurrence of the next visible event. To do so, we introduce additional modes (called waiting modes) for the continuous-time observer. We therefore extend the set $\mathcal{Q}$ of modes, by defining the set of observer modes as

$$\hat{\mathcal{Q}} := \mathcal{Q} \cup \left\{ v_l + \frac{1}{2} \,\middle|\, l = 1, 2, \ldots, V \right\},$$

where $v_l + \frac{1}{2}$, $l = 1, 2, \ldots, V$ are labels to denote the waiting modes. Furthermore, we define the mode transition map $\hat{\sigma} : \hat{\mathcal{Q}} \to \hat{\mathcal{Q}}$, as

$$\hat{\sigma}(\hat{q}) := \begin{cases} \hat{q} + \frac{1}{2} & \text{if } \hat{q} \in \mathcal{V} \\ \sigma(\hat{q}) & \text{if } \hat{q} \in \mathcal{Q} \setminus \mathcal{V} \\ \sigma\left(\hat{q} - \frac{1}{2}\right) & \text{if } \hat{q} \in \hat{\mathcal{Q}} \setminus \mathcal{Q} \end{cases}$$

The waiting modes will only be used after an expected visible event which has not yet happened, i.e., when $\hat{t}_{j+1}^v \leq t < t_{j+1}^v$. For the additional waiting modes of the observer, we have to determine a drift vector for the state estimate. Notice from the observer (16) that at the visible event times we update the state estimate according to (16a). That is, we add $L_{q(t_j^{v-})}$ times the amount of time that the actual visible event is later than predicted. From this we can derive a continuous evolution of the state estimate keeping $\hat{x}_0$ constant (at zero) and using a drift vector of $L_{q(t_j^{v-})}$ for the remaining state, i.e., $\dot{\hat{x}} = [0\ L_{q(t_j^{v-})}^\top]^\top$, to avoid the need to reset the states at t_{j+1}^v, $j \in \mathbb{N}_{\geq 1}$ (although other choices guaranteeing (17) are fine as well). Furthermore, the output in the waiting mode is identical to the output of the preceding mode, i.e., $h_{\hat{q}} = h_{\hat{q}-\frac{1}{2}}$ for $\hat{q} \in \hat{\mathcal{Q}} \setminus \mathcal{Q}$.

Using the above reasoning, we can deduce a continuous-time observer in the form of a jump-flow system [13]. The discrete-time observer (16) will be embedded in the observer by including the state variables $\tilde{x} \in \mathbb{R}^N$ and $\tilde{t} \in \mathbb{R}$ in the continuous-time observer, which will satisfy $\tilde{x}(t) = \hat{\bar{x}}(t_j^v)$, $t \in [t_j^v, t_{j+1}^v)$ (solutions to (16)), and $\tilde{t}(t) = t_j^v$, $t \in [t_j^v, t_{j+1}^v)$. In between visible events $\tilde{x}$ and $\tilde{t}$ will be constant. This leads to the following flow expressions for the continuous-time observer. When the measurement information equals the estimated output, i.e., $y = \hat{y}$ with $y = h_q$ and $\hat{y} = h_{\hat{q}}$ see (3), the flow expressions are given by

$$\left.\begin{aligned} \dot{\hat{x}} &= f_{\hat{q}} \\ \dot{\hat{q}} &= 0 \\ \dot{\tilde{x}} &= 0 \\ \dot{\tilde{t}} &= 0 \\ \hat{y} &= h_{\hat{q}} \end{aligned}\right\} \quad \text{if } e_{k_{\hat{q}}}^\top \hat{x} \geq 0 \wedge \hat{q} \in \mathcal{Q} \wedge y = \hat{y}, \tag{18a}$$

$$\left.\begin{aligned} \dot{\hat{x}} &= \begin{bmatrix} 0 \\ L_{\sigma_v^{-1}(\hat{q}-\frac{1}{2})} \end{bmatrix} \\ \dot{\hat{q}} &= 0 \\ \dot{\tilde{x}} &= 0 \\ \dot{\tilde{t}} &= 0 \\ \hat{y} &= h_{\hat{q}} \end{aligned}\right\} \quad \text{if } \hat{q} \in \hat{\mathcal{Q}} \setminus \mathcal{Q} \wedge y = \hat{y}. \tag{18b}$$

Note that (18a) describes the normal flow predictions based on model (3), while (18b) corresponds to the predictions in the waiting modes. The jump expressions for the normal flow predictions are given by

$$\left.\begin{aligned} \hat{x}^+ &= \hat{x} + \alpha_{\hat{q}} e_1 \\ \tilde{x}^+ &= \tilde{x} \\ \hat{q}^+ &= \hat{\sigma}(\hat{q}) \\ \tilde{t}^+ &= \tilde{t} \end{aligned}\right\} \quad \text{if } e_{k_{\hat{q}}}^\top \hat{x} = 0 \wedge \hat{q} \in \mathcal{Q} \wedge y = \hat{y}, \tag{18c}$$

following the normal jump expressions based on model (3). Furthermore, (18c) also describes that if a predicted visible event occurs before the actual visible event ($\hat{t}_{j+1}^v < t_{j+1}^v$), the discrete state jumps to a waiting mode.

If the measurement information differs from the estimated output, i.e., $y \neq \hat{y}$, a jump is required, since the system is in a different mode than the observer. Apart from an initial mode mismatch, the measurement difference $y \neq \hat{y}$ occurs via a change in y. In this case, a distinction is made between jumping from a mode without measurement information to a mode with measurement information, i.e., $y \neq \hat{y}$ and $y \neq 0$, and jumping from a mode with measurement information to a mode without measurement information, i.e., $y \neq \hat{y}$ and $y = 0$. In the former case, the measurement information provides information about the mode after the jump ($\hat{q}^+ = y$). In the latter case, the mode after the jump is derived by using the cyclic mode transitions ($\hat{q}^+ = \sigma(\hat{y})$). The jump expressions are given by

$$\left.\begin{aligned} \xi &= \tilde{t} + C_{p(y)}\tilde{x} + c_{p(y)} \\ \hat{x}^+ &= \begin{bmatrix} 0 \\ A_{p(y)} \end{bmatrix}\tilde{x} + \begin{bmatrix} \alpha_{\sigma^{-1}(y)} \\ a_{p(y)} \end{bmatrix} + \begin{bmatrix} 0 \\ L_{p(y)} \end{bmatrix}(t-\xi) \\ \hat{q}^+ &= y \\ \tilde{x}^+ &= \hat{\bar{x}}^+ \\ \tilde{t}^+ &= t \end{aligned}\right\} \ldots$$
$$\ldots\text{if } y \neq \hat{y} \wedge y \neq 0, \tag{18d}$$

$$\left.\begin{aligned} \xi &= \tilde{t} + C_{\sigma^{-1}(\hat{y})}\tilde{x} + c_{\sigma^{-1}(\hat{y})} \\ \hat{x}^+ &= \begin{bmatrix} 0 \\ A_{\sigma^{-1}(\hat{y})} \end{bmatrix}\tilde{x} + \begin{bmatrix} \alpha_{\sigma(\hat{y})} \\ a_{\sigma^{-1}(\hat{y})} \end{bmatrix} + \begin{bmatrix} 0 \\ L_{\sigma^{-1}(\hat{y})} \end{bmatrix}(t-\xi) \\ \hat{q}^+ &= \sigma(\hat{y}) \\ \tilde{x}^+ &= \hat{\bar{x}}^+ \\ \tilde{t}^+ &= t \end{aligned}\right\} \ldots$$
$$\ldots\text{if } y \neq \hat{y} \wedge y = 0, \tag{18e}$$

where we denoted $p(y) = \sigma_v^{-1}(\sigma^{-1}(y))$ and introduced ξ for ease of exposition. Recall that $\hat{\bar{x}}(t) = [0\ I]\hat{x}(t)$. Note that (18d) and (18e) describe the jump expressions at occurrence of a visible event. Based on some knowledge of initial conditions of the original system (3), we initialize the observer as $\hat{x}(0) \in \mathbb{R}_+^{N+1}$ with $\hat{x}_0(0) = \alpha_Q$, $\hat{q}(0) = 1$, $\tilde{x}(0) = \hat{\bar{x}}(0)$ and $\tilde{t}(0) = 0$.

PROPOSITION 4.2. *Suppose Assumption 4.1 holds. Then the continuous-time observer* (18) *asymptotically reconstructs the states* $\bar{x}$ *of the system* (3)*, i.e.,*

$$\lim_{t\to\infty} \|\bar{x}(t) - \hat{\bar{x}}(t)\| = 0.$$

PROOF. Since the continuous-time observer was designed to satisfy $\|[0\ I]x(t_j^{v+}) - [0\ I]\hat{x}(t_j^{v+})\| = \|\bar{x}(t_j^v) - \hat{\bar{x}}(t_j^v)\|$, $j \in \mathbb{N}_{\geq 1}$, it suffices to show that there exists a constant P such that $\|\bar{x}(t) - \hat{\bar{x}}(t)\| \leq P\|\bar{x}(t_j^v) - \hat{\bar{x}}(t_j^v)\|$ for $t \in [t_j^v, t_{j+1}^v)$, $j \in \mathbb{N}_{\geq 1}$. The latter will follow from the observation that the observation error only changes when the observer is in a different mode than the actual system, as we will show below.

Notice that (11) describes the evolution of the mode changes of the system, i.e., the state updates at the switching times t_k. Let $\hat{t}_k$ denote the switching times as predicted by the continuous-time observer. Then we have from (11b):

$$t_{k+1} - \hat{t}_{k+1} = t_k - \hat{t}_k + \tilde{C}_{q(t_k^-)}[x(t_k^-) - \hat{x}(t_k^-)]. \tag{19}$$

Since at time t_j^v we have $\hat{t}_{k(j)} = t_{k(j)}$, it follows from (19) that the duration of the mode difference is a linear function of the initial observer error, i.e., and so is the sum of the durations of mode differences. Finally, since during mode differences the rate of increase is bounded, and the number of invisible modes between two consecutive visible modes is finite, the observation error $\|\bar{x}(t) - \hat{\bar{x}}(t)\|$ on the interval $t \in [t_j, t_{j+1})$ can be upperbounded by $\|\bar{x}(t) - \hat{\bar{x}}(t)\| \leq P\|\bar{x}(t_j^v) - \hat{\bar{x}}(t_j^v)\|$.

Notice that from (19) we also have that $\lim_{k\to\infty} |t_k - \hat{t}_k| = 0$. □

REMARK 4.3. *Using a deadbeat observer the complete state can be asymptotically reconstructed, i.e.,* $\lim_{t\to\infty}\|x(t) - \hat{x}(t)\| = 0$*. For non-deadbeat observers peaking occurs in* $\|x_0(t) - \hat{x}_0(t)\|$ *due to mismatch in event times combined with system jumps* (2)*. However, in most manufacturing and traffic applications one is interested in the buffer sizes or queue lengths, i.e.,* $\hat{\bar{x}}(t)$*.*

REMARK 4.4. *As we observed in Lemma 2.11 and Remark 3.1, we are essentially dealing with a positive system. However, the observer* (16) *does not necessarily guarantee positivity of the state estimates* $\hat{x}(t)$*,* $t \in \mathbb{R}$*. Though the observer* (18) *is well defined and asymptotically recovers the state of the original system, from a physical point of view it would be better to have non-negative state estimates. As we show in the next section by means of an example, it is possible to derive observers that respect the positivity property by generating non-negative state estimates. Note that observer gains satisfying* $(A_v - L_vC_v) \geq 0$ *and* $L_vC_v \geq 0$*,* $v \in \mathcal{V}$*, suffice since*

$$\begin{aligned} \hat{\bar{x}}(t_{j+1}^{v-}) =& (A_{q(t_j^{v-})} - L_{q(t_j^{v-})}C_{q(t_j^{v-})})\bar{x}(t_j^{v-}) + \ldots \\ & \ldots L_{q(t_j^{v-})}C_{q(t_j^{v-})}x(t_j^{v-}) + a_{q(t_j^{v-})}. \end{aligned}$$

In fact, designing directly positive observers is one of the questions for future research.

One direction to pursue in this context is considering the dynamics (14) *only once every* J *time-instances and lift the system (see, for instance, [10]) leading to a linear time-invariant positive system. The positive observation problem for linear discrete time-invariant positive systems has been dealt with in [1], where a necessary and sufficient condition for the existence and the design of a positive linear observer of Luenberger form has been given by means of the feasibility of a linear program (LP). This could form an interesting starting point to obtain positive discrete-time observers of the form* (16a) *(which is doable under certain assumptions). The step towards a continuous-time observers could follow then* mutatis mutandis *the line of reasoning as indicated above.*

An alternative solution leading to positive estimates is to use a projection of $\hat{\bar{x}}(t)$ *on the positive cone* $\Omega = \mathbb{R}_+^N$ *as the estimated state instead of* $\hat{\bar{x}}(t)$*, i.e., use* $P_\Omega\hat{\bar{x}}(t)$ *with* $P_\Omega : \mathbb{R}^N \to \mathbb{R}_+^N$*, given for* $z \in \mathbb{R}^N$

$$(P_\Omega z)_i = \max(0, z_i).$$

This projected estimate also asymptotically recovers the true state, i.e.,

$$\lim_{t\to\infty} \|P_\Omega\hat{\bar{x}}(t) - \bar{x}(t)\| = 0.$$

5. TRAFFIC INTERSECTION

To demonstrate the observer design, we introduce a signalized T-junction consisting of three flows of cars, see Figure 2. Each flow can go into two directions. For this example we assume that each flow has a single signal, i.e., if a car receives a green light it can move in two different directions. Therefore, it is not possible to give multiple flows a green light simultaneously. The flows are served in order 1, 2, 3, and then back to flow 1 after which the cycle is repeated. The intersection uses a clearing policy, i.e., it completely empties the queue of a flow before switching to serve the next flow. Switching to serve cars from flow i requires a clearing/setup time γ_i to make sure all vehicles from the previously served flow have cleared the intersection. Vehicles arrive at flow i with arrival rate λ_i and are served with process rates μ_i, $i = 1, 2, 3$. A sensor measures the crossing of vehicles in lane 1.

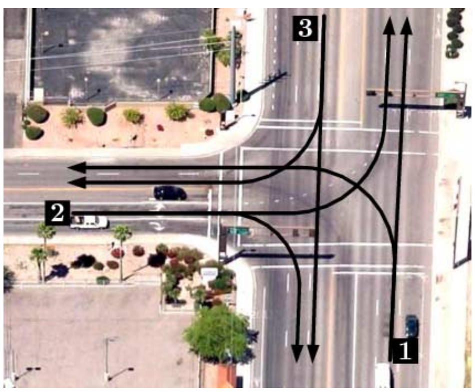

Figure 2: Signalized T-junction containing three vehicle flows 1-3.

The dynamics can be written in the form (3) with

$$f_1 = f_3 = f_5 = \begin{bmatrix} -1 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix},$$

$$f_2 = \begin{bmatrix} 0 \\ \lambda_1 - \mu_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}, \quad f_4 = \begin{bmatrix} 0 \\ \lambda_1 \\ \lambda_2 - \mu_2 \\ \lambda_3 \end{bmatrix}, \quad f_6 = \begin{bmatrix} 0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 - \mu_3 \end{bmatrix}$$

$$\alpha_1 = \alpha_3 = \alpha_5 = 0, \quad \alpha_2 = \gamma_2, \quad \alpha_4 = \gamma_3, \quad \alpha_6 = \gamma_1,$$
$$k_1 = k_3 = k_5 = 1, \quad k_2 = 2, \quad k_4 = 3, \quad k_6 = 4,$$
$$h_1 = h_3 = h_4 = h_5 = h_6 = 0, \quad h_2 = 2.$$

The modes 1,3 and 5 denote setting up to serve flow 1,2 and 3, respectively. Modes 2,4 and 6 denote serving flow 1,2 and 3, respectively. Writing this system in the form (14) gives

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\lambda_2}{\mu_1 - \lambda_1} & 1 & 0 \\ \frac{\lambda_3}{\mu_1 - \lambda_1} & 0 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} \frac{1}{\mu_1 - \lambda_1} & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & \frac{\lambda_1 \mu_3}{(\mu_2 - \lambda_2)(\mu_3 - \lambda_3)} & \frac{\lambda_1}{\mu_3 - \lambda_3} \\ 0 & \frac{\lambda_2 \lambda_3}{(\mu_2 - \lambda_2)(\mu_3 - \lambda_3)} & \frac{\lambda_2}{\mu_3 - \lambda_3} \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & \frac{\mu_3}{(\mu_2 - \lambda_2)(\mu_3 - \lambda_3)} & \frac{1}{\mu_3 - \lambda_3} \end{bmatrix}.$$

we omitted a_i and c_i as they cancel out in the observer error dynamics. Note that though the pairs (A_1, C_1) and (A_2, C_2) are unobservable, we can build a periodic deadbeat observer by using the observer gains

$$L_1 = \begin{bmatrix} 0 & \lambda_2 & \lambda_3 \end{bmatrix}^\top, \tag{20a}$$

$$L_2 = \begin{bmatrix} \lambda_1 & \frac{\lambda_2 \lambda_3}{\mu_3} & 0 \end{bmatrix}^\top, \tag{20b}$$

Using these gains, the matrices $A_v - L_v C_v$ and $L_v C_v$ ($v = 1, 2$) are positive matrices, yielding a positive observer. In this example the observer estimation starts at $t = 50$. For parameters

$$\lambda = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top, \qquad \mu = \begin{bmatrix} 8 & 10 & 12 \end{bmatrix}^\top,$$
$$\gamma = \begin{bmatrix} 5 & 10 & 15 \end{bmatrix}^\top,$$

and initial estimated state $\hat{x}(50) = \begin{bmatrix} 70 & 20 & 30 \end{bmatrix}^\top$ and mode $\hat{q}(50) = 4$, i.e., serving vehicles from flow 2, a simulation result is presented in Figure 3. The queue lengths at the intersection are presented by dashed lines and the estimated queue lengths are presented by solid lines. Since the measurement information equals the estimated output, i.e., $y(50) = \hat{y}(50)$, the observer dynamics are given by (18a)–(18e) until measurement and estimated output differ. This occurs at $t = 70.7$, when the system starts serving vehicles from queue 1. At this time instant $y \neq \hat{y}$ and the system jumps according to (18d). According to the initial estimated state $\hat{x}(50)$, this event was predicted at $\tilde{t} = 81.7$ with $\tilde{x}(\tilde{t}) = \begin{bmatrix} 101.7 & 58.3 & 15 \end{bmatrix}^\top$. Therefore, $\hat{x}(70.7^+) = \begin{bmatrix} 90.7 & 52.8 & 15 \end{bmatrix}^\top$, causing jumps in $\hat{x}_2$ and $\hat{x}_3$. Also, the estimated mode changes to serving products from flow 1.

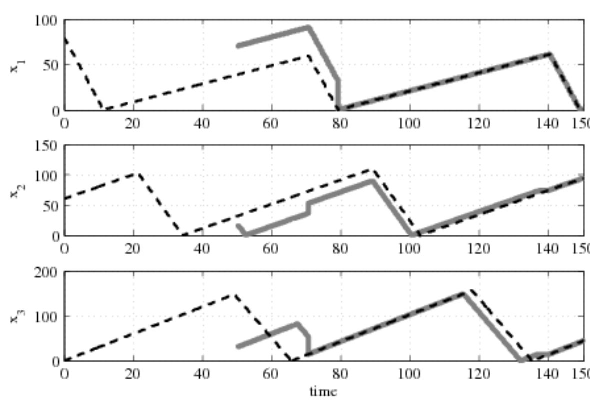

Figure 3: Actual (dashed lines) and estimated (solid lines) queue lengths of the 3-flow T-junction.

The second visible event occurs at $t = 79.2$ when $x_1 = 0$, also earlier than predicted since $\hat{x}_1$ is estimated too large. At this moment $y \neq \hat{y}$ and $y = 0$, therefore jump (18e) results in $\hat{x}_1(79.2^+) = 0$. Note that at this moment both $\hat{x}_1(t)$ and $\hat{x}_3(t)$ are correctly estimated. The observer predicts the next visible event at $\tilde{t} = 137$. However, since $\hat{x}_2(79.2^+) < x_2(79.2^+)$ the actual event occurs later. Therefore the observer switches to the additional waiting mode (18b) via jump (18c). At $t = 140.3$, detection of the next visible event, the estimated buffer levels have converged exactly to the actual buffer levels, due to the deadbeat observer.

6. CONCLUSIONS

This paper presented a methodology to design observers for a special class of piecewise affine hybrid systems (PWAHS), being highly relevant in the context of manufacturing and traffic applications. Although all subsystems are unobservable and not all events are visible, a continuous-time observer was constructed which guarantees that the estimate converges to the state of the plant under suitable conditions. One of the main ideas in the construction of the continuous-time observer was sampling of the system at the visible events leading to a discrete-time periodic linear system, for which an observer can be designed using standard techniques from control theory. If this step is successfully performed, a continuous-time observer that asymptotically recovers the true state of the original hybrid systems can be synthesized. Indeed, the discrete-time observer can then be used as a blueprint for the continuous-time observer, where besides the plant dynamics additional 'waiting' modes are assigned to the observer. Occurrence of visible events before the time that the event was predicted to occur results in a discrete state switch and an update of the continuous states. The observer switches to a 'waiting' mode if an event occurs later than predicted. These principles are formally shown to result in a successful observer design.

The results in this paper lead to various other questions that will be considered in future work. First of all, it would be of interest to take the positivity of the state variable into account leading to observers that always create positive state estimates as well (positive observers). Some first hints were already provided in this direction. In addition, it is of interest to formulate necessary and sufficient conditions in terms of the data of the original PWAHS system when the proposed design is indeed successful and how this relates to fundamental observability and detectability properties. Finally, it is of interest to investigate to what extent the observer design principles put forward in this paper can be applied to more general classes of hybrid systems. As such, this paper provides ideas that might be fruitfully exploited into various future research directions.

7. ACKNOWLEDGMENTS

This work is supported by the Netherlands Organization for Scientific Research (NWO-VIDI grant 639.072.072).

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