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Controllability of a class of bimodal discrete-time piecewise linear systems ☆

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ABSTRACT

In this paper we will provide full algebraic necessary and sufficient conditions for the controllability/reachability/null controllability of a class of bimodal discrete-time piecewise linear systems including several instances of interest that are not covered by existing works which focus primarily on the planar case. In particular, the class is characterized by a continuous right-hand side, a scalar input and a transfer function from the control input to the switching variable with at most two zeroes whereas the state can be of any dimension. To prove the main result, we will make use of geometric control theory for linear systems and a novel result on controllability for input-constrained linear systems with non-convex constraint sets.

1. Introduction

Controllability has always played an important role in modern control theory. Kalman and Hautus studied this notion for linear systems and gave complete characterizations in algebraic forms. In the presence of input constraints, characterizations for the controllability of discrete-time linear systems were given in [1–3]. In case of hybrid dynamical systems, such as piecewise linear systems, such complete characterizations of null controllability, reachability or controllability are hard to come by. Indeed, it is known from [4] that certain controllability problems for discrete-time piecewise linear systems in a general setting are undecidable. However, several results were obtained on the controllability of different subclasses of piecewise linear systems. In [5,6], Xu and Xie give characterizations for the controllability of discrete-time *planar* bimodal piecewise linear systems. Brogliato obtains necessary and sufficient conditions for the controllability of a class of continuous-time piecewise linear systems in [7], but his results only apply to planar systems as well. Bemporad et al. [8] propose an algorithmic approach based on optimization tools. Although this approach makes it possible to check controllability of a given discrete-time system, it does not allow for drawing conclusions about any general class of systems. Arapostathis and Broucke give a fairly complete treatment of stability and controllability of continuous-time *planar* conewise linear systems in [9], which are piecewise linear systems for which the regions are convex cones. In [10], Lee and Arapostathis provide a characterization of controllability for a class of continuous-time piecewise linear systems but they assume, among other things, that the number of inputs in each subsystem is one less than the number of states. In [11], Camlibel et al. give algebraic necessary and sufficient conditions for the controllability of a class of continuous-time conewise linear systems, while stabilizability characterizations for bimodal piecewise linear systems are given in [12].

In this paper we present algebraic necessary and sufficient conditions for the controllability, reachability and null controllability of a class of bimodal discrete-time piecewise linear systems. The class is characterized by the property that the dynamics is continuous across the switching plane and that the input is scalar. We impose no restriction on the dimension of the state variable as opposed to the earlier mentioned works [5–7,9] that study the controllability of planar piecewise linear systems. Also, we do not adopt the assumption in [10], namely that the number of inputs is one less than the number of states. However, we assume that

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the transfer function from the control input to the switching variable, i.e. the variable that determines the active mode, has at most two zeroes. This seemingly rather odd assumption forms a complexity boundary in solving this problem for larger classes of PWL systems. Allowing three or more zeroes in the transfer function necessitates solving the controllability problem for discrete-time linear systems with non-convex input constraint sets, which is known to be a very hard problem. In spite of this assumption, our results are the first to provide algebraic necessary and sufficient conditions for the controllability for the indicated class of piecewise linear systems that contains several instances of interest not covered by the existing results. At the end of the paper we will demonstrate this fact with an example whose controllability could indeed not have been checked by earlier works.

The paper is organized as follows. We lay the groundwork to solve the controllability problem in Section 2 and we define the class of systems we are interested in Section 3. In Sections 4–6 we consider two different problems whose solutions are needed to tackle the main problem. In Section 7 we present our main results. Conclusions are presented in Section 8.

2. Definitions

2.1. Definitions

Consider the discrete-time system

$$x[k+1] = g(x[k], u[k]) \qquad y[k] = h(x[k]) \tag{1}$$

where $x[k] \in \mathbb{R}^n$, $u[k] \in \mathbb{R}^m$, $y[k] \in \mathbb{R}^p$ are the state, the input and the output variable, respectively, at discrete time $k \in \mathbb{N}$. Here, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are given functions. Given an initial state $x_0 \in \mathbb{R}^n$ and an input sequence $\mathbf{u} = \{u[0], \ldots, u[N]\}$ with $N \in \mathbb{N}$, we will denote the state trajectory corresponding to $\mathbf{u}$ and the initial state $x[0] = x_0$ by $x_{x_0,\mathbf{u}}$. Likewise, we will denote the corresponding output sequence by $y_{x_0,\mathbf{u}}$.

Definition 2.1. Consider the system (1). Let $\mathbf{y} = \{y[0], \ldots, y[N]\}$ be a sequence with $y[k] \in \mathbb{R}^p$, $k = 0, \ldots, N$. Given an initial state $x_0 \in \mathbb{R}^n$, we say that x_0 is compatible with $\mathbf{y}$, if there exists an input sequence $\mathbf{u} = \{u[0], \ldots, u[N-1]\}$ such that $y_{x_0,\mathbf{u}}[k] = y[k]$ for $k = 0, \ldots, N$. Likewise, given an input sequence $\bar{\mathbf{u}} = \{\bar{u}[0], \ldots, \bar{u}[N-1]\}$, we say that $\bar{\mathbf{u}}$ is compatible with $\mathbf{y}$, if there exists an initial state $\bar{x}_0 \in \mathbb{R}^n$ such that $y_{\bar{x}_0,\bar{\mathbf{u}}}[k] = y[k]$ for $k = 0, \ldots, N$.

Definition 2.2. Consider the system (1). We say that (1) is

- null controllable if for all $x_0 \in \mathbb{R}^n$ there exist an $N \in \mathbb{N}$ and an input sequence $\mathbf{u} = \{u[0], \ldots, u[N-1]\}$ such that $x_{x_0,\mathbf{u}}[N] = 0$.
- reachable if for all $x_f \in \mathbb{R}^n$ there exist an $N \in \mathbb{N}$ and an input sequence $\mathbf{u} = \{u[0], \ldots, u[N-1]\}$ such that $x_{0,\mathbf{u}}[N] = x_f$.
- controllable if for all $x_0, x_f \in \mathbb{R}^n$ there exist an $N \in \mathbb{N}$ and an input sequence $\mathbf{u} = \{u[0], \ldots, u[N-1]\}$ such that $x_{x_0,\mathbf{u}}[N] = x_f$.

2.2. Classical results

Consider the linear system

$$x[k+1] = Ax[k] + Bu[k] \tag{2}$$

with $x[k] \in \mathbb{R}^n$ being the state and $u[k] \in \mathbb{R}^m$ being the input, together with the input constraint

$$u[k] \in \mathcal{U}, \quad k \in \mathbb{N} \tag{3}$$

where $\mathcal{U} \subseteq \mathbb{R}^m$ is a solid polyhedral closed cone, i.e. there exists a matrix $U \in \mathbb{R}^{l\times m}$ for some $l \in \mathbb{N}$ such that $\mathcal{U} = \{u \in \mathbb{R}^m \mid Uu \geq 0\}$ and $\mathcal{U}$ has a nonempty interior. The inequalities in $Uu \geq 0$ are to be interpreted component-wise. The definitions of null controllability, reachability and controllability as in Definition 2.2 are similar for the constrained system (2)–(3) with the understanding that (3) should hold for the input sequence.

Definition 2.3. Let $\mathcal{U} \subseteq \mathbb{R}^m$ be a nonempty set. We define the dual cone $\mathcal{U}^*$ of $\mathcal{U}$ as

$$\mathcal{U}^* := \{v \in \mathbb{R}^m \mid v^\top u \geq 0 \, \forall u \in \mathcal{U}\}.$$

The following lemma can be based on [1,2,13].

Lemma 2.4. *Let $\mathcal{U}$ be a solid polyhedral closed cone. The constrained system (2)–(3) is*

- *null controllable if, and only if, the following implications hold:*

$$\left.\begin{array}{ll} \lambda \in \mathbb{C}\setminus\{0\}, & z \in \mathbb{C}^n, \\ z^\top A = \lambda z^\top, & z^\top B = 0 \end{array}\right\} \Rightarrow z = 0 \tag{4a}$$

$$\left.\begin{array}{ll} \lambda \in (0,\infty), & z \in \mathbb{R}^n, \\ z^\top A = \lambda z^\top, & B^\top z \in \mathcal{U}^* \end{array}\right\} \Rightarrow z = 0 \tag{4b}$$

- *reachable/controllable if, and only if, the following implications hold:*

$$\left.\begin{array}{ll} \lambda \in \mathbb{C}, & z \in \mathbb{C}^n, \\ z^\top A = \lambda z^\top, & z^\top B = 0 \end{array}\right\} \Rightarrow z = 0 \tag{5a}$$

$$\left.\begin{array}{ll} \lambda \in [0,\infty), & z \in \mathbb{R}^n, \\ z^\top A = \lambda z^\top, & B^\top z \in \mathcal{U}^* \end{array}\right\} \Rightarrow z = 0. \tag{5b}$$

3. Problem definition

In this paper we are interested in the null controllability/reachability/controllability of bimodal piecewise linear (PWL) systems that are of the form

$$x[k+1] = \begin{cases} A_1x[k] + B_1u[k] & y[k] \geq 0 \\ A_2x[k] + B_2u[k] & y[k] \leq 0 \end{cases} \tag{6}$$

$$y[k] = C^\top x[k]$$

where $x[k] \in \mathbb{R}^n$ is the state, $u[k] \in \mathbb{R}$ is the scalar input, $y[k] \in \mathbb{R}$ is a variable determining the active mode at discrete time $k \in \mathbb{N}$ and the matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, $B_1, B_2, C \in \mathbb{R}^n$ are given.

We assume $C \neq 0$ and that the right-hand side of (6) is continuous. This is equivalent to the existence of a vector $E \in \mathbb{R}^n$ such that

$$A_2 = A_1 + EC^\top \quad \text{and} \quad B_1 = B_2 =: B. \tag{7}$$

To show that this controllability problem is far from being trivial and that the controllability of (6) cannot be inferred from the controllability properties of the subsystems only, we would like to give a motivating example.

Example 3.1. Consider the PWL system given by

$$x_1[k+1] = \begin{cases} x_1[k] - x_2[k] & x_2[k] \geq 0 \\ x_1[k] + x_2[k] & x_2[k] \leq 0 \end{cases} \tag{8a}$$

$$x_2[k+1] = u[k]. \tag{8b}$$

In this example, both of the subsystems are controllable as linear systems, as is easily verified. However, the piecewise linear system is not controllable as $x_1[k] \leq x_1[0]$, $k \in \mathbb{N}$. This shows that controllability of a PWL system cannot be characterized only in terms of the controllability of its subsystems.

Define the transfer functions $G_i(z) = C^\top(zI - A_i)^{-1}B$ for $i = 1, 2$. It follows from (7) that σ is a zero of $G_1(z)$ if and only if it is a zero of $G_2(z)$ and that $G_1(z) \equiv 0$ if and only if $G_2(z) \equiv 0$. In the rest of the paper we will assume that $G_1(z) \not\equiv 0$, as otherwise the system (6) would not be controllable, and that it has at most two zeroes. We will provide insights on the need of this rather odd requirement later. Let $\mathcal{V}_i^*$ be the largest (A_i, B)-invariant subspace contained in $\ker C^\top$ for $i = 1, 2$, i.e. $\mathcal{V}_i^*$ is the largest of the subspaces, $\mathcal{V}_i$, that satisfy $(A_i + BF_i)\mathcal{V}_i \subseteq \mathcal{V}_i \subseteq \ker C^\top$ for some $F_i \in \mathbb{R}^{1\times n}$, $i = 1, 2$. We will denote the set $\{F_i \in \mathbb{R}^{1\times n} \mid (A_i + BF_i)\mathcal{V}_i^* \subseteq \mathcal{V}_i^* \subseteq \ker C^\top\}$ by $\mathcal{F}_i^*$. Likewise, let $\mathcal{Z}_i^*$ be the smallest $(C^\top, A_i)$-invariant subspace that contains $\operatorname{im} B$ for $i = 1, 2$, i.e. $\mathcal{Z}_i^*$ is the smallest of the subspaces, $\mathcal{Z}_i$, that satisfy $(A_i + K_iC^\top)\mathcal{Z}_i \subseteq \mathcal{Z}_i$ and $\operatorname{im} B \subseteq \mathcal{Z}_i$ for some $K_i \in \mathbb{R}^{n\times 1}$, $i = 1, 2$. See [14–16] for a detailed discussion on these particular subspaces. Since $G_i(z) \not\equiv 0, i = 1, 2, G_i(z)$ is invertible as a rational function. Therefore, $\mathbb{R}^n$ admits the decomposition $\mathbb{R}^n = \mathcal{V}_i^* \oplus \mathcal{Z}_i^*$ for $i = 1, 2$. See [14] for the proof of this implication. Due to (7), it holds that $\mathcal{V}_1^* = \mathcal{V}_2^* =: \mathcal{V}^*$, $\mathcal{Z}_1^* = \mathcal{Z}_2^* =: \mathcal{Z}^*$ and $\mathcal{F}_1^* = \mathcal{F}_2^* =: \mathcal{F}^*$. First, we apply the pre-compensating state feedback $u[k] = Fx[k] + v[k]$ to (6) with $F \in \mathcal{F}^*$. Due to (7) we have that $(A_1 + BF)|_{\mathcal{V}^*} = (A_2 + BF)|_{\mathcal{V}^*}$ Then, we apply the similarity transformation, $\tilde{x} = T^{-1}x$, with $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ where $\operatorname{im} T_1 = \mathcal{V}^*$ and $\operatorname{im} T_2 = \mathcal{Z}^*$. For ease of exposition, we will not use a different symbol for the new state vector and we will denote $v[k]$ by $u[k]$. Then, we obtain the following representation of (6) that is easier to deal with for characterizing null controllability, reachability and controllability:

$$x_1[k+1] = Hx_1[k] + \begin{cases} g_1 y[k] & y[k] \geq 0 \\ g_2 y[k] & y[k] \leq 0 \end{cases} \tag{9a}$$

$$x_2[k+1] = \begin{cases} J_1 x_2[k] + bu[k] & y[k] \geq 0 \\ J_2 x_2[k] + bu[k] & y[k] \leq 0 \end{cases} \tag{9b}$$

$$y[k] = c^\top x_2[k]$$

in which $x_1[k], g_1, g_2 \in \mathbb{R}^{n_1}$, $x_2[k], b, c \in \mathbb{R}^{n_2}$ for $k \in \mathbb{N}$ and $H \in \mathbb{R}^{n_1\times n_1}$, $J_1, J_2 \in \mathbb{R}^{n_2\times n_2}$ where $n_1 = \dim \mathcal{V}^*$, $n_2 = \dim \mathcal{Z}^*$ and $n_1 + n_2 = n$. Due to the assumption we made on the number of zeroes of $G_1(z)$, we have that $0 \leq n_1 \leq 2$.

Obviously, controllability properties are invariant under pre-compensating feedbacks and similarity transformations. Hence null controllability/reachability/controllability of (9) is equivalent to null controllability/reachability/controllability of (6). We will now study particular properties of the subsystems (9a) and (9b) that are useful for the main developments. The subsystem (9b) belongs to a special class of systems which we will introduce and analyze in the next section. In Section 6 we will analyze the subsystem (9a), which we call a push–pull type of system. Note that $\tilde{G}_i(z) = c^\top(zI - J_i)^{-1}b \not\equiv 0$ for $i = 1, 2$. Let $\tilde{\mathcal{V}}_i^*$ denote the largest (J_i, b)-invariant subspace contained in $\ker c^\top$ for $i = 1, 2$. Then, $\tilde{\mathcal{V}}_1^* = \tilde{\mathcal{V}}_2^* = \{0\}$. Let $\tilde{\mathcal{Z}}_i^*$ denote the smallest $(c^\top, J_i)$-invariant subspace that contains $\operatorname{im} b$. Then, $\tilde{\mathcal{Z}}_1^* = \tilde{\mathcal{Z}}_2^* = \mathbb{R}^{n_2}$. Due to (7), we also have that $J_2 = J_1 + e_2c^\top$ and

$$g_2 = g_1 + e_1 \tag{10}$$

where $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = T^{-1}E$ and $C^\top T = \begin{bmatrix} 0 & c^\top \end{bmatrix}$.

We would like to underline that what made this decomposition possible is the continuity assumption (7). As we will see in Section 7, the main results follow from the results we derive on the subsystems (9a) and (9b). Hence, the continuity assumption is one of the cornerstones of this work. Its omittance would require an entirely different treatment of the subject than the one in this paper, which is currently an open interesting problem.

4. Observability and invertibility of a special class of bimodal discrete-time piecewise linear systems

Consider the discrete-time piecewise linear system

$$x[k+1] = \begin{cases} A_1 x[k] + Bu[k] & y[k] \geq 0 \\ A_2 x[k] + Bu[k] & y[k] \leq 0 \end{cases} \tag{11a}$$

$$y[k] = C^\top x[k] \tag{11b}$$

where $x[k] \in \mathbb{R}^n$, $u[k], y[k] \in \mathbb{R}$ for $k \in \mathbb{N}$ and $A_1, A_2 \in \mathbb{R}^{n\times n}$, $B, C \in \mathbb{R}^{n\times 1}$. We will study a special class of the system (11) as we will need these results later in proving our main result. In particular, we adopt the following assumptions.

Assumption 4.1. The following statements hold.

1. The transfer functions $G_i(z) = C^\top(zI - A_i)^{-1}B \not\equiv 0$ for $i = 1, 2$;
2. The right-hand side of (11a) is continuous, i.e. $A_2 = A_1 + EC^\top$ for some vector $E \in \mathbb{R}^n$;
3. The largest (A_i, B)-invariant subspace contained in $\ker C^\top$, $\mathcal{V}_i^*$, is $\{0\}$ for $i = 1, 2$.
4. The smallest $(C^\top, A_i)$-invariant subspace containing $\operatorname{im} B$, $\mathcal{Z}_i^*$, is $\mathbb{R}^n$ for $i = 1, 2$.

We would like to point out that the system (9b) belongs to this special class of discrete-time piecewise linear systems.

Corollary 4.2. *Consider the system (11) and suppose Assumption 4.1 holds. Then, the following statements hold.*

1. $(C^\top, A_i)$ *is observable,*
2. (A_i, B) *is controllable,*
3. $C^\top B = C^\top A_i B = \cdots = C^\top A_i^{n-2} B = 0,$
4. $C^\top A_i^{n-1} B \neq 0$

for $i = 1, 2$.

Definition 4.3. We say that (11) is both observable and invertible if for any output sequence $\mathbf{y} = \{y[0], y[1], \ldots\}$ of infinite length there exist a unique initial state $x_0 \in \mathbb{R}^n$ and a unique input sequence of infinite length, $\{u[0], u[1], \ldots\}$, that are both compatible with $\mathbf{y}$.

Proposition 4.4. *Consider the system (11) for which Assumption 4.1 holds. Then, it is both observable and invertible. In particular, the first n entries of the output sequence $\{y[0], \ldots, y[n-1]\}$ uniquely determine the initial state x_0.*

Proof. Let

$$i_k = \begin{cases} 1 & y[k] > 0 \\ 2 & y[k] \leq 0 \end{cases} \quad k \in \mathbb{N}.$$

Then one can write for the system (11)

$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[n-1] \end{bmatrix} = Qx[0] \quad \text{with } Q = \begin{bmatrix} C^\top \\ C^\top A_{i0} \\ \vdots \\ C^\top A_{i_{n-2}} \cdots A_{i1} A_{i0} \end{bmatrix}.$$

Based on the second statement in Assumption 4.1 one can factorize Q in the following way.

$$Q = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & f_{3,2} & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & f_{n,2} & \cdots & f_{n,n-1} & 1 \end{bmatrix} \begin{bmatrix} C^\top \\ C^\top A_{i0} \\ \vdots \\ C^\top A_{i0}^{n-1} \end{bmatrix}.$$

The left factor is a lower triangular matrix whose off-diagonal terms vary based on the values of $i_0, \ldots, i_{n-2}$. The right factor is the observability matrix generated by $(C^\top, A_{i1})$. Since the pair $(C^\top, A_{i1})$ is observable it follows that Q is nonsingular. Thus, for any sequence $\{y[0], \ldots, y[n-1]\}$ there exists a unique $x[0]$ that produces this output sequence.

Furthermore, $y[n] = C^\top A_{in-1}A_{in-2}\cdots A_{i0}x[0] + Su[0]$ where $S = C^\top A_{in-1}A_{in-2}\cdots A_{i1}B$. Based on Corollary 4.2, we have that $S = C^\top {A_{i1}}^{n-1}B \neq 0$. Since $x[0]$ is uniquely determined by $\{y[0], \ldots, y[n-1]\}$, $u[0]$ is uniquely determined by $\{y[0], \ldots, y[n]\}$. By repeating the same argument, $u[1]$ is uniquely determined by $\{y[0], \ldots, y[n+1]\}$ and so on. Thus, (11) is both observable and invertible. □

5. Insights on the complexity of the controllability problem

Lemma 5.1. *The system (6) for which (7) holds with $G_i(z) = C^\top(zI - A_i)^{-1}B \not\equiv 0$, $i = 1, 2$, is controllable if and only if the push–pull system (9a) is controllable.*

Proof. Since the controllability of (6) is equivalent to the controllability of (9), the proof will be complete when we show (9) is controllable if and only if (9a) is controllable.

The necessity of this condition is clear.

To show its sufficiency we take two arbitrary states, $(x_{10}^\top, x_{20}^\top)^\top \in \mathbb{R}^n$ and $(x_{1f}^\top, x_{2f}^\top)^\top \in \mathbb{R}^n$. We would like to demonstrate how to steer $(x_{10}^\top, x_{20}^\top)^\top$ to $(x_{1f}^\top, x_{2f}^\top)^\top$. From (9b) we produce an output sequence $\{y[0], \ldots, y[n_2-1]\}$ that is compatible with x_{20} and apply it to (9a). Let $x_{1m} = x_1[n_2]$ denote the state the system (9a) is steered to upon the application of this sequence. Since (9a) is controllable there must exist $r \geq n_2$ such that $x_1[n_2] = x_{1m}$ and $x_1[r] = x_{1f}$ for some sequence $\{y[n_2], \ldots, y[r-1]\}$. Then, we extend the output sequence with $\{y[r], y[r+1], \ldots\}$, which is chosen to be compatible with x_{2f}. Due to Proposition 4.4, there is a unique input sequence $\{u[0], u[1], \ldots\}$ compatible with $\{y[0], y[1], \ldots\}$ for which it necessarily holds that $x_2[0] = x_{20}$, $x_2[r] = x_{2f}$. Thus we have constructed an input that satisfies $x_1[r] = x_{1f}$, $x_2[r] = x_{2f}$ with $x_1[0] = x_{10}$, $x_2[0] = x_{20}$, which completes the proof. □

Corollary 5.2. *The system (6) for which (7) holds with $G_i(z) = C^\top(zI - A_i)^{-1}B \not\equiv 0$, $i = 1, 2$, is null controllable if and only if the push–pull system (9a) is null controllable.*

Proof. The proof is obtained in exactly the same way as in the proof of Lemma 5.1 by taking $(x_{1f}^\top, x_{2f}^\top)^\top = 0$. □

Now that we have established the equivalency of the controllabilities of (6) and (9a), we embark upon the task of characterizing the controllability of the push–pull system (9a). Although we will give results on the controllability of push–pull systems under some conditions in the next section, first we would like to discuss the complexities inherent in this problem. We start with the framework under which we will discuss the controllability of push–pull systems.

A bimodal discrete-time push–pull system with a scalar input is given by

$$x[k+1] = Ax[k] + \begin{cases} B_+u[k] & u[k] \geq 0 \\ B_-u[k] & u[k] \leq 0 \end{cases} \tag{12}$$

where $x[k] \in \mathbb{R}^n$, $u[k] \in \mathbb{R}$ for $k \in \mathbb{N}$ and $A \in \mathbb{R}^{n\times n}$, $B_+, B_- \in \mathbb{R}^n$. Note that (9a) fits in this framework with y considered as the "input".

The system (12) can easily be written as the following equivalent input-constrained linear system.

$$x[k+1] = Ax[k] + [B_+ \; -B_-]v[k] \quad \text{with } v[k] \in \tilde{\mathcal{U}} \tag{13}$$

where $\tilde{\mathcal{U}} = [0, \infty) \times \{0\} \bigcup \{0\} \times [0, \infty)$. $\tilde{\mathcal{U}}$ is clearly a non-convex set. Therefore, solving the controllability problem for the PWL system (6) is equivalent to solving the controllability problem for the *discrete-time* linear system (13) with a particular non-convex input constraint set. Even though controllability of *continuous-time* linear systems with non-convex input constraint sets can be solved by convexification of the input constraint set through incorporating all the Filippov solutions (see *e.g.* [11,17]), clearly this cannot be done for discrete-time systems. Therefore, we need to resort to another method.

In the next section we will show how we obtained the controllability characterization for (13) under the state dimension restriction $n \in \{1, 2\}$ through a case-by-case analysis. Solving the controllability problem for (13) without imposing any restriction on the dimension of the state would allow us to remove the assumption we made on the maximum number of zeroes of $G_1(z)$. However, at higher dimensions, $n \geq 3$, case-by-case analysis is a very complicated, if not impossible task. Thus, a structural approach in characterizing controllability of (13) is indispensable if one wants to solve the controllability problem for (6) in a broader generality, that is without limitation on the maximum number of zeroes of $G_1(z)$ as long as the number of zeroes is finite. But, to the best of our knowledge, controllability of discrete-time linear systems with non-convex input constraint sets has not been solved so far and it is known to be a very hard problem. That is why in this paper we only deal with systems of the form (6) having at most two zeroes. Despite this, our results are still the first in characterizing controllability for the class of systems described in Section 3 and this class is not captured by any of the works in the literature.

Now we are ready to state our results on the controllability of push–pull systems of the form (12) for which $n = 1, 2$ holds. We would like to stress that these results are novel findings on the controllability of discrete-time linear systems with non-convex input constraint sets.

6. Controllability of bimodal discrete-time push–pull systems

Theorem 6.1. *Consider the system (12) with $n \in \{1, 2\}$. The following statements are equivalent.*

(i) *(12) is reachable.*
(ii) *The linear system given by*

$$x[k+1] = Ax[k] + [B_+ \; -B_-]v[k] \quad \text{with } v[k] \in \bar{\mathcal{U}} \tag{14}$$

where $\bar{\mathcal{U}} = \mathbb{R}^2_+$ is reachable.

Proof. (i) ⇒ (ii): Recall that (12) and (13) are equivalent systems. Since $\tilde{\mathcal{U}} \subset \bar{\mathcal{U}}$, the implication follows.

(ii) ⇒ (i): First, we consider the $n = 1$ case. Note that in this case the matrices A, B_+, B_- are scalars. Based on Lemma 2.4, reachability of the input-constrained linear system (14) dictates that if $A \geq 0$, then $B_+B_- > 0$ and if $A < 0$, then either $B_+ \neq 0$ or $B_- \neq 0$. Now, we will show that these conditions are sufficient for the reachability of (13). For the $A \geq 0$ case, it is easy to see that if $B_+B_- > 0$ then any state can be reached in one step by using the appropriate control input. For the $A < 0$ case, since either $B_+ \neq 0$ or $B_- \neq 0$, the sign of the terms in one of the sequences $\{B_+, AB_+, A^2B_+, \ldots\}$ and $\{B_-, AB_-, A^2B_-, \ldots\}$ is alternating, which is sufficient to conclude that any state can be reached in at most two steps.

Now we turn our attention to the case where $n = 2$. Note that under similarity transformation reachability properties of (13) and (14) remain invariant. First, we apply a similarity transformation to the system (14) to obtain

$$z[k+1] = Mz[k] + [N_1 \; N_2]v[k] \quad \text{with } v[k] \in \bar{\mathcal{U}} \tag{15}$$

where M is in Jordan form. Since (14) is reachable, (15) is reachable as well. Thus, the implications in Lemma 2.4 hold for (15). We will show that these implications are sufficient for the reachability of the system

$$z[k+1] = Mz[k] + [N_1\ N_2]v[k] \quad \text{with } v[k] \in \tilde{\mathcal{U}}. \tag{16}$$

This will imply the reachability of (13) and thus the reachability of (12).

We will make a case-by-case analysis of the implications in Lemma 2.4. Based on the Jordan form of M we distinguish three possible main cases. We adopt the following notation $N_1 = [p_1\ p_2]^\top$, $N_2 = [q_1\ q_2]^\top$. We point out that the state of the system (16) at the kth time instant starting from zero initial condition is given by

$$z[k] = [N_1\ N_2\ MN_1\ MN_2\ \cdots\ M^{k-1}N_1\ M^{k-1}N_2]\vec{v}_k$$

where $\vec{v}_k \in \tilde{\mathcal{U}}^k$. It is easy to see that for reachability of (16) it suffices to find a finite set of cones generated by vector pairs $\{M^iN_l, M^jN_\ell\}$ with $i \neq j$ and $(l, \ell) \in \{1, 2\} \times \{1, 2\}$ such that their union is $\mathbb{R}^2$. In the rest of this proof we will employ this idea to show reachability of (16).

1. $M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ with $\lambda_1, \lambda_2 \in \mathbb{R}$.
   (a) $\lambda_1 = \lambda_2 \geq 0$
   In this case, we have from implications (5) that the system (15) can never be reachable. It is easy to see that (16) can never be reachable in this case either. Hence, this case is irrelevant.
   (b) $\lambda_1 > 0$ and $\lambda_2 < 0$
   From implication (5a) we have that either $p_2 \neq 0$ or $q_2 \neq 0$. From implication (5b) we have $p_1q_1 < 0$. We consider two situations in this case, $p_2 = 0, q_2 \neq 0$ and $p_2 \neq 0, q_2 \neq 0$. In the former, the union of the three cones generated by the vector pairs $\{N_1, MN_2\}$, $\{N_1, M^2N_2\}$, $\{MN_2, M^2N_2\}$ is $\mathbb{R}^2$. For the latter, we consider two possible situations, $p_2q_2 > 0$ and $p_2q_2 < 0$. If $p_2q_2 < 0$, then the union of the four cones generated by the vector pairs $\{N_1, MN_1\}$, $\{N_1, MN_2\}$, $\{N_2, MN_1\}$, $\{N_2, MN_2\}$ is $\mathbb{R}^2$. If $p_2q_2 > 0$ and $|\lambda_1| < |\lambda_2|$, then the union of the five cones generated by $\{N_1, MN_1\}$, $\{MN_1, M^3N_2\}$, $\{M^3N_2, N_2\}$, $\{N_2, M^2N_2\}$, $\{M^2N_2, N_1\}$ is $\mathbb{R}^2$. If $p_2q_2 > 0$ and $|\lambda_1| \geq |\lambda_2|$, the union of the four cones generated by $\{N_1, MN_1\}$, $\{M^2N_2, M^3N_2\}$, $\{N_1, M^2N_2\}$, $\{MN_1, M^3N_2\}$ is $\mathbb{R}^2$.
   (c) $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_1 \neq \lambda_2$
   From implication (5b) we have $p_1q_1 < 0$ and $p_2q_2 < 0$. We consider two possible configurations; $\text{rank}([N_1\ N_2]) = 1$ and $\text{rank}([N_1\ N_2]) = 2$. In the former, the union of the cones generated by the vector pairs $\{N_1, MN_1\}$, $\{N_2, MN_2\}$, $\{N_1, MN_2\}$ and $\{N_2, MN_1\}$ is $\mathbb{R}^2$. In the latter, one can always find sufficiently large $j, l \in \mathbb{N}$ such that the union of the four cones generated by the vector pairs $\{N_1, M^jN_1\}$, $\{N_2, M^jN_1\}$, $\{N_2, M^lN_2\}$ and $\{N_1, M^lN_2\}$ is $\mathbb{R}^2$.
   (d) $\lambda_1 = 0, \lambda_2 > 0$
   From implication (5b), we have $p_1q_1 < 0$ and $p_2q_2 < 0$. Under these conditions, the union of the four cones generated by the vector pairs $\{N_1, MN_1\}$, $\{N_1, MN_2\}$, $\{N_2, MN_1\}$, $\{N_2, MN_2\}$ is $\mathbb{R}^2$.
   (e) $\lambda_1 = 0, \lambda_2 < 0$
   From implication (5b), we get $p_1q_1 < 0$ and from implication (5a) we get either $p_2 \neq 0$ or $q_2 \neq 0$. We consider two situations, $p_2 \neq 0, q_2 = 0$ and $p_2 \neq 0, q_2 \neq 0$. In the former, the union of the four cones generated by $\{N_1, MN_1\}$, $\{N_2, MN_1\}$, $\{N_1, M^2N_1\}$, $\{N_2, M^2N_1\}$ is $\mathbb{R}^2$. In the latter, we again consider two situations, $p_2q_2 > 0$ and $p_2q_2 < 0$. If $p_2q_2 > 0$, then the union of the four cones generated by $\{N_1, MN_1\}$, $\{N_2, MN_2\}$, $\{N_1, M^2N_1\}$, $\{N_2, M^2N_2\}$ is $\mathbb{R}^2$. If $p_2q_2 < 0$, the union of the four cones generated by $\{N_1, MN_1\}$, $\{N_2, MN_1\}$, $\{N_1, MN_2\}$, $\{N_2, MN_2\}$ is $\mathbb{R}^2$.
   (f) $\lambda_1 = \lambda_2 < 0$
   From implication (5a) we get $\text{rank}[N_1\ N_2] = 2$. Under this condition, the union of the four cones generated by $\{N_1, MN_2\}$, $\{N_2, MN_1\}$, $\{MN_2, M^3N_1\}$, $\{N_1, M^2N_2\}$ is $\mathbb{R}^2$.
   (g) $\lambda_1 < 0, \lambda_2 < 0$ and $\lambda_1 \neq \lambda_2$
   From implication (5a) we have either $p_1 \neq 0$ or $q_1 \neq 0$ and either $p_2 \neq 0$ or $q_2 \neq 0$. First, we consider the case where $p_1 \neq 0, q_2 \neq 0, p_2 = q_1 = 0$. In this case, the union of the four cones generated by $\{N_1, MN_2\}$, $\{N_2, MN_1\}$, $\{N_1, M^2N_2\}$, $\{MN_1, M^3N_2\}$ is $\mathbb{R}^2$. The case where $p_2 \neq 0, q_1 \neq 0$ and $p_1 = q_2 = 0$ can be treated in a similar way. Next, we take $p_1 \neq 0, p_2 \neq 0$ and diverge from our usual approach here to proceed as follows. Consider the following system where $w[k]$ is the scalar non-negative input.
   $$z[k+1] = Mz[k] + N_1w[k] \quad w[k] \geq 0. \tag{17}$$
   By invoking Lemma 2.4 under the conditions $p_1 \neq 0, p_2 \neq 0$, we can immediately conclude that (17) is reachable. We can recast (17) as a two-input system by constraining the second input to be zero as follows.
   $$z[k+1] = Mz[k] + [N_1\ N_2]v[k] \quad v[k] \in \mathcal{U}^* \tag{18}$$
   where $\mathcal{U}^* = [0, \infty) \times \{0\}$. Note that (17) and (18) are equivalent in terms of reachability. Comparing the systems (16) and (18), we see that since $\mathcal{U}^* \subset \tilde{\mathcal{U}}$ and (18) is reachable, (16) is reachable as well.

2. $M = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$
   (a) $\lambda \geq 0$
   From implication (5b), we have $p_2q_2 < 0$. When $\lambda = 0$, the union of the four cones generated by $\{N_1, MN_1\}$, $\{N_1, MN_2\}$, $\{N_2, MN_1\}$, $\{N_2, MN_2\}$ is $\mathbb{R}^2$. For $\lambda > 0$ and $p_1 = q_1 = 0$, the union of the four cones generated by $\{N_1, MN_1\}$, $\{N_1, MN_2\}$, $\{N_2, MN_1\}$, $\{N_2, MN_2\}$ is $\mathbb{R}^2$. To analyze the cases in which $\lambda > 0$ and either $p_1 \neq 0$ or $q_1 \neq 0$, we observe the evolutions of M^kN_1 and M^kN_2, which are given as $\lambda^{k-1}\begin{bmatrix} \lambda p_1 + kp_2 \\ \lambda p_2 \end{bmatrix}$ and $\lambda^{k-1}\begin{bmatrix} \lambda q_1 + kq_2 \\ \lambda q_2 \end{bmatrix}$, respectively. It becomes clear from this formulation that one can always find large enough $j, l \in \mathbb{N}$ such that the union of the four cones generated by $\{N_1, M^lN_1\}$, $\{N_2, M^lN_1\}$, $\{N_1, M^jN_2\}$, $\{N_2, M^jN_2\}$ is $\mathbb{R}^2$.
   (b) $\lambda < 0$
   From implication (5a), we have either $p_2 \neq 0$ or $q_2 \neq 0$. We take $p_2 \neq 0$ and consider the system (17). Note that the M matrix here is different than the one in the subcase 1(g). By invoking Lemma 2.4 under the condition $p_2 \neq 0$, we can immediately conclude that (17) is reachable. By following the same reasoning as in the subcase 1(g), we can arrive at the conclusion that (16) is reachable.

3. $M = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$ with $\omega \neq 0$. This case corresponds to M having a pair of complex conjugate eigenvalues, $\sigma \mp j\omega$. From implication (5a), we have that $\text{rank}([N_1\ N_2]) \geq 1$. We take $N_1 \neq 0$ and consider the system (17). Note that the M matrix here is different than the one in the subcase 1(g). By invoking Lemma 2.4 under the condition $N_1 \neq 0$, we immediately conclude that (17) is reachable. By following the same reasoning as in the subcase 1(g), we deduce that (16) is reachable. □

Corollary 6.2. *Consider the system (12) with $n \in \{1, 2\}$. The following statements are equivalent.*

(i) *The system (12) is null controllable.*
(ii) *The system (14) is null controllable with the input constraint $v[k] \in \bar{\mathcal{U}} := \mathbb{R}^2_+$.*

Proof. (i) ⇒ (ii): This implication is obvious.

(ii) ⇒ (i): First we consider the case $n = 1$. If $A = 0$, then the system (13) is clearly always null controllable. We will now consider the reverse-time versions of (13) and (14), given by

$$x[k] = A^{-1}x[k+1] + [-A^{-1}B_+ \; A^{-1}B_-]v[k] \quad v[k] \in \tilde{\mathcal{U}} \tag{19}$$

and

$$x[k] = A^{-1}x[k+1] + [-A^{-1}B_+ \; A^{-1}B_-]v[k] \quad v[k] \in \bar{\mathcal{U}} \tag{20}$$

respectively. Whenever A^{-1} exists, these reverse-time versions of (13) and (14) exist. If (14) is null controllable, then (20) is reachable. From Theorem 6.1, we deduce that (19) is reachable as well. Hence, its forward-time version, (13) is null controllable.

Thus, we have established the null controllability of (13) for the case $A \neq 0$ and hence for the case $n = 1$.

Now, we turn our attention to the case $n = 2$. The cases where A is nonsingular are immediately covered by the reasoning above. For the remaining cases, we will build a case-by-case analysis based on the transformed system (16).

1. $M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ with $\lambda_1, \lambda_2 \in \mathbb{R}$.
   (a) $\lambda_1 = \lambda_2 = 0$
   In this case the system (16) is clearly always null controllable.
   (b) $\lambda_1 = 0$ and $\lambda_2 \neq 0$
   We can expand the systems (15) and (16) as

$$\begin{pmatrix} z_1[k+1] \\ z_2[k+1] \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{pmatrix} z_1[k] \\ z_2[k] \end{pmatrix} + [N_1 \; N_2]v[k] \tag{21}$$

with $v[k] \in \bar{\mathcal{U}}$ and

$$\begin{pmatrix} z_1[k+1] \\ z_2[k+1] \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{pmatrix} z_1[k] \\ z_2[k] \end{pmatrix} + [N_1 \; N_2]v[k] \tag{22}$$

with $v[k] \in \tilde{\mathcal{U}}$, respectively. Next, we extract the z_2 dynamics of these systems as

$$z_2[k+1] = \lambda_2 z_2[k] + [p_2 \; q_2]v[k] \quad v[k] \in \bar{\mathcal{U}} \tag{23}$$

and

$$z_2[k+1] = \lambda_2 z_2[k] + [p_2 \; q_2]v[k] \quad v[k] \in \tilde{\mathcal{U}} \tag{24}$$

respectively. If (14) is null controllable, (15) and (21) are null controllable. Hence, (23) is null controllable as well. Based on the analysis for the $n = 1$ case, we can conclude (24) is null controllable. Considering the system (22), since (24) is null controllable we can first steer $z_2[0]$ to the origin with the appropriate control input and apply zero input afterwards to steer z_1 to the origin. Therefore, we can steer any $z[0] = [z_1[0] \; z_2[0]]^\top$ to the origin, which shows (22) and (16) are null controllable.

2. $M = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$
   (a) $\lambda = 0$
   As $M^2 = 0$, the system (16) is always null controllable. □

Corollary 6.3. *Consider the system (12) with $n \in \{1, 2\}$. The following statements are equivalent.*

(i) *The system (12) is controllable.*
(ii) *The system (14) is controllable with the input constraint $v[k] \in \bar{\mathcal{U}} := \mathbb{R}^2_+$.*

Proof. (i) ⇒ (ii): This is obvious.

(ii) ⇒ (i): Since (14) is controllable, it is reachable and null controllable. From Theorem 6.1 and Corollary 6.2, we have that (12) is reachable and null controllable. Therefore, (12) is controllable. □

7. Main results

We are now in a position to present the main result of this paper.

Theorem 7.1. *Consider the system (6) for which (7) holds with $G_i(z) = C^\top(zI - A_i)^{-1}B \not\equiv 0$ and $G_i(z)$ having at most two zeroes, $i = 1, 2$. Then the following statements hold.*

1. *The system (6) is null controllable if and only if the following implications hold:*

$$\left.\begin{matrix} \lambda \in \mathbb{C} \setminus \{0\}, & z \in \mathbb{C}^n, \\ z^\top A_i = \lambda z^\top, & z^\top B = 0 \end{matrix} \quad i = 1, 2 \right\} \Rightarrow z = 0 \tag{25a}$$

$$\left.\begin{matrix} \lambda \in (0, \infty), & z \in \mathbb{R}^n, \\ w_1 \leq 0, & w_2 \geq 0 \\ \begin{bmatrix} z^\top & w_i \end{bmatrix} \begin{bmatrix} A_i - \lambda I & B \\ C^\top & 0 \end{bmatrix} = 0 & \end{matrix} \quad i = 1, 2 \right\} \Rightarrow z = 0. \tag{25b}$$

2. *The system (6) is controllable if and only if the following implications hold.*

$$\left.\begin{matrix} \lambda \in \mathbb{C}, & z \in \mathbb{C}^n, \\ z^\top A_i = \lambda z^\top, & z^\top B = 0 \end{matrix} \quad i = 1, 2 \right\} \Rightarrow z = 0 \tag{26a}$$

$$\left.\begin{matrix} \lambda \in [0, \infty), & z \in \mathbb{R}^n, \\ w_1 \leq 0, & w_2 \geq 0 \\ \begin{bmatrix} z^\top & w_i \end{bmatrix} \begin{bmatrix} A_i - \lambda I & B \\ C^\top & 0 \end{bmatrix} = 0 & \end{matrix} \quad i = 1, 2 \right\} \Rightarrow z = 0. \tag{26b}$$

Proof. We will prove the second statement only. The proof of the first can be obtained in a similar way and is therefore omitted.

From Lemma 5.1, we know that the controllability of the push–pull system (9a) is equivalent to the controllability of (9) and thus to the controllability of (6). Using Lemma 2.4 and Corollary 6.3 we can give a characterization of the controllability of (9a), which is as follows

$$\left.\begin{matrix} \lambda \in \mathbb{C}, & z_1 \in \mathbb{C}^{n_1}, \\ z_1^\top H = \lambda z_1^\top, & z_1^\top g_1 = z_1^\top g_2 = 0 \end{matrix} \right\} \Rightarrow z_1 = 0 \tag{27a}$$

$$\left.\begin{matrix} \lambda \in [0, \infty), & z_1 \in \mathbb{R}^{n_1}, \\ z_1^\top H = \lambda z_1^\top, & z_1^\top g_1 \geq 0, z_1^\top g_2 \leq 0 \end{matrix} \right\} \Rightarrow z_1 = 0. \tag{27b}$$

Our aim is now to show the equivalence of (27) and (26). Observe that the condition (27a) is equivalent to the controllability of $(H, [g_1 \; g_2])$ as a linear system without constraints. Due to (7), (26a) is equivalent to the controllability of $(A_1, [B \; E])$ as a linear system without constraints. Therefore we need to demonstrate that controllability of $(H, [g_1 \; g_2])$ and controllability of $(A_1, [B \; E])$, as linear systems without constraints, are equivalent. For this purpose we will use the Hautus' test, which states that the controllability of $(A_1, [B \; E])$ as a linear system without constraints is equivalent to the implication

$$\left.\lambda \in \mathbb{C}, z \in \mathbb{C}^n, \quad z^\top [A_1 - \lambda I \; B \; E] = 0 \right\} \Rightarrow z = 0. \tag{28}$$

Condition (28) can also be written for the system (9) since the controllability properties do not change under similarity transformation and pre-compensating feedback. Thus, instead of (28) we will use the following implication

$$\left.\begin{matrix} \lambda \in \mathbb{C}, \quad z \in \mathbb{C}^n, \\ z^\top \begin{bmatrix} H - \lambda I & g_1 c^\top & 0 & e_1 \\ 0 & J_1 - \lambda I & b & e_2 \end{bmatrix} = 0 \end{matrix} \right\} \Rightarrow z = 0. \tag{29}$$

First we take $z^\top = (z_1^\top, z_2^\top)$, then observe that when

$$\begin{bmatrix} z_1^\top & z_2^\top \end{bmatrix} \begin{bmatrix} H - \lambda I & g_1 c^\top & 0 & e_1 \\ 0 & J_1 - \lambda I & b & e_2 \end{bmatrix} = 0 \tag{30}$$

for some $\lambda \in \mathbb{C}$, it holds that

$$\begin{bmatrix} z_1^\top g_1 & z_2^\top \end{bmatrix} \begin{bmatrix} c^\top & 0 \\ J_1 - \lambda I & b \end{bmatrix} = 0. \quad (31)$$

We have that the largest (J_1, b)-invariant subspace contained in $\ker c^\top$, $\tilde{\mathcal{V}}_1^*$, is $\{0\}$. Then it follows from Proposition II.3 in [11] that $z_1^\top g_1 = 0$ and $z_2 = 0$. Substituting $z_2 = 0$ into (30), we obtain $z_1^\top e_1 = 0$, thus $z_1^\top g_2 = 0$ due to (10). Hence, assuming that (27a) holds, it now dictates that $z_1 = 0$ and hence $z = 0$. Therefore, $(A_1, [B\ E])$ is controllable as a linear system if $(H, [g_1\ g_2])$ is controllable as a linear system. To show how the controllability of $(A_1, [B\ E])$ implies the controllability of $(H, [g_1\ g_2])$, we take $z^\top = (z_1^\top, z_2^\top) = (z_1^\top, 0)$ in (30). Then, assuming that (29) holds, we recover the implication (27a).

To prove the equivalence of the implications (26b) and (27b) we will again use the transformed system (9). We take $z^\top = (z_1^\top, z_2^\top)$. Then, the equality in the condition (26b),

$$\begin{bmatrix} z_1^\top & z_2^\top & w_i \end{bmatrix} \begin{bmatrix} H - \lambda I & g_i c^\top & 0 \\ 0 & J_i - \lambda I & b \\ 0 & c^\top & 0 \end{bmatrix} = 0, \quad i = 1, 2 \quad (32)$$

can be rewritten as

$$z_1^\top H = \lambda z_1^\top, \qquad [z_2^\top w_1 + z_1^\top g_1] \begin{bmatrix} J_1 - \lambda I & b \\ c^\top & 0 \end{bmatrix} = 0,$$
$$[z_2^\top w_2 + z_1^\top g_2] \begin{bmatrix} J_2 - \lambda I & b \\ c^\top & 0 \end{bmatrix} = 0. \quad (33)$$

Since the largest (J_i, b)-invariant subspace contained in $\ker c^\top$, $\tilde{\mathcal{V}}_i^*$, is $\{0\}$ for $i = 1, 2$, it follows from Proposition II.3 in [11] that $z_2 = 0$, $z_1^\top g_1 = -w_1 \geq 0$ and $z_1^\top g_2 = -w_2 \leq 0$. Due to (27b), $z_1 = 0$, hence $z = 0$. We have shown that (27b) implies (26b). Conversely, if (26b) holds, by taking $z_2 = 0$, $w_1 = -z_1^\top g_1 \leq 0$ and $w_2 = -z_1^\top g_2 \geq 0$ in (26b) for the transformed system (9) the implication (27b) follows. Thus, we have proven the equivalence of the implications (26b) and (27b), which completes the proof. □

Clearly, due to the complexity boundary indicated in Section 5 the necessary and sufficient conditions for controllability apply to a particular class of piecewise linear systems. To indicate that this class is not captured by the previous works, we provide an example below. We would like to stress that the example below is the simplest one that could have been chosen within the class of systems our results apply to and beyond the application range of the existing results.

Example 7.2. Consider the PWL system (6) given by the matrices

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

This system has three states and one input, thus it does not fall under the framework of the results in [5,10]. Note that the transfer functions $G_i(z) = C^\top(zI - A_i)^{-1}B$, $i = 1, 2$, have no zeroes, which means that this PWL system is covered by the results in this paper. Indeed, using Theorem 7.1, it is easily verified that this PWL system is controllable.

Now, we will show with hand calculation that this system is controllable. First, we take $x[0] = 0$ and we want to steer the state to some arbitrary x_f in finite time, say $r \in \mathbb{N}$, i.e. $x[r] = x_f$. It is easy to see that $x[1] = \begin{bmatrix} u[0] & u[0] & u[0] \end{bmatrix}^\top$. As $y[1] \equiv 0$, $x[2]$ is deterministically given as

$$x[2] = \begin{bmatrix} -u[0] + u[1] & u[1] & u[0] + u[1] \end{bmatrix}^\top.$$

Since $y[2] \equiv 0$, we have

$$x[3] = \begin{bmatrix} u[0] - u[1] + u[2] & u[2] & u[0] + u[1] + u[2] \end{bmatrix}^\top.$$

We can write this last equality as

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ u[2] \end{bmatrix} = x[3]. \quad (34)$$

Due to the invertibility of the 3×3 matrix in the above equation, for any given $x[3] = x_f$ one can find the appropriate input sequence $\{u[0], u[1], u[2]\}$ to steer the state of the system from the origin to x_f. Thus, we have established that this PWL system is reachable.

By a similar reasoning and hand calculation, one can also show null controllability of this system. Since this system is reachable and null controllable, we conclude that it is controllable.

8. Conclusion

In this paper we presented algebraic characterizations of the controllability, reachability and null controllability of a class of discrete-time bimodal piecewise linear systems. By using geometric control theory we showed that the controllability problem is equivalent to the controllability of a particular subsystem of the original system, which is in the form of a so-called push–pull system. By studying the controllability of linear systems with a particular non-convex input constraint set, we derived conditions for the controllability of push–pull systems up to a certain dimension, hence leading to our main result. Future research in this topic will be the characterization of controllability for discrete-time push–pull systems of arbitrary dimension.

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