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Disturbance decoupling of switched linear systems

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ABSTRACT

In this paper, we consider disturbance decoupling problems for switched linear systems. We will provide necessary and sufficient conditions for three different versions of disturbance decoupling, which differ based on which signals are considered to be the disturbance. In the first version, the exogenous input is considered as the disturbance, in the second, the switching signal and in the third both of them are considered as disturbances. All three versions of disturbance decoupling have direct counterparts for linear parameter-varying (LPV) systems, while the latter instance of the problem is relevant for disturbance decoupling of piecewise linear systems, as we will show. The solutions of the three disturbance decoupling problems will be based on geometric control theory for switched linear systems and will entail both mode-dependent and mode-independent static state feedback.

1. Introduction

Geometric control theory for linear time-invariant systems has a long and rich history, as is evidenced by the availability of various textbooks on the topic [1–3]. In particular, for solving disturbance decoupling problems (DDPs) for linear systems, the usage of geometric theory turned out to be extremely powerful. Also solutions to various problems for smooth nonlinear systems using a nonlinear geometric approach are available in the literature; see, e.g., [4–6]. However, outside the context of linear or smooth nonlinear control systems, the number of results on DDPs is rather limited. This is specifically surprising for hybrid dynamical systems or subclasses such as switched systems [7], as they have been studied extensively over the past two decades.

Only a few results are available on geometric control theory and solutions to DDPs for switched systems. In [8], the largest controlled invariant set for switched linear systems (SLSs) is studied in which both the switching (discrete control input) and the continuous input can be manipulated as control inputs. In the context of linear parameter-varying (LPV) systems, various parameter-varying (controlled and conditioned) invariant subspaces are introduced in [9] and various algorithms are presented to compute them. Based on [9] first results in the direction of applying these concepts to DDPs with respect to continuous disturbances are given in [10] using a parameter-dependent state feedback. Recently, in [11], the DDP for switched linear systems using mode-dependent state feedback control is solved and combined with results on quadratic stabilizability [12,13]. Also for reachability problems for SLSs, invariant subspaces played an important role. In particular, in [14,15], it was shown that the reachable set of an SLS is equal to the smallest invariant set containing the subspace spanned by all the input matrices of the individual subsystems. For switched nonlinear systems, the only work known to the authors is [16]. In [16], local versions of DDP with respect to continuous disturbances are solved using both mode-dependent and mode-independent static state feedback.

The objectives of this paper are to provide complete answers to DDPs for SLSs using various invariant subspaces for SLSs. Building upon the preliminary version of this paper [17], we first assume that the control input is absent and analyze the disturbance decoupling properties of an SLS. In contrast with the above mentioned references, which only study disturbance decoupling (DD) with respect to exogenous disturbances, we consider three variants of DD as will be formally defined in Section 2, namely DD with respect to the disturbances being either (i) the exogenous disturbances, (ii) the switching signal, or (iii) both the exogenous disturbances and the switching signal. We will show that the latter instance of the problem is relevant for disturbance decoupling of piecewise linear systems. In Section 3, we will fully characterize these three DD properties and show the equivalence of these properties with their counterparts for linear parameter-varying

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(LPV) systems. In Section 4, we will add control inputs to the problem and solve the DDP using state feedback controllers that may be both mode-dependent and mode-independent. We will allow for direct feedthrough terms of the control input into the to-be-decoupled output variable, a situation that was not considered in the aforementioned references. Note also that variants (ii) and (iii) are defined in the present paper for the first time. Before stating the conclusions, we provide algorithms to compute the largest *common* controlled invariant subspaces using both mode-dependent and mode-independent feedbacks in Section 5, which can be used to verify the characterizations of the solvability of the DDPs provided in Section 4 and also to construct feedbacks solving the DDPs.

2. Problem formulation

A switched linear system (SLS) without control inputs is described by the following equations

$$\dot{x}(t) = A_{\sigma(t)}x(t) + E_{\sigma(t)}d(t) \tag{1a}$$

$$z(t) = H_{\sigma(t)}x(t) \tag{1b}$$

where $x(t) \in \mathbb{R}^{n_x}$, $d(t) \in \mathbb{R}^{n_d}$ and $z(t) \in \mathbb{R}^{n_z}$ denote the state variable, the exogenous disturbance input and output, respectively, at time $t \in \mathbb{R}_+ := [0, \infty)$. For each $i \in \{1, \ldots, M\}$, $A_i \in \mathbb{R}^{n_x \times n_x}$, $E_i \in \mathbb{R}^{n_x \times n_d}$ and $H_i \in \mathbb{R}^{n_z \times n_x}$ are matrices describing a linear subsystem. Switching between subsystems (modes) is orchestrated by the switching signal σ. We assume that σ lies in the set $\mathcal{S}$ of right-continuous functions $\mathbb{R}_+ \to \{1, \ldots, M\}$ that are piecewise constant with a finite number of discontinuities in a finite length interval. Particular switching signals are the constant ones $\sigma^i \in \mathcal{S}, i = 1, \ldots, M$, which are defined as $\sigma^i(t) = i$ for all $t \in \mathbb{R}_+$. We assume that the exogenous signal d is locally integrable, i.e. $d \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$. Clearly, the SLS (1) has for each $d \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$, initial condition $x(0) = x_0 \in \mathbb{R}^{n_x}$ and switching signal $\sigma \in \mathcal{S}$, a unique solution $x_{x_0,\sigma,d}$ and a corresponding output $z_{x_0,\sigma,d}$.

We will consider the following three variants of disturbance decoupling, which differ based on which signals are considered to be the disturbance.

Definition 2.1. The SLS (1) is called disturbance decoupled (DD) with respect to d if

$$z_{x_0,\sigma,d_1} = z_{x_0,\sigma,d_2} \tag{2}$$

for all $x_0 \in \mathbb{R}^{n_x}$, $\sigma \in \mathcal{S}$ and $d_1, d_2 \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$.

Definition 2.2. The SLS (1) is called disturbance decoupled (DD) with respect to σ if

$$z_{x_0,\sigma_1,d} = z_{x_0,\sigma_2,d} \tag{3}$$

for all $x_0 \in \mathbb{R}^{n_x}$, $\sigma_1, \sigma_2 \in \mathcal{S}$ and $d \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$.

Definition 2.3. The SLS (1) is called disturbance decoupled (DD) with respect to both σ and d if

$$z_{x_0,\sigma_1,d_1} = z_{x_0,\sigma_2,d_2} \tag{4}$$

for all $x_0 \in \mathbb{R}^{n_x}$, $\sigma_1, \sigma_2 \in \mathcal{S}$ and $d_1, d_2 \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$.

It immediately follows from these definitions that the SLS (1) is DD with respect to both σ and d if and only if it is DD with respect to σ and DD with respect to d.

DD with respect to disturbance d is commonly studied and well motivated within the context of linear systems [1–3] and nonlinear systems [4–6]. DD with respect to σ or both σ and d is typical for switched systems as studied here. These variants of DD are relevant in situations where the switching signal σ is uncontrolled and we would like to design a (closed-loop) system in which σ does not influence certain important performance variables z. In particular, when σ models certain faults in the system such as breakage of pipes, actuators, sensors, etc. and thus each mode $i \in \{1, \ldots, M\}$ corresponds to one of these discrete fault scenarios, it would be desirable to decouple z from σ (and possibly other disturbances d). Hence, as such these variants of DD constitute fundamental problems in the area of fault-tolerant control [18]. Another motivation for DD with respect to both σ and d are DD problems for piecewise linear (PWL) systems and linear parameter-varying (LPV) systems; see Sections 3.4 and 3.5 below.

3. Disturbance decoupling characterizations

To provide characterizations for the above mentioned DD properties, we need to introduce some concepts and a technical lemma. We call a subspace $\mathcal{V} \in \mathbb{R}^{n_x}$ A-invariant for $A \in \mathbb{R}^{n_x \times n_x}$, if $A\mathcal{V} \subseteq \mathcal{V}$. We call a subspace $\{A_1, \ldots, A_M\}$-invariant $A_i \in \mathbb{R}^{n_x \times n_x}$, $i = 1, \ldots, M$, if $A_i\mathcal{V} \subseteq \mathcal{V}$ for all $i = 1, \ldots, M$. Given a matrix $A \in \mathbb{R}^{n_x \times n_x}$ and a subspace $\mathcal{W} \in \mathbb{R}^{n_x}$, let $\langle A|\mathcal{W}\rangle$ denote the smallest A-invariant subspace that contains $\mathcal{W}$, i.e.,

$$\langle A|\mathcal{W}\rangle = \mathcal{W} + A\mathcal{W} + \cdots + A^{n_x-1}\mathcal{W}. \tag{5}$$

For a set of matrices $\{A_1, \ldots, A_M\}$ and a subspace $\mathcal{W}$, the smallest $\{A_1, \ldots, A_M\}$-invariant subspace that contains $\mathcal{W}$, denoted by $\mathcal{V}_s(\mathcal{W})$, is uniquely defined by the following three properties:

(1) $\mathcal{W} \subseteq \mathcal{V}_s(\mathcal{W})$;
(2) $\mathcal{V}_s(\mathcal{W})$ is $\{A_1, \ldots, A_M\}$-invariant;
(3) For any subspace $\mathcal{V}$ being $\{A_1, \ldots, A_M\}$-invariant with $\mathcal{W} \subseteq \mathcal{V}$, it holds that $\mathcal{V}_s(\mathcal{W}) \subseteq \mathcal{V}$.

Calculation of $\mathcal{V}_s(\mathcal{W})$ can be done using the recurrence relation

$$\mathcal{V}_1 = \mathcal{W}; \qquad \mathcal{V}_{i+1} = \sum_{j=1}^{M} \langle A_j|\mathcal{V}_i\rangle.$$

Since $\mathcal{V}_i \subseteq \mathcal{V}_{i+1}$ for $i = 1, 2, \ldots$ and $\mathcal{V}_p = \mathcal{V}_{p+1}$ implies $\mathcal{V}_q = \mathcal{V}_p$ for all $q \geq p$, it holds that $\mathcal{V}_q = \mathcal{V}_s(\mathcal{W})$ for all $q \geq n_x$; see, e.g., [9,15].

The reachable set of (1) is defined as $\mathcal{R} := \{x_{0,\sigma,d}(T) \mid T \in \mathbb{R}_+, \sigma \in \mathcal{S}$ and $d \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})\}$ being the set of states that can be reached from the origin in finite time for some σ and d.

Lemma 3.1. *For the SLS (1),*

$$\mathcal{R} = \mathcal{V}_s\left(\sum_{i=1}^{M} \operatorname{im} E_i\right).$$

See [14,15] for the proof of this lemma.

3.1. Disturbance decoupling with respect to d

In this section, we consider DD with respect to d. Before giving the main result of this subsection, we would like to give a motivating example which shows that this is not a trivial problem.

Example 3.2. Consider a two-mode SLS with the 1st subsystem described as

$$\dot{x}_1(t) = 0 \qquad \dot{x}_2(t) = d(t) \qquad z(t) = x_1(t) \tag{6}$$

and the 2nd subsystem as

$$\dot{x}_1(t) = d(t) \qquad \dot{x}_2(t) = 0 \qquad z(t) = x_2(t). \tag{7}$$

It is obvious that both the linear subsystems are DD with respect to d. However, under the switching signal $\sigma(t)$ described as

$$\sigma(t) = \begin{cases} 1 & 0 \leq t < t_1 \\ 2 & t_1 \leq t \end{cases} \tag{8}$$

the output at t_1 is given by

$$z(t_1) = x_{20} + \int_0^{t_1} d(\tau)d\tau.$$

Therefore, one can observe that it is not sufficient that the subsystems of an SLS are DD with respect to d for the SLS itself to be DD with respect to d. This observation is consistent with the following theorem.

Theorem 3.3. *The SLS (1) is DD with respect to d if and only if there exists an $\{A_1, \ldots, A_M\}$-invariant subspace $\mathcal{V}$ such that*

$$\sum_{i=1}^{M} \operatorname{im} E_i \subseteq \mathcal{V} \subseteq \ker \begin{bmatrix} H_1 \\ \vdots \\ H_M \end{bmatrix}. \tag{9}$$

Proof. Once σ is fixed, the SLS (1) reduces to a linear time-varying system. Using the resulting linearity properties, it can be shown that (2) is equivalent to

$$z_{0,\sigma,d} = 0 \quad \forall \sigma \in \mathcal{S}\ \forall d \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d}). \tag{10}$$

Necessity. Since each $\bar{x} \in \mathcal{R}$ can be reached for some σ and d, i.e. $\bar{x} = x_{0,\sigma,d}(T)$ for some $T \in \mathbb{R}_+$, we can take $\overline{\sigma}^i(t) = \begin{cases} \sigma(t) & 0 \le t < T \\ \sigma^i(t) & t \ge T \end{cases}$ for $i = 1, \ldots, M$ in (10) to get that $H_i\bar{x} = z_{0,\overline{\sigma}^i,d}(T) = 0$. Hence (9) holds for $\mathcal{V} = \mathcal{R}$ due to Lemma 3.1.

Sufficiency. Since $x_{0,\sigma,d}(t) \in \mathcal{R}$ for all $t \in \mathbb{R}_+$, it follows from Lemma 3.1 that $x_{0,\sigma,d}(t) \in \mathcal{V}$ for all $t \in \mathbb{R}_+$ as $\mathcal{R} = \mathcal{V}_s\left(\sum_{i=1}^M \operatorname{im} E_i\right) \subseteq \mathcal{V}$. Hence, due to (9), $z_{0,\sigma,d}(t) = H_{\sigma(t)}x_{0,\sigma,d}(t) = 0$ and thus (10) is satisfied. □

Remark 3.4. Revisiting the SLS in Example 3.2, one can see that

$$\sum_{i=1}^{2} \operatorname{im} E_i = \mathbb{R}^2 \qquad \ker \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \{0\}.$$

Clearly, there cannot be a subspace $\mathcal{V}$ that satisfies

$$\sum_{i=1}^{2} \operatorname{im} E_i \subseteq \mathcal{V} \subseteq \ker \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

Therefore, the SLS in Example 3.2 is not DD with respect to d according to Theorem 3.3, which agrees with our previous observation based on computing the output of the SLS explicitly for a particular switching signal.

3.2. Disturbance decoupling with respect to σ

In this section, we consider DD with respect to σ. We denote the unobservable subspace corresponding to the pair (H_i, A_i) by $\mathcal{N}_i$, i.e. $\mathcal{N}_i = \ker H_i \cap \ker H_iA_i \cap \cdots \cap \ker H_iA_i^{n_x-1}$. Note that $\mathcal{N}_i$ is also the largest A_i-invariant subspace contained in $\ker H_i$, $i \in \{1, \ldots, M\}$.

Lemma 3.5. *The SLS (1) is DD with respect to σ if and only if for all $(i, j) \in \{1, \ldots, M\} \times \{1, \ldots, M\}$ the following conditions hold*

(i) $H_i = H_j$;
(ii) $\mathcal{N}_i = \mathcal{N}_j$;
(iii) $\operatorname{im}(A_i - A_j) \subseteq \mathcal{N}_i$;
(iv) $\operatorname{im}(E_i - E_j) \subseteq \mathcal{N}_i$.

Proof. *Necessity*. To prove the necessity of the conditions, take $\sigma_1 = \sigma^i$ and $\sigma_2 = \sigma^j$. Let $\bar{d}(t) = d_0$ for all $t \in \mathbb{R}_+$. Then, one gets

$$z_{x_0,\sigma^i,\bar{d}} = z_{x_0,\sigma^j,\bar{d}}$$

and hence

$$\begin{aligned} &H_ie^{A_it}x_0 + \left(\int_0^t H_ie^{A_i(t-s)}E_i\,ds\right)d_0 = \\ &H_je^{A_jt}x_0 + \left(\int_0^t H_je^{A_j(t-s)}E_j\,ds\right)d_0 \end{aligned} \tag{11}$$

for all $t \in \mathbb{R}_+$. Since x_0 and d_0 are both arbitrary, one gets

$$H_iA_i^k = H_jA_j^k \tag{12}$$

$$H_iA_i^kE_i = H_jA_j^kE_j \tag{13}$$

for all $k \in \mathbb{N}$ by differentiating (11) with respect to time and evaluating at $t = 0$. Condition (i) follows from (12) for $k = 0$, and (ii) follows from $k = 0, 1, \ldots, n_x - 1$. To see that (iii) holds, note that

$$\begin{aligned} H_iA_i^\ell(A_i - A_j) &= H_iA_i^{\ell+1} - H_iA_i^\ell A_j \\ &\overset{(12)}{=} H_iA_j^{\ell+1} - H_iA_i^\ell A_j \\ &= H_i(A_j^\ell - A_i^\ell)A_j \\ &\overset{(12)}{=} 0 \end{aligned}$$

for all $\ell \in \mathbb{N}$. For condition (iv), observe that

$$H_iA_i^\ell E_i \overset{(13)}{=} H_jA_j^\ell E_j \overset{(12)}{=} H_iA_i^\ell E_j$$

and hence $H_iA_i^\ell(E_i - E_j) = 0$ for all $\ell \in \mathbb{N}$.

Sufficiency. First, we will show that

$$H_iA_i^k = H_jA_j^k \tag{14}$$

for all $(i, j) \in \{1, 2, \ldots, M\} \times \{1, 2, \ldots, M\}$ and $k \in \mathbb{N}$ by induction. Note that (14) holds for $k = 0$ due to condition (i). Assume that it holds for some k. Then

$$\begin{aligned} H_iA_i^{k+1} - H_jA_j^{k+1} &= H_iA_i^{k+1} - H_iA_i^kA_j \\ &= H_iA_i^k(A_i - A_j) \\ &\overset{\text{(iii)}}{=} 0. \end{aligned}$$

Also note that

$$\begin{aligned} H_iA_i^kE_i - H_jA_j^kE_j &\overset{(14)}{=} H_iA_i^kE_i - H_iA_i^kE_j \\ &= H_iA_i^k(E_i - E_j) \\ &\overset{\text{(iv)}}{=} 0 \end{aligned}$$

for all $k \in \mathbb{N}$. Thus, we get

$$H_iA_i^kE_i = H_jA_j^kE_j \tag{15}$$

for all $(i, j) \in \{1, 2, \ldots, M\} \times \{1, 2, \ldots, M\}$ and $k \in \mathbb{N}$. Now, let

$$\tilde{z} = z_{x_0,\sigma_1,d} - z_{x_0,\sigma_2,d} \tag{16}$$

$$\tilde{x} = x_{x_0,\sigma_1,d} - x_{x_0,\sigma_2,d} \tag{17}$$

for some x_0, σ_1, σ_2, and d. It follows from (14) for $k = 0$ that

$$\tilde{z}(t) = H_{\sigma_1(t)}\tilde{x}(t).$$

By differentiating and using (14) for $k = 1$ and (15) for $k = 0$, one gets

$$\dot{\tilde{z}}(t) = H_{\sigma_1(t)}A_{\sigma_1(t)}\tilde{x}(t)$$

for all $t \in \mathbb{R}_+$. Repeating the same argument, one obtains

$$\tilde{z}^{(k)}(t) = H_{\sigma_1(t)}A_{\sigma_1(t)}^k\tilde{x}(t)$$

for all $k \in \mathbb{N}$ and for all $t \in \mathbb{R}_+$. Due to (14), this yields

$$\tilde{z}^{(k)}(t) = H_iA_i^k\tilde{x}(t)$$

for all $t \in \mathbb{R}_+$, in which $i \in \{1,\ldots,M\}$ can be selected arbitrarily. Then, there exists a nonzero polynomial $p(\lambda)$ (e.g., the characteristic polynomial of A_i for any i) such that

$$p\left(\frac{d}{dt}\right)\tilde{z} = 0$$

by the Cayley–Hamilton theorem. Since $\tilde{x}(0) = 0$, one has $\tilde{z}^{(k)}(0) = 0$ for all $k \in \mathbb{N}$. Therefore, $\tilde{z}(t) = 0$ for all $t \in \mathbb{R}_+$. In other words,

$$z_{x_0,\sigma_1,d} = z_{x_0,\sigma_2,d}$$

for all x_0, σ_1, σ_2, and d. □

Based on Lemma 3.5, we can also derive an alternative characterization of DD with respect to σ, which is more geometric in nature.

Theorem 3.6. *The SLS (1) is DD with respect to σ if and only if there exists an $\{A_1,\ldots,A_M\}$-invariant subspace $\mathcal{V}$ such that for all $(i,j) \in \{1,\ldots,M\} \times \{1,\ldots,M\}$*

(i) $H_i = H_j$;
(ii) $\operatorname{im}(A_i - A_j) \subseteq \mathcal{V} \subseteq \ker H_i$;
(iii) $\operatorname{im}(E_i - E_j) \subseteq \mathcal{V}$.

Proof. *Necessity*. The necessity of these conditions follows directly from Lemma 3.5 by taking $\mathcal{V} = \mathcal{N}_i$.

Sufficiency. As $\mathcal{N}_i$ is the largest A_i-invariant subspace that is contained in $\ker H_i$ for all $i \in \{1,\ldots,M\}$, condition (ii), together with the $\{A_1,\ldots,A_M\}$-invariance of $\mathcal{V}$, implies that $\mathcal{V} \subseteq \mathcal{N}_i$. This fact together with (i) implies that condition (i), (iii) and (iv) of Lemma 3.5 are satisfied. To show that also (ii) of Lemma 3.5 is satisfied, we observe that statement (ii) of this theorem implies that $H_iA_i = H_iA_j$ for all $(i,j) \in \{1,\ldots,M\} \times \{1,\ldots,M\}$. Due to $\operatorname{im}(A_i - A_j) \subseteq \mathcal{V}$ and $\mathcal{V}$ being $\{A_1,\ldots,A_M\}$-invariant, it must hold that $A_i^k \operatorname{im}(A_i - A_j) \subseteq \mathcal{V} \subseteq \ker H_i$ for all $k \in \mathbb{N}$ and thus $H_iA_i^{k+1} = H_iA_i^kA_j$ for all $(i,j) \in \{1,\ldots,M\} \times \{1,\ldots,M\}$. We will now prove that

$$H_iA_i^k = H_jA_j^k \tag{18}$$

for all $k \in \mathbb{N}$ using induction. Clearly, it holds for $k = 0$. Suppose it holds for k, then

$$H_jA_j^{k+1} \overset{(18)}{=} H_iA_i^kA_j = H_iA_i^{k+1}.$$

Hence, (18) holds for all $k \in \mathbb{N}$ and thus $\mathcal{N}_i = \mathcal{N}_j$ for all $(i,j) \in \{1,\ldots,M\} \times \{1,\ldots,M\}$. As such, we recovered the conditions of Lemma 3.5, which completes the proof. □

3.3. Disturbance decoupling with respect to σ and d

Using the above results, we will now characterize DD with respect to σ and d.

Theorem 3.7. *The following statements are equivalent.*

(I) *The SLS (1) is DD with respect to σ and d.*
(II) *The conditions*
  (i) $H_i = H_j = H$
  (ii) $\mathcal{N}_i = \mathcal{N}_j = \mathcal{N}$
  (iii) $\operatorname{im}(A_i - A_j) \subseteq \mathcal{N}$
  (iv) $\operatorname{im} E_i \subseteq \mathcal{N}$
  hold for all $(i,j) \in \{1,\ldots,M\} \times \{1,\ldots,M\}$.
(III) *There exists an $\{A_1,\ldots,A_M\}$-invariant subspace $\mathcal{V}$ such that*
  (i) $H_i = H_j = H$
  (ii) $\operatorname{im}(A_i - A_j) \subseteq \mathcal{V} \subseteq \ker H$
  (iii) $\operatorname{im} E_i \subseteq \mathcal{V}$
  for all $(i,j) \in \{1,\ldots,M\} \times \{1,\ldots,M\}$.

Proof. (I) ⇒ (II): If the SLS (1) is DD with respect to σ and d, the hypotheses of Theorem 3.3 and Lemma 3.5 must be true, i.e. inclusion (9) holds along with the conditions in Lemma 3.5. Observe that for all $x \in \mathcal{V}$ one can write

$$HA_i^kx = 0$$

for all $k \in \mathbb{N}$ and $i = 1,\ldots,M$. This shows that $x \in \mathcal{N}$. Therefore $\mathcal{V} \subseteq \mathcal{N}$. Furthermore, $\sum_{j=1}^M\sum_{i=1}^M \operatorname{im}(E_i - E_j) \subseteq \sum_{i=1}^M \operatorname{im} E_i$. Then, combining the conditions of Theorem 3.3 and Lemma 3.5 gives (II).

(II) ⇒ (III): Since $\mathcal{N}_i$ is an A_i-invariant subspace contained in $\ker H$, one can write

$$A_i\mathcal{N} \subseteq \mathcal{N}$$

for all $i = 1,\ldots,M$. Thus, it follows that $\mathcal{N}$ is an $\{A_1,\ldots,A_M\}$-invariant subspace contained in $\ker H$.

(III) ⇒ (I): It follows from (III) that $\sum_{i=1}^M \operatorname{im} E_i \subseteq \mathcal{V} \subseteq \ker H$. This recovers the condition of Theorem 3.3. One can also recover the conditions of Lemma 3.5 by following the steps shown in the sufficiency part of Theorem 3.6. Thus, (I) follows. □

3.4. Disturbance decoupling of piecewise linear systems

Based on the above results, we can show the importance of DD with respect to σ and d for disturbance decoupling of piecewise linear (PWL) systems of the form

$$\dot{x}(t) = A_ix(t) + E_id(t) \quad \text{if } x(t) \in \mathcal{X}_i \tag{19a}$$

$$z(t) = Hx(t), \tag{19b}$$

in which $\mathcal{X}_i \subset \mathbb{R}^{n_x}$, $i \in \{1,\ldots,M\}$, are polyhedral regions with non-empty interiors that form a partitioning of the state space $\mathbb{R}^{n_x}$, i.e., $\bigcup_{i=1}^M \mathcal{X}_i = \mathbb{R}^{n_x}$ and the interiors of different regions have an empty intersection. Since the right-hand side of a PWL system can be discontinuous, solutions will be interpreted in the sense of Filippov [19], which includes possible sliding motions at the boundaries of the regions. See [19] for more details and exact definitions of Filippov solutions. To avoid any ambiguity in the definition of the output as in (19b) during sliding motions, we assume that the output matrix H is independent of i. Note that this is a necessary condition for the SLS (1) corresponding to (19) to be DD with respect to both σ and d.

Definition 3.8. The PWL system (19) is called disturbance decoupled (DD) with respect to d if for all $x_0 \in \mathbb{R}^{n_x}$, $d_1, d_2 \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$, it holds that

$$z_1 = z_2 \tag{20}$$

where $z_j = Hx_j$, $j = 1, 2$ and x_j, $j = 1, 2$, is some Filippov solution to (19) for initial state $x(0) = x_0$ and disturbance input d_j, $j = 1, 2$.

Note that due to possible non-uniqueness of Filippov solutions, multiple solutions might correspond to one initial condition and one disturbance input.

Theorem 3.9. *The PWL (19) system is DD with respect to d, if the corresponding SLS (1) is DD with respect to σ and d.*

Proof. Let x_j be Filippov solutions to (19) for x_0 and d_j and $z_j = Hx_j$ the corresponding outputs for $j = 1, 2$. Then we have

$$\dot{x}_j(t) \in \operatorname{conv}\{A_ix_j + E_id_j \mid i = 1,\ldots,M\}$$

for almost all $t \in \mathbb{R}_+$, with conv denoting the convex hull. Defining $\tilde{z}(t) := z_1(t) - z_2(t)$, we can write

$$\begin{aligned}\dot{\tilde{z}}(t) &= H(\dot{x}_1(t) - \dot{x}_2(t)) \\ &\in \operatorname{conv}\{HA_ix_1 + HE_id_1 \mid i = 1,\ldots,M\} \\ &\quad - \operatorname{conv}\{HA_ix_2 + HE_id_2 \mid i = 1,\ldots,M\}.\end{aligned}$$

Since the corresponding SLS is DD with respect to σ and d, we have the implications $HA_i^k = HA_j^k$ and $HA_i^k E_j = 0$ for all $(i, j, k) \in \{1, \ldots, M\} \times \{1, \ldots, M\} \times \mathbb{N}$. Then it follows that

$$\dot{\tilde{z}}(t) = HA_1(x_1(t) - x_2(t)). \tag{21}$$

Differentiating (21) repeatedly with respect to time, we get the following identity

$$\tilde{z}^{(k)}(t) = HA_1^k(x_1(t) - x_2(t)) \quad \forall k \in \mathbb{N}.$$

Due to the Cayley–Hamilton theorem there exists a polynomial p such that

$$p\left(\frac{d}{dt}\right)\tilde{z}(t) = 0.$$

As $x_1(0) = x_2(0)$, we have $\tilde{z}(t) \equiv 0$. □

This theorem demonstrates the relevance of DD with respect to σ and d for SLSs in the context of DD with respect to d for PWL systems. As such, when control inputs u are present, disturbance decoupling problems (DDPs), i.e., designing controllers that render the closed-loop SLS DD with respect to σ and d, can be used also for solving DDPs of PWL systems with respect to d. DDPs for SLSs will be considered in the next section. However, before doing so, we would like to show that although DD of the SLS with respect to σ and d is a sufficient condition for DD of a PWL system with respect to d, it is not necessary.

Example 3.10. Consider the PWL system

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(t) \quad x_1(t) \geq 0$$

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} d(t) \quad x_1(t) < 0$$

$$z = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

It is clear that the corresponding SLS is not disturbance decoupled with respect to σ and d. Yet, the PWL system described above is disturbance decoupled with respect to d. This example demonstrates the fact that the condition in Theorem 3.9 is indeed sufficient but not necessary.

It is also worth pointing out that for a PWL system to be DD with respect to d, it is not sufficient that the corresponding SLS is DD with respect to d only. For a PWL system, an initial condition x_0 and disturbance d_1 lead to a certain (Filippov) solution to (19a) with a corresponding switching signal denoted by σ_1. Likewise, the same initial condition x_0 and a different disturbance d_2 lead to a possibly different (Filippov) solution to (19a) with a corresponding switching signal denoted by σ_2. Since $\sigma_1 \neq \sigma_2$ in general, DD with respect to d of the corresponding SLS is therefore not sufficient. The following example demonstrates this observation.

Example 3.11. Consider the PWL system

$$\dot{x}(t) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d(t) \quad x_1(t) \geq 0$$

$$\dot{x}(t) = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d(t) \quad x_1(t) \leq 0$$

$$z = \begin{bmatrix} 0 & 1 \end{bmatrix} x.$$

It is easy to check that the corresponding SLS is DD with respect to d but not DD with respect to σ. However, setting the initial condition to $x_0 = (0\ 1)^\top$ and taking two disturbance signals d_1, d_2 such that the solution $x_{x_0,d_1}(t)$ lies in the region $x_1(t) > 0$ and the solution $x_{x_0,d_2}(t)$ lies in the region $x_1(t) < 0$ for all $t > 0$, we obtain the outputs

$$z_{x_0,d_1} = e^{2t} \qquad z_{x_0,d_2} = e^{t}$$

which are clearly not identical. Thus, the PWL system is not DD with respect to d.

3.5. Disturbance decoupling of linear parameter-varying systems

Another important class of dynamical systems is formed by linear parameter-varying (LPV) systems. In this section, we will show how DD for SLS is relevant for DD for LPV systems. Consider the LPV system of the form

$$\dot{x}(t) = A(w(t))x(t) + E(w(t))d(t) \tag{22a}$$

$$z(t) = H(w(t))x(t) \tag{22b}$$

with $x(t)$, $d(t)$ and $z(t)$ as defined before and $w(t) \in \mathbb{W}$ denotes an (uncertain) parameter at time $t \in \mathbb{R}_+$ with $\mathbb{W}$ being the set $\{w \in \mathbb{R}^m \mid \sum_{i=1}^M w_i = 1 \text{ and } w_i \geq 0,\ i = 1, \ldots, M\}$. The matrices $A(w)$, $E(w)$ and $H(w)$ are given by

$$A(w) = \sum_{i=1}^M w_i A_i \qquad E(w) = \sum_{i=1}^M w_i E_i$$

$$H(w) = \sum_{i=1}^M w_i H_i.$$

The parameter w is time-varying and we assume that $w : \mathbb{R}_+ \to \mathbb{W}$ belongs to the set $\mathcal{PC}(\mathbb{R}_+, \mathbb{W})$ of piecewise continuous functions with only a finite number of discontinuities in a finite length interval. We will denote the set $\mathcal{PC}(\mathbb{R}_+, \mathbb{W})$ as $\mathcal{PC}$ in the rest of this section. Note that the behavior of the SLS (1) is a subset of the behavior of the LPV system (22) as any $\sigma \in \mathcal{S}$ leads to a corresponding $w \in \mathcal{PC}$ that takes values at the extreme points of $\mathbb{W}$.

In what follows, we will characterize the disturbance decoupling properties of the LPV system (22).

Definition 3.12. The LPV system is called disturbance decoupled (DD) with respect to d if

$$z_{x_0,w,d_1} = z_{x_0,w,d_2}$$

for all $x_0 \in \mathbb{R}^{n_x}$, $w \in \mathcal{PC}$ and $d_1, d_2 \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$.

Definition 3.13. The LPV system is called disturbance decoupled (DD) with respect to w if

$$z_{x_0,w^1,d} = z_{x_0,w^2,d}$$

for all $x_0 \in \mathbb{R}^{n_x}$, $w^1, w^2 \in \mathcal{PC}$ and $d \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$.

Definition 3.14. The LPV system is called disturbance decoupled (DD) with respect to d and w if

$$z_{x_0,w^1,d_1} = z_{x_0,w^2,d_2}$$

for all $x_0 \in \mathbb{R}^{n_x}$, $w^1, w^2 \in \mathcal{PC}$ and $d_1, d_2 \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$.

Theorem 3.15. *The LPV system (22) is DD with respect to*

(i) d
(ii) w
(iii) *both d and w*

if and only if the SLS (1) is DD with respect to

(1) d
(2) σ
(3) *both d and σ,*

respectively.

Proof. We will show the equivalences (i) ⇔ (1) and (ii) ⇔ (2). The equivalence (iii) ⇔ (3) immediately follows then from combining (i) ⇔ (1) and (ii) ⇔ (2).

(i) ⇔ (1): It is obvious that (i) ⇒ (1). To show the reverse implication we take an initial state x_0, $w \in \mathcal{PC}$ and disturbances d_1, d_2 and let x_1 and x_2 denote the state trajectories corresponding to x_0, w, d_1 and x_0, w, d_2, respectively. Furthermore, we denote the corresponding outputs by z_1 and z_2. Then, $\tilde{x} := x_1 - x_2$ with $\tilde{d} := d_1 - d_2$ satisfies

$$\dot{\tilde{x}}(t) = A(w(t))\tilde{x}(t) + E(w(t))\tilde{d}(t). \tag{23}$$

We will now study system (23). From Theorem 3.3, we have that there exists an $\{A_1, \dots, A_M\}$-invariant subspace $\mathcal{V}$ satisfying

$$\sum_{i=1}^{M} \operatorname{im} E_i \subseteq \mathcal{V} \subseteq \ker \begin{bmatrix} H_1 \\ \vdots \\ H_M \end{bmatrix}.$$

Then it follows that

$$A(w)\mathcal{V} \subseteq \mathcal{V} \qquad \operatorname{im} E(w) \subseteq \mathcal{V} \subseteq \ker H(w) \quad \forall w \in \mathbb{W}. \tag{24}$$

For arbitrary $\tilde{x}_0$, w and $\tilde{d}$, the unique solution to the linear time-varying system (23) is given by the Peano–Baker formula (see [20]) as

$$\tilde{x}(t) = \Phi(t, 0)\tilde{x}_0 + \int_0^t \Phi(t, \tau)E(w(\tau))\tilde{d}(\tau)d\tau \tag{25}$$

where the state transition matrix, $\Phi(t, \tau)$, is given as

$$\Phi(t, \tau) = \sum_{n=0}^{\infty} \Psi_n(t)$$

with

$$\Psi_{n+1}(t) = \int_\tau^t A(w(s))\Psi_n(s)ds \qquad \Psi_0(t) = I.$$

Assuming $w \in \mathcal{PC}$ and $\tilde{d} \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$, we will now prove the following implication for system (23).

$$\tilde{x}_0 \in \mathcal{V} \Rightarrow \tilde{x}(t) \in \mathcal{V} \quad \forall t \in \mathbb{R}_+.$$

First we show that $\Phi(t, \tau)\mathcal{V} \subseteq \mathcal{V}$ for all $\tau, t \in \mathbb{R}_+$ by induction. Take a vector $x \in \mathcal{V}$ and note that $\Psi_0(t)x \in \mathcal{V}$. Now assume that $\Psi_n(t)x \in \mathcal{V}$ for all $t \in \mathbb{R}_+$. Then, by using the relation $A(w)\mathcal{V} \subseteq \mathcal{V}$ for all $w \in \mathbb{W}$, it is easy to see that $\Psi_{n+1}(t)x = \int_0^t A(w(s))\Psi_n(s)xds \in \mathcal{V}$ for all $t \in \mathbb{R}_+$. Thus, $\Phi(t, \tau)\mathcal{V} \subseteq \mathcal{V}$ for all $\tau, t \in \mathbb{R}_+$.

Next, observe that $\operatorname{im} E(w) \subseteq \mathcal{V}$. Then by the same induction argument as above one can show for any $\tilde{d} \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$ and $w \in \mathcal{PC}$ that

$$\int_0^t \Phi(t, \tau)E(w(\tau))\tilde{d}(\tau)d\tau \in \mathcal{V} \quad \forall t \in \mathbb{R}_+.$$

Since $\tilde{x}(0) = 0$, we can easily infer that $\tilde{x}(t) \in \mathcal{V}$ for all $t \in \mathbb{R}_+$. As $\mathcal{V} \subseteq \ker H(w)$ for all $w \in \mathbb{W}$ from (24), we conclude that $\tilde{z}(t) := z_1(t) - z_2(t) = H(w(t))\tilde{x}(t) = 0$ for all $t \in \mathbb{R}_+$, which shows that $z_{x_0,w,d_1} = z_{x_0,w,d_2}$.

(ii) ⇔ (2): Clearly, (ii) ⇒ (2). To show the reverse implication we take an initial state x_0, $w^1, w^2 \in \mathcal{PC}$ and disturbance d and let x_1 and x_2 denote the state trajectories corresponding to x_0, w^1, d and x_0, w^2, d, respectively. Furthermore, we denote the corresponding outputs by z_1 and z_2. Then for $\tilde{x} := x_1 - x_2$ we have

$$\begin{aligned}\dot{\tilde{x}}(t) &= A(w^1(t))x_1(t) - A(w^2(t))x_2(t) \\ &\quad + (E(w^1(t)) - E(w^2(t)))d(t).\end{aligned} \tag{26}$$

From Theorem 3.6 we have that

$$H_i = H_j := H$$
$$HA_i^l = HA_j^l$$
$$HA_k^l E_i = HA_k^l E_j$$

for all $(i, j, k, l) \in \{1, \dots, M\} \times \{1, \dots, M\} \times \{1, \dots, M\} \times \mathbb{N}$. We define $\tilde{z} = z_1 - z_2$ and pre-multiply (26) by H to obtain

$$\dot{\tilde{z}}(t) = HA_1\tilde{x}(t). \tag{27}$$

Differentiating (27) repeatedly leads to

$$z^{(k)}(t) = HA_1^k\tilde{x}(t)$$

for all $k \in \mathbb{N}$. Invoking now the Cayley–Hamilton theorem, we obtain

$$p\left(\frac{d}{dt}\right)\tilde{z} = 0$$

where p is the characteristic polynomial of A_1. Since $\tilde{x}(0) = 0$ and thus $z^{(k)}(0) = 0$ for $k \in \mathbb{N}$, we infer that $z_{x_0,w^1,d} = z_{x_0,w^2,d}$. □

4. DDP by state feedback

In the previous section, we provided full characterizations of DD properties. Now we will consider if and how we should choose control inputs in order to render an SLS disturbance decoupled in some sense. In order to do so, consider the SLS

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + E_{\sigma(t)}d(t) \tag{28a}$$

$$z(t) = H_{\sigma(t)}x(t) + J_{\sigma(t)}u(t) \tag{28b}$$

where we included now a control input $u(t) \in \mathbb{R}^{n_u}$ at time $t \in \mathbb{R}_+$. As before, we denote the solution corresponding to $x_0 \in \mathbb{R}^{n_x}$, $\sigma \in \mathcal{S}$, $d \in \mathcal{L}_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$ and $u \in \mathcal{L}_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_u})$ by $x_{x_0,\sigma,d,u}$ and the corresponding output by $z_{x_0,\sigma,d,u}$. We are now interested in finding conditions under which controllers can be found such that the closed-loop system is DD with respect to d, σ, or both. We start with mode-dependent static state feedback controllers.

4.1. Solution of DDP with respect to d by mode-dependent state feedback

Problem 4.1. The disturbance decoupling problem with respect to d (DDPd) by mode-dependent state feedback for SLS (28) amounts to finding $F_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \dots, M$ such that

$$\dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)}F_{\sigma(t)})x(t) + E_{\sigma(t)}d(t) \tag{29a}$$

$$z(t) = (H_{\sigma(t)} + J_{\sigma(t)}F_{\sigma(t)})x(t) \tag{29b}$$

is DD with respect to d.

Note that the SLS (29) results from putting system (28) in closed loop with $u(t) = F_{\sigma(t)}x(t)$, which requires knowledge of the active mode $\sigma(t)$ at time $t \in \mathbb{R}_+$.

Definition 4.2. Consider the SLS (28) with $d = 0$. A subspace $\mathcal{V}$ is called output-nulling $\{(A_1, B_1), \dots, (A_M, B_M)\}$-invariant if for any $x_0 \in \mathcal{V}$ and $\sigma \in \mathcal{S}$ there exists a control input $u \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_u})$ such that $x_{x_0,\sigma,0,u}(t) \in \mathcal{V}$ and $z_{x_0,\sigma,0,u}(t) = 0$ for all $t \in \mathbb{R}_+$.

Sometimes an output-nulling $\{(A_1, B_1), \dots, (A_M, B_M)\}$-invariant subspace is called a *common output-nulling controlled invariant subspace* for (28).

Definition 4.3. Given a linear subspace $\mathcal{V} \subseteq \mathbb{R}^{n_x}$, we define the extended subspace $e(\mathcal{V}) \subseteq \mathbb{R}^{n_x+n_z}$ as

$$e(\mathcal{V}) = \mathcal{V} \times \{0\}.$$

Theorem 4.4. *Consider the SLS (28) with $d = 0$. Let $\mathcal{V}$ be a subspace of $\mathbb{R}^{n_x}$. The following statements are equivalent.*

(i) *$\mathcal{V}$ is common output-nulling controlled invariant.*
(ii) $\begin{bmatrix} A_j \\ H_j \end{bmatrix} \mathcal{V} \subseteq e(\mathcal{V}) + \operatorname{im} \begin{bmatrix} B_j \\ J_j \end{bmatrix}$ *for all $j = 1, \ldots, M$.*
(iii) *There exist $F_j \in \mathbb{R}^{n_u \times n_x}$, $j = 1, \ldots, M$, such that $(A_j + B_jF_j)\mathcal{V} \subseteq \mathcal{V} \subseteq \ker(H_j + J_jF_j)$ for all $j = 1, \ldots, M$.*

Proof. The proof is similar in nature to the (more complicated) proof of Theorem 4.10, which we will provide below, and is therefore omitted. □

Let $\{\mathcal{V}_j \mid j \in \mathcal{J}\}$ be a collection of common output-nulling controlled invariant subspaces for the SLS (28). It follows from Definition 4.2 that $\sum_{j\in\mathcal{J}} \mathcal{V}_j$ is common output-nulling controlled invariant. Therefore, the set of all common output-nulling controlled invariant subspaces admits a largest element. The largest common output-nulling controlled invariant subspace for a given SLS plays a crucial role in the solution of DDPd by mode-dependent state feedback.

Definition 4.5. Consider the SLS (28) with $d = 0$. We define $\mathcal{V}^*_{md}$ as the largest common output-nulling controlled invariant subspace for the SLS (28) that is

(i) $\mathcal{V}^*_{md}$ is common output-nulling controlled invariant;
(ii) if $\mathcal{V}$ is a common output-nulling controlled invariant subspace for the SLS (28), then $\mathcal{V} \subseteq \mathcal{V}^*_{md}$.

Corollary 4.6. *Consider the SLS (28). DDPd by mode-dependent state feedback is solvable if and only if*

$$\sum_{i=1}^{M} \operatorname{im} E_i \subseteq \mathcal{V}^*_{md}. \tag{30}$$

Proof. The sufficiency of this condition is obvious. For the necessity, we have from Theorem 3.3 that there must exist a subspace $\mathcal{V}$ satisfying

$$(A_j + B_jF_j)\mathcal{V} \subseteq \mathcal{V} \subseteq \ker(H_j + J_jF_j) \quad \text{for } j = 1, \ldots, M$$

such that

$$\sum_{i=1}^{M} \operatorname{im} E_i \subseteq \mathcal{V}.$$

Then, it follows from Theorem 4.4 that $\mathcal{V}$ is common output-nulling controlled invariant. By Definition 4.5, we have $\mathcal{V} \subseteq \mathcal{V}^*_{md}$, which completes the proof. □

In Section 5, we will present a means to verify the hypothesis of this theorem providing an algorithm to compute the largest common output-nulling controlled invariant subspace for a given SLS.

Remark 4.7. For the special case that $J_i = 0$, $i = 1, \ldots, M$, this problem was solved also in [11]. In [11], the DDP with respect to d by mode-dependent state feedback was combined with the question of quadratic stability. Sufficient conditions were given exploiting known results for quadratic stabilization as in [12,13]. These stability conditions can also be added to the theorems that we present here, but unfortunately, they are, just as in [11], not so trivial to verify, certainly for a high number of subsystems. Indeed, the sufficient conditions that guarantee solvability of DDP with quadratic stability (DDPQS) according to Theorem 3.2 in [11] are the existence of F_j, $j = 1, \ldots, M$, such that $(A_j + B_jF_j)\mathcal{V}^*_{md} \subseteq \mathcal{V}^*_{md} \subseteq \ker(H_j + J_jF_j)$ for all $j = 1, \ldots, M$ and there exists a convex combination $\sum \alpha_j(A_j + B_jF_j)$ with $\sum_{j=1}^{M} \alpha_j = 1$ and $\alpha_j \geq 0$, $j = 1, \ldots, M$, being a Hurwitz matrix. Since a parameterization of all F_j, $j = 1, \ldots, M$, satisfying $(A_j+B_jF_j)\mathcal{V}^*_{md} \subseteq \mathcal{V}^*_{md} \subseteq \ker(H_j+J_jF_j)$ is hard to come by in the first place, and one has to search for both α_j and F_j, $j = 1, \ldots, M$, which is a non-convex problem, these conditions are not easy to verify. Also, for the satisfaction of the stabilization condition, it is unclear if using $\mathcal{V}^*_{md}$ instead of another common output-nulling controlled invariant subspace containing $\sum_{i=1}^{M} \operatorname{im} E_i$ is introducing conservatism into the conditions. Other differences with respect to [11], are that DDP with respect to d by mode-independent state feedback and DDP with respect to σ were not considered in [11], while we treat these problems in the next sections.

4.2. Solution of DDP with respect to d by a mode-independent state feedback

Problem 4.8. The disturbance decoupling problem with respect to d (DDPd) by mode-independent state feedback for SLS (28) amounts to finding $F \in \mathbb{R}^{n_u \times n_x}$ such that

$$\dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)}F)x(t) + E_{\sigma(t)}d(t) \tag{31a}$$

$$z(t) = (H_{\sigma(t)} + J_{\sigma(t)}F)x(t) \tag{31b}$$

is DD with respect to d.

Note that the SLS (31) results from putting system (28) in closed loop with $u(t) = Fx(t)$. The latter state feedback controller does not require knowledge of the active mode $\sigma(t)$ at time $t \in \mathbb{R}_+$.

Definition 4.9. Consider the SLS (28) with $d = 0$. A subspace $\mathcal{V}$ is called output-nulling $\{(A_1, B_1), \ldots, (A_M, B_M)\}$-invariant under mode-independent control if for any $x_0 \in \mathcal{V}$ there exists a control input $u \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_u})$ such that $x_{x_0,\sigma,0,u}(t) \in \mathcal{V}$ and $z_{x_0,\sigma,0,u}(t) = 0$ for all $\sigma \in \mathcal{S}$ and for all $t \in \mathbb{R}_+$.

Sometimes a subspace that is output-nulling $\{(A_1, B_1), \ldots, (A_M, B_M)\}$-invariant under mode-independent control is called a *common output-nulling controlled invariant subspace under mode-independent control* for (28).

Theorem 4.10. *Consider the SLS (28) with $d = 0$. Let $\mathcal{V}$ be a subspace of $\mathbb{R}^{n_x}$. Define the matrices $A_s \in \mathbb{R}^{M(n_x+n_z)\times n_x}$ and $B_s \in \mathbb{R}^{M(n_x+n_z)\times n_u}$*

$$A_s = \begin{bmatrix} A_1 \\ H_1 \\ \vdots \\ A_M \\ H_M \end{bmatrix}, \qquad B_s = \begin{bmatrix} B_1 \\ J_1 \\ \vdots \\ B_M \\ J_M \end{bmatrix} \tag{32}$$

and $e(\mathcal{V})^M$ as $e(\mathcal{V})^M = \overbrace{e(\mathcal{V}) \times e(\mathcal{V}) \times \cdots \times e(\mathcal{V})}^{M \text{ times}}$. The following statements are equivalent.

(i) *$\mathcal{V}$ is common output-nulling controlled invariant under mode-independent control.*
(ii) $A_s\mathcal{V} \subseteq e(\mathcal{V})^M + \operatorname{im} B_s$.
(iii) *There exists $F \in \mathbb{R}^{n_u \times n_x}$ such that $(A_j+B_jF)\mathcal{V} \subseteq \mathcal{V} \subseteq \ker(H_j + J_jF)$ for all $j = 1, \ldots, M$.*

Proof. (i) ⇒ (ii). Assume that $\sigma(t) = \sigma^j$ for some $j \in \{1, \ldots, M\}$. Let $x_0 \in \mathcal{V}$ and u be a control input such that $x_{x_0,\sigma^j,0,u}(t) = x^j(t) \in \mathcal{V}$ and $z_{x_0,\sigma^j,0,u}(t) = z^j(t) = 0$ for all $j = 1, \ldots, M$ and $t \in \mathbb{R}_+$. Then one can write

$$\begin{pmatrix} x^1(t) \\ \vdots \\ x^M(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ \vdots \\ x_0 \end{pmatrix} + \int_0^t \left(\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_M \end{bmatrix} \begin{pmatrix} x^1(s) \\ \vdots \\ x^M(s) \end{pmatrix} + \begin{bmatrix} B_1 \\ \vdots \\ B_M \end{bmatrix} u(s) \right) ds. \tag{33}$$

Furthermore, $z_{x_0,\sigma^j,0,u}(t) = H_j x^j(t) + J_j u(t) = 0$ for all $j = 1, \ldots, M$ and for all $t \in \mathbb{R}_+$. Thus, one can also write

$$\int_0^t \begin{pmatrix} z^1(s) \\ \vdots \\ z^M(s) \end{pmatrix} ds = 0. \tag{34}$$

Combining (33) with (34), we get

$$\begin{bmatrix} A_1 & & 0 \\ H_1 & & \\ & \ddots & \\ & & A_M \\ 0 & & H_M \end{bmatrix} \int_0^t \begin{pmatrix} x^1(s) \\ \vdots \\ x^M(s) \end{pmatrix} ds = \underbrace{\begin{pmatrix} x^1(t) - x_0 \\ 0 \\ \vdots \\ x^M(t) - x_0 \\ 0 \end{pmatrix}}_{\in e(\mathcal{V})^M} - \underbrace{\begin{bmatrix} B_1 \\ J_1 \\ \vdots \\ B_M \\ J_M \end{bmatrix} \int_0^t u(s) ds}_{\in \operatorname{im} B_s}. \tag{35}$$

Now we divide the left-hand side of (35) by t and take the limit to obtain

$$\lim_{t \to 0} \frac{1}{t} \begin{bmatrix} A_1 & & 0 \\ H_1 & & \\ & \ddots & \\ & & A_M \\ 0 & & H_M \end{bmatrix} \int_0^t \begin{pmatrix} x^1(s) \\ \vdots \\ x^M(s) \end{pmatrix} ds = \begin{bmatrix} A_1 & & 0 \\ H_1 & & \\ & \ddots & \\ & & A_M \\ 0 & & H_M \end{bmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_0 \end{pmatrix} = \begin{bmatrix} A_1 \\ H_1 \\ \vdots \\ A_M \\ H_M \end{bmatrix} x_0. \tag{36}$$

Equality (36) holds because $x^j(t)$ is continuous for all $j = 1, \ldots, M$. The rest of the proof is clear from here on.

(ii) ⇒ (iii): Choose a basis $\{q_1, \ldots, q_{n_v}, q_{n_v+1}, \ldots, q_{n_x}\}$ for $\mathbb{R}^{n_x}$ such that $\{q_1, \ldots, q_{n_v}\}$ is a basis for $\mathcal{V}$. For $k = 1, \ldots, n_v$ there exist vectors $\bar{q}_{j,k} \in \mathcal{V}$ and $u_k \in \mathbb{R}^{n_u}$ such that $A_j q_k = \bar{q}_{j,k} + B_j u_k$ and $H_j q_k = J_j u_k$ for all $j = 1, \ldots, M$. For $k = 1, \ldots, n_v$ define $F q_k = -u_k$ and for $k = n_v + 1, \ldots, n_x$ let $F q_k$ be arbitrary vectors in $\mathbb{R}^{n_x}$. Then for $(j, k) \in \{1, \ldots, M\} \times \{1, \ldots, n_v\}$ we have $(A_j + B_j F) q_k = \bar{q}_{j,k} \in \mathcal{V}$ and $(H_j + J_j F) q_k = 0$. Hence $(A_j + B_j F)\mathcal{V} \subseteq \mathcal{V} \subseteq \ker(H_j + J_j F)$ for all $j = 1, \ldots, M$.

(iii) ⇒ (i): Let $x_0 \in \mathcal{V}$ and apply the feedback law $u(t) = Fx(t)$ to obtain the closed loop system $\dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)} F)x(t), z(t) = (H_{\sigma(t)} + J_{\sigma(t)} F)x(t)$. Clearly, $x_{x_0,\sigma,0,u}(t) \in \mathcal{V}$ and $z_{x_0,\sigma,0,u}(t) = 0$ for all $t \in \mathbb{R}_+$. Thus, (i) follows. □

Definition 4.11. Consider the SLS (28) with $d = 0$. We define $\mathcal{V}^*_{mi}$ as the largest common output-nulling controlled invariant subspace under mode-independent control for the SLS (28), that is,

1. $\mathcal{V}^*_{mi}$ is common output-nulling controlled invariant under mode-independent control;
2. if $\mathcal{V}$ is common output-nulling controlled invariant under mode-independent control for the SLS (28), then $\mathcal{V} \subseteq \mathcal{V}^*_{mi}$.

Corollary 4.12. *Consider the SLS (28). DDPd by mode-independent state feedback is solvable if and only if*

$$\sum_{i=1}^{M} \operatorname{im} E_i \subseteq \mathcal{V}^*_{mi}. \tag{37}$$

Proof. Following the line of reasoning in the proof of Corollary 4.6, the proof can be obtained from Theorems 3.3, 4.10 and Definition 4.11. □

In Section 5, we will present an algorithm to compute the largest common output-nulling controlled invariant subspace under mode-independent control for a given SLS.

4.3. Solution of DDP with respect to σ by mode-dependent state feedback

Problem 4.13. Consider the SLS (28). The disturbance decoupling problem with respect to σ (DDPσ) by mode-dependent state feedback amounts to finding $F_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, M$ such that the SLS (29) is DD with respect to σ.

Before giving the theorem for the solvability of Problem 4.13, we need to introduce the following lemma.

Lemma 4.14. *Consider the SLS (28). DDPσ by mode-dependent feedback is solvable if and only if there exist a common output-nulling controlled invariant subspace $\mathcal{V}$ and $F_k \in \mathbb{R}^{n_u \times n_x}$, $k = 1, \ldots, M$, with $(A_k + B_k F_k)\mathcal{V} \subseteq \mathcal{V} \subseteq \ker(H_k + J_k F_k)$, $k = 1, \ldots, M$, such that the following three conditions hold for all $(i, j) \in \{1, \ldots, M\} \times \{1, \ldots, M\}$:*

(i) $H_i + J_i F_i = H_j + J_j F_j$,
(ii) $\operatorname{im}(A_i + B_i F_i - A_j - B_j F_j) \subseteq \mathcal{V}$,
(iii) $\operatorname{im}(E_i - E_j) \subseteq \mathcal{V}$.

Proof. The proof of the lemma directly follows from Theorems 3.6 and 4.4. □

Theorem 4.15. *Consider the SLS (28). DDPσ by mode-dependent feedback is solvable if and only if there exist $G_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, M$, such that for all $(i, j) \in \{1, \ldots, M\} \times \{1, \ldots, M\}$ it holds that*

(i) $H_i + J_i G_i = H_j + J_j G_j$,
(ii) $\operatorname{im}(A_i + B_i G_i - A_j - B_j G_j) \subseteq \mathcal{V}^*_{md}$,
(iii) $\operatorname{im}(E_i - E_j) \subseteq \mathcal{V}^*_{md}$.

In case the above conditions are satisfied, a mode-dependent state feedback $u(t) = F_{\sigma(t)} x(t)$ that renders (29) DD with respect to σ can be constructed by letting $\{v_1, v_2, \ldots, v_{nx}\}$ be a basis for $\mathbb{R}^{n_x}$ such that $\{v_1, \ldots, v_q\}$ is a basis for $\mathcal{V}^*_{md}$ and defining

$$F_i v_k = \begin{cases} \tilde{F}_i v_k & k \in \{1, 2, \ldots, q\} \\ G_i v_k & k \in \{q + 1, q + 2, \ldots, n_x\} \end{cases}$$

in which $\tilde{F}_i$ satisfies $(A_i + B_i \tilde{F}_i)\mathcal{V}^*_{md} \subseteq \mathcal{V}^*_{md} \subseteq \ker(H_i + J_i \tilde{F}_i)$, $i \in \{1, \ldots, M\}$.

Proof. *Necessity*. For a subspace $\mathcal{V}$ we will use the notation

$$\mathcal{F}_i(\mathcal{V}) = \{F | (A_i + B_i F)\mathcal{V} \subseteq \mathcal{V} \subseteq \ker(H_i + J_i F)\}.$$

If DDPσ is solvable, then as a result of Lemma 4.14, there exists a common output-nulling controlled invariant subspace, $\mathcal{V}$, such that $H_i + J_i F'_i = H_j + J_j F'_j$, for some $F'_k \in \mathcal{F}_k(\mathcal{V})$ $k = 1, \ldots, M$, for all $(i, j) \in \{1, \ldots, M\} \times \{1, \ldots, M\}$. It also holds that

$\text{im}\,(A_i + B_iF_i' - A_j - B_jF_j') \subseteq \mathcal{V} \subseteq \ker(H_i + J_iF_i')$
$\text{im}\,(E_i - E_j) \subseteq \mathcal{V}$.

Let $\tilde{F}_i \in \mathcal{F}_i(\mathcal{V}^*_{md})$ for all $i = 1, \ldots, M$. Since $\mathcal{V} \subseteq \mathcal{V}^*_{md}$, one can choose a basis $\{v_1, v_2, \ldots, v_{nx}\}$ for $\mathbb{R}^{n_x}$ such that $\{v_1, \ldots, v_p\}$ is a basis for $\mathcal{V}$ and $\{v_1, \ldots, v_q\}$ is a basis for $\mathcal{V}^*_{md}$. Define

$$G_iv_k = \begin{cases} \tilde{F}_iv_k & k \in \{1, 2, \ldots, q\} \\ F_i'v_k & k \in \{q+1, q+2, \ldots, n_x\}. \end{cases}$$

Clearly, $G_i \in \mathcal{F}_i(\mathcal{V}^*_{md})$ and $H_i + J_iG_i = H_j + J_jG_j$ for all $(i, j) \in \{1, \ldots, M\} \times \{1, \ldots, M\}$. We claim that $\text{im}\,(A_i+B_iG_i-A_j-B_jG_j) \subseteq \mathcal{V}^*_{md}$. To see this, note that

$$[(A_i + B_iG_i) - (A_j + B_jG_j)]v_k \in \begin{cases} \mathcal{V}^*_{md} & k \in \{1, 2, \ldots, q\} \\ \mathcal{V} & k \in \{q+1, q+2, \ldots, n_x\}. \end{cases}$$

Since $\mathcal{V} \subseteq \mathcal{V}^*_{md}$, it immediately follows that $\text{im}\,(A_i + B_iG_i - A_j - B_jG_j) \subseteq \mathcal{V}^*_{md}$. Note that

$\text{im}\,(E_i - E_j) \subseteq \mathcal{V} \subseteq \mathcal{V}^*_{md}$

and thus also (iii) holds.

Sufficiency. Let $\{v_1, v_2, \ldots, v_{nx}\}$ be a basis for $\mathbb{R}^{n_x}$ such that $\{v_1, \ldots, v_q\}$ is a basis for $\mathcal{V}^*_{md}$. Furthermore, let $\tilde{F}_i \in \mathcal{F}_i(\mathcal{V}^*_{md})$, $i = 1, \ldots, M$. Define

$$F_iv_k = \begin{cases} \tilde{F}_iv_k & k \in \{1, 2, \ldots, q\} \\ G_iv_k & k \in \{q+1, q+2, \ldots, n_x\}. \end{cases}$$

Note that $F_i \in \mathcal{F}_i(\mathcal{V}^*_{md})$ for all $i = 1, \ldots, M$. It is easy to see that $\text{im}\,(A_i + B_iF_i - A_j - B_jF_j) \subseteq \mathcal{V}^*_{md}$ and $H_i + J_iF_i = H_j + J_jF_j$ for all $(i, j) \in \{1, \ldots, M\} \times \{1, \ldots, M\}$. Thus we recovered the conditions of Lemma 4.14, thereby completing the proof. □

4.4. Solution of DDP with respect to d and σ (DDPdσ) by a mode-dependent state feedback

Problem 4.16. Consider the SLS (28). The disturbance decoupling problem with respect to d and σ (DDPdσ) by mode-dependent state feedback amounts to finding $F_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, M$ such that the SLS (29) is DD with respect to d and σ.

Theorem 4.17. *Consider the SLS (28). DDPdσ by mode-dependent state feedback is solvable if and only if there exist $G_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, M$ such that for all $(i, j) \in \{1, \ldots, M\} \times \{1, \ldots, M\}$ it holds that*

(i) $H_i + J_iG_i = H_j + J_jG_j$,
(ii) $\text{im}\,(A_i + B_iG_i - A_j - B_jG_j) \subseteq \mathcal{V}^*_{md}$,
(iii) $\text{im}\,E_i \subseteq \mathcal{V}^*_{md}$.

In case the above conditions are satisfied, a mode-dependent state feedback $u(t) = F_{\sigma(t)}x(t)$ that renders (29) DD with respect to d and σ can be constructed by letting $\{v_1, v_2, \ldots, v_{nx}\}$ be a basis for $\mathbb{R}^{n_x}$ such that $\{v_1, \ldots, v_q\}$ is a basis for $\mathcal{V}^*_{md}$ and defining

$$F_iv_k = \begin{cases} \tilde{F}_iv_k & k \in \{1, 2, \ldots, q\} \\ G_iv_k & k \in \{q+1, q+2, \ldots, n_x\} \end{cases}$$

in which $\tilde{F}_i$ satisfies $(A_i + B_i\tilde{F}_i)\mathcal{V}^*_{md} \subseteq \mathcal{V}^*_{md} \subseteq \ker(H_i + J_i\tilde{F}_i)$, $i \in \{1, \ldots, M\}$.

Proof. The proof of the theorem is obtained along similar lines as in the proof of Theorem 4.15. □

Remark 4.18. In Theorems 4.15 and 4.17, if $\mathcal{V}^*_{md}$ is replaced with $\mathcal{V}^*_{mi}$ and $G_i = G_j$ for all $(i, j) \in \{1, \ldots, M\} \times \{1, \ldots, M\}$, then the solutions to DDPσ and DDPdσ by mode-independent state feedback, respectively, are obtained.

5. Algorithms to test the hypotheses of the solutions to the DDP

5.1. Algorithm for Corollary 4.6

In this subsection, we will present an algorithm to find the largest common output-nulling controlled invariant subspace for the SLS (28).

Algorithm 5.1.

$$\mathcal{V}_0 = \mathbb{R}^{n_x}; \tag{38a}$$

$$\mathcal{V}_{i+1} = \bigcap_{j=1}^{M} \{x \mid \exists u\; A_jx + B_ju \in \mathcal{V}_i \text{ and } H_jx + J_ju = 0\}. \tag{38b}$$

From this recurrence relation, it follows that $\mathcal{V}_{i+1} \subseteq \mathcal{V}_i$ for all $i = 0, 1, \ldots,$ and if $\mathcal{V}_k = \mathcal{V}_{k+1}$ for some k, then $\mathcal{V}_i = \mathcal{V}_k$ for all $i \geq k$. Let q be the smallest $k \in \mathbb{N}$ such that $\mathcal{V}_k = \mathcal{V}_{k+1}$. Obviously, $q \leq n_x$. We claim that $\mathcal{V}_q = \mathcal{V}^*_{md}$.

Theorem 5.2. *Consider the SLS (28) and Algorithm 5.1. Then, $\mathcal{V}_q = \mathcal{V}^*_{md}$ with $q := \min\{k \in \mathbb{N} \mid \mathcal{V}_k = \mathcal{V}_{k+1}\} \leq n_x$.*

Proof. We will first show $\mathcal{V}_{i+1} \subseteq \mathcal{V}_i$ for $i \in \mathbb{N}$ by induction. It is obvious that $\mathcal{V}_1 \subseteq \mathcal{V}_0$. Suppose now that $\mathcal{V}_{i+1} \subseteq \mathcal{V}_i$.

$$\begin{aligned} \mathcal{V}_{i+2} &= \bigcap_{j=1}^{M} \{x \mid \exists u\; A_jx + B_ju \in \mathcal{V}_{i+1}, H_jx + J_ju = 0\} \\ &\subseteq \bigcap_{j=1}^{M} \{x \mid \exists u\; A_jx + B_ju \in \mathcal{V}_i, H_jx + J_ju = 0\} \\ &= \mathcal{V}_{i+1} \end{aligned}$$

from which $\mathcal{V}_{i+2} \subseteq \mathcal{V}_{i+1}$ follows. Therefore, $\mathcal{V}_{i+1} \subseteq \mathcal{V}_i$ for $i \in \mathbb{N}$.

Suppose that for some k, $\mathcal{V}_{k+1} = \mathcal{V}_k$. Then

$$\begin{aligned} \mathcal{V}_{k+2} &= \bigcap_{j=1}^{M} \{x \mid \exists u\; A_jx + B_ju \in \mathcal{V}_{k+1}, H_jx + J_ju = 0\} \\ &= \bigcap_{j=1}^{M} \{x \mid \exists u\; A_jx + B_ju \in \mathcal{V}_k, H_jx + J_ju = 0\} \\ &= \mathcal{V}_{k+1}. \end{aligned}$$

Hence, $\mathcal{V}_i = \mathcal{V}_k$ for all $i \geq k$. Since $\mathcal{V}_k \subseteq \mathcal{V}_{k-1} \subseteq \cdots \subseteq \mathcal{V}_0$, we have $\mathcal{V}_q = \mathcal{V}_{q+1}$ for some $q \leq n_x$.

$$\mathcal{V}_q = \bigcap_{j=1}^{M} \{x \mid \exists u\; A_jx + B_ju \in \mathcal{V}_q,\; H_jx + J_ju = 0\}.$$

This shows that

$$\begin{bmatrix} A_j \\ H_j \end{bmatrix} \mathcal{V}_q \subseteq e(\mathcal{V}_q) + \text{im} \begin{bmatrix} B_j \\ J_j \end{bmatrix}$$

for all $j = 1, \ldots, M$. By Theorem 4.4, $\mathcal{V}_q$ is indeed common output-nulling controlled invariant.

To show that $\mathcal{V}_q$ is the largest common output-nulling controlled invariant subspace, we consider a common output-nulling controlled invariant subspace $\tilde{\mathcal{V}}$ for the SLS (28).

$\tilde{\mathcal{V}} \subseteq \mathcal{V}_0$, thus the following:

$$\begin{aligned} \tilde{\mathcal{V}} &= \bigcap_{j=1}^{M} \{x \in \tilde{\mathcal{V}} \mid \exists u\; A_jx + B_ju \in \tilde{\mathcal{V}},\; H_jx + J_ju = 0\} \\ &\subseteq \bigcap_{j=1}^{M} \{x \mid \exists u\; A_jx + B_ju \in \mathcal{V}_0,\; H_jx + J_ju = 0\} \\ &= \mathcal{V}_1. \end{aligned}$$

Hence, $\tilde{\mathcal{V}} \subseteq \mathcal{V}_1$. Repeating the procedure above, we arrive at the conclusion $\tilde{\mathcal{V}} \subseteq \mathcal{V}_q$. □

5.2. Algorithm for Corollary 4.12

In this subsection, we will present the algorithm to find the largest common output-nulling controlled invariant subspace under mode-independent control for the SLS (28).

Algorithm 5.3. Define the matrices A_s and B_s in the same way as in Theorem 4.10.

$$\mathcal{V}_0 = \mathbb{R}^{n_x}; \qquad \mathcal{V}_{i+1} = \{x \mid A_s x \in e(\mathcal{V}_i)^M + \operatorname{im} B_s\}. \tag{39}$$

As in the previous algorithm, it holds that $\mathcal{V}_{i+1} \subseteq \mathcal{V}_i$ for all $i = 0, 1, \ldots$ and if $\mathcal{V}_k = \mathcal{V}_{k+1}$ for some k, then $\mathcal{V}_i = \mathcal{V}_k$ for all $i \geq k$. Let q be the smallest $k \in \mathbb{N}$ such that $\mathcal{V}_k = \mathcal{V}_{k+1}$. Obviously, $q \leq n_x$. We claim that $\mathcal{V}_q = \mathcal{V}^*_{mi}$ in the next theorem of which the proof can be obtained following a similar line of reasoning as in the proof of Theorem 5.2.

Theorem 5.4. *Consider the SLS (28) and Algorithm 5.3. Then,* $\mathcal{V}_q = \mathcal{V}^*_{mi}$ *with* $q := \min\{k \in \mathbb{N} \mid \mathcal{V}_k = \mathcal{V}_{k+1}\} \leq n_x$.

Note that by defining A_s as in (32) and redefining $B_s \in \mathbb{R}^{M(n_x+n_z)\times Mn_u}$ as

$$B_s = \begin{bmatrix} B_1 & 0 & \ldots & 0 \\ J_1 & 0 & \ldots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & \ldots & 0 & B_M \\ 0 & \ldots & 0 & J_M \end{bmatrix}.$$

Algorithm 5.3 yields the largest common output-nulling controlled invariant subspace (under mode-dependent control).

Remark 5.5. The algorithm of Section 5.1 to obtain the largest common controlled invariant subspace (with mode-dependent feedback) for an SLS inside another subspace is related to the algorithm presented before in [9,10] for LPV systems for the special case that $J_i = 0$, $i = 1, \ldots, M$. The algorithm in Section 5.2 for mode-independent feedback was not presented in the literature before.

6. Conclusions

In this paper, three different disturbance decoupling (DD) properties for switched linear systems were analyzed. The difference between the three properties is induced by which signals are considered as the disturbances: (i) the exogenous signal, (ii) the switching signal, or (iii) both the exogenous signal and the switching signal. The latter variant of DD is relevant in the context of fault-tolerant control and piecewise linear systems, as we motivated in the paper. In particular, DD of a switched linear system with respect to the switching signal and the exogenous disturbance signal implies DD of corresponding piecewise linear systems with respect to the continuous disturbance signal. In addition, all the mentioned DD properties for switched linear systems can be transformed directly into corresponding properties for linear parameter-varying (LPV) systems. Complete geometric characterizations for these properties were given, which were used to solve also disturbance decoupling problems (DDPs) by a suitable choice of controllers. Both mode-dependent and mode-independent static state feedback based controllers were considered. We used common controlled invariant subspaces (using both mode-dependent and mode-independent control) to characterize the solvability of the DDPs. Algorithms to compute these subspaces were provided as well, so that these results can be applied by straightforward computations.

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