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Output-feedback Control of Lur'e-type Systems with Set-valued Nonlinearities: a Popov-criterion Approach

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Abstract— **This paper presents an output-feedback controller design for Lur'e-type systems with set-valued nonlinearities in the feedback loop based on a generalization of a Popov-like criterion. Hereto, we introduce the concept of absolute input-to-state stability (ISS) that generalizes the well-known absolute stability property. The latter concept is used to design a state-feedback controller that renders the closed-loop system absolutely ISS and, therewith, robust to uncertainties in the nonlinearities and disturbances, such as measurement noise. Furthermore, an output-feedback controller design is constructed by exploiting the ISS property, where a model-based observer is used to estimate the system state. The control strategy is applied to a mechanical motion system with non-collocation of actuation and dry friction for which well-known strategies such as direct friction compensation fail. The effectiveness of the proposed output-feedback control strategy is shown in simulations.**

I. Introduction

In this paper, we present an output-feedback controller design for Lur'e-type systems with set-valued nonlinearities in the feedback loop. Supporting such a controller design, we propose a generalization of a Popov-like criterion (as developed in the scope of absolute stability theory), in the sense that it is applicable to systems with set-valued nonlinearities. An important class of engineering systems that can be described in this form is linear mechanical motion systems with set-valued friction laws.

Existing stabilization techniques based on absolute stability theory for locally Lipschitzian systems with slope-restricted nonlinearities are discussed in [1], [2]. However, due to the set-valued nature of the nonlinearities considered here, these results are not applicable. A generalized circle criterion, that is suitable for systems with set-valued nonlinearities, is discussed in [3]. Unfortunately, the conditions of the circle criterion are rather restrictive for typical motion control applications as will be indicated in this paper.

The proposed extension to the Popov-criterion also guarantees input-to-state stability (ISS) (instead of only asymptotic stability) with respect to perturbations on the system (e.g. measurement noise). In analogy with the "absolute stability" property, obtained by satisfaction of the conventional Popov-criterion, see e.g. [4], [5], one might call this property "absolute ISS". The fact that the satisfaction of such an adapted Popov criterion guarantees absolute ISS implies robustness with respect to both uncertainties in the set-valued nonlinearities and measurement errors. Moreover, the ISS property is used to construct an observer-based output-feedback controller and we provide a separation principle for the output-feedback controlled system.

The notion of ISS [6] is a useful property in the field of control, which ensures, roughly speaking, that the state of the system is bounded for a bounded input. In [7], a proof of ISS for Lur'e-type systems is given, which is not applicable to systems with set-valued nonlinearities. Next to the fact that the usual work on ISS considers continuous systems, they focus typically on the use of smooth ISS Lyapunov functions (see e.g. [7]–[9] to mention just a few). In case of extending the Popov-criterion to the discontinuous systems as considered here, one has to adopt non-smooth (ISS) Lyapunov functions. The reason of non-smoothness is that the Lyapunov function contains a term consisting of an integral of the nonlinearity. Despite some recent work, see [10]–[12], to bring ISS concepts to the realm of discontinuous and switched systems, none of these papers can be used in the present context.

The concept of absolute ISS is used for the design of a stabilizing feedback controller for a mechanical motion system with set-valued frictional nonlinearities that can be described by a Lur'e-type system. An advantage of the proposed controller design is that it is applicable to systems with non-collocation of actuation and set-valued friction laws. Additionally, the absolute ISS property implies robustness with respect to uncertainties in the friction and measurement errors. A common approach to tackle motion control problems for systems with (set-valued) friction is the application of direct friction compensation techniques, see e.g. [13]–[15] and many others. Such friction compensation schemes are typically applied when the actuation and friction are collocated, meaning that the friction force and the actuation force act at the same place and the friction can be compensated directly (if an accurate friction model is available). Here, the presented approach leads to feedback controllers for motion systems with non-collocation of actuation and friction, that are inherently robust to uncertainties in the friction characteristic.

The structure of this paper is as follows. Notations and definitions used in the paper are introduced in Section II. The controller designs are presented in Section III, where we discuss the state-feedback controller design with the generalization of the Popov criterion and the absolute ISS property, the observer design and the output-feedback controller design. A rotor dynamic system is presented in Section IV as an example of a mechanical motion system with non-collocated actuation and set-valued friction laws and the

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results of the application of the output-feedback controller design are shown in simulations. We finish this paper with conclusions in Section V. The proofs will be omitted due to lack of space and can be found in [16].

II. Notations and definitions

A function $u : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is piecewise continuous if on every bounded interval the function has only a finite number of points at which it is discontinuous. We will regard every piecewise continuous function u to be right continuous, i.e. $\lim_{t\downarrow\tau} u(t) = u(\tau)$ for all $\tau \in \mathbb{R}_+$. With $\|\cdot\|$ we will denote the usual Euclidean norm for vectors in $\mathbb{R}^n$. A function $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded, i.e. $\gamma(s) \rightarrow \infty$ as $s \rightarrow \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is of class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for each fixed $s \in \mathbb{R}_+$, the function $\beta(s, \cdot)$ is decreasing and tends to zero at infinity. A differential inclusion is given by an expression of the form

$$\dot{x}(t) \in F(x(t), e(t)), \tag{1}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a set-valued mapping and with the state $x \in \mathbb{R}^n$ and the input $e \in \mathbb{R}^m$. An absolutely continuous function x is considered to be a solution of the differential inclusion (1) given a piecewise continuous input e if (1) is satisfied almost everywhere.

Definition 1 ([6])
The system (1) is said to be input-to-state stable (ISS) if there exist a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that for each initial condition $x(0) = x_0$ and each piecewise continuous bounded input function e defined on $[0, \infty)$,

- *all solutions x of the system (1) exist on $[0, \infty)$, and*
- *all solutions satisfy*

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma \sup_{\tau\in[0,t]} (\|e(\tau)\|), \quad \forall t \geq 0. \tag{2}$$

The system is called globally asymptotically stable (GAS) if the above holds for $e = 0$.
Consider the following linear system

$$\begin{aligned} \dot{x} &= Ax + Gw \\ z &= Hx + Dw, \end{aligned} \tag{3}$$

with the state $x \in \mathbb{R}^n$, input $w \in \mathbb{R}^p$ and output $z \in \mathbb{R}^p$.

Definition 2
The system (3) or the quadruple (A, G, H, D) is said to be strictly passive if there exist an $\varepsilon > 0$ and a matrix $P = P^T > 0$ such that

$$\begin{bmatrix} A^\top P + PA + \varepsilon I & PG - H^\top \\ G^\top P - H & -D - D^\top \end{bmatrix} \leq 0. \tag{4}$$

III. Control of Lur'e-type Systems with Set-valued Nonlinearities

In this section, we consider systems of the form

$$\dot{x} = Ax + Gw + Bu \tag{5a}$$

$$z = Hx \tag{5b}$$

$$y = Cx \tag{5c}$$

$$w \in -\varphi(z), \tag{5d}$$

where $x \in \mathbb{R}^n$ is the system state, $w \in \mathbb{R}^p$ and $z \in \mathbb{R}^p$ are the respective input and output of a set-valued nonlinearity φ, $u \in \mathbb{R}^m$ is the control input and $y \in \mathbb{R}^\kappa$ is the measured output. The controller design proposed in this section aims at the stabilization of the origin $x = 0$. We first state the following assumptions on the properties of the set-valued nonlinearity $\varphi(z)$ in (5d).

Assumption 1
The set-valued nonlinearity $\varphi : \mathbb{R}^p \rightarrow \mathbb{R}^p$ satisfies

- $0 \in \varphi(0)$;
- *φ is upper semicontinuous (see [17]);*
- *φ is decomposed as $\varphi(z) = [\varphi_1(z_1), ..., \varphi_p(z_p)]^\top$, $z = [z_1, ..., z_p]^\top$ and $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$, for $i = 1, ...p$;*
- *φ_i, $i = 1, \ldots, p$, are set-valued on a countable set (of Lebesgue measure zero) of isolated points;*
- *for all $z_i \in \mathbb{R}$ the set $\varphi_i(z_i) \subseteq \mathbb{R}$, $i = 1, ..., p$, is non-empty, convex, closed and bounded;*
- *each φ_i satisfies the $[0, \infty]$ sector condition in the sense that $z_i w_i \leq 0$ for all $w_i \in -\varphi_i(z_i)$ for $i = 1, ..., p$;*
- *there exist positive constants γ_1 and γ_2 such that for $w \in -\varphi(z)$ and for any $z \in \mathbb{R}^p$ it holds that*

$$\|w\| \leq \gamma_1 \|z\| + \gamma_2. \tag{6}$$

The input functions $u(\cdot)$ are assumed to be in the space of piecewise continuous bounded functions from $[0, \infty)$ to $\mathbb{R}^m$, denoted by $\mathbb{PC}$. Clearly, the nonlinear function $(t, x) \mapsto Ax - G\varphi(Hx) + Bu(t)$ is upper semicontinuous on intervals, where u is continuous and attains non-empty, convex, closed and bounded set-values. From [17, p. 98] or [18, § 7], it follows that local existence of solutions is guaranteed given an initial state x_0 at initial time 0. Due to the growth condition (6), finite escape times are prevented and thus any solution to (5) is globally defined on $[0, \infty)$. Hence, solutions $x(\cdot)$ and also $z(\cdot) = Hx(\cdot)$ are absolutely continuous functions. Note that $0 \in \varphi(0)$ implies that the origin $x = 0$ is an equilibrium of system (5) for input $u = 0$.

A. State-feedback control

To stabilize the origin $x = 0$ of system (5), we propose a linear static state-feedback law (assuming $C = I$, i.e. $y = x$):

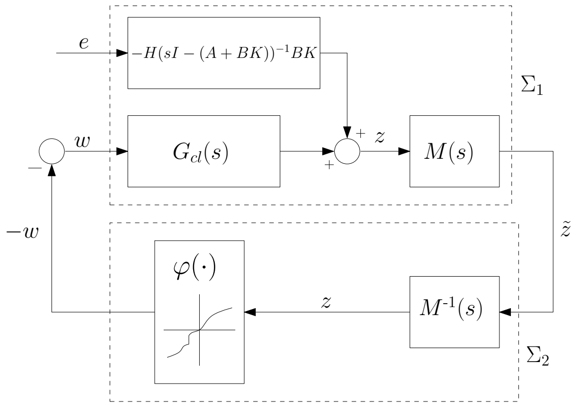
$$u = K(x - e). \tag{7}$$

where we take the measurement error e into account, which is piecewise continuous and bounded. Here, $K \in \mathbb{R}^{m\times n}$ is the control gain matrix. Consequently, the resulting closed-loop system is described by the following differential inclusion:

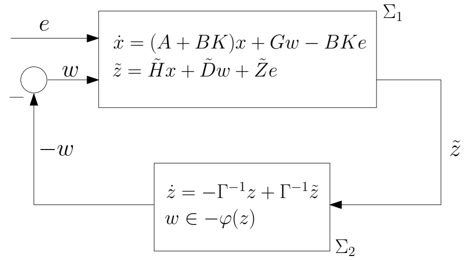
$$\begin{aligned} \dot{x} &= (A + BK)x + Gw - BKe \\ z &= Hx \end{aligned} \tag{8a}$$

$$w \in -\varphi(z). \tag{8b}$$

The transfer function $G_{cl}(s)$ from the input w to the output z of system (8) is given by $G_{cl}(s) = H(sI - (A + BK))^{-1}G,\ s \in \mathbb{C}$. The intended control goal here is to render the closed-loop system (8) "absolutely ISS" with respect to e, as formalized below, by means of a proper choice of the control gain K.



(a) Cascade representation of system (8) using the dynamic multiplier (9).



(b) Closed-loop system (10) after transformation with dynamic multiplier (9).

Fig. 1: Block diagrams of the closed-loop system (10).

Definition 3
We call a system (8) *absolutely ISS with respect to input* e*, if the system* (8) *is ISS with respect to input* e*, as in Definition 1, for any* φ *satisfying Assumption 1.*

To obtain sufficient conditions to guarantee that system (8) is absolutely ISS, we use, as in [4] for smooth systems, a so-called dynamic multiplier with transfer function $M(s)$ given by

$$M(s) = I + \Gamma s, \;\; s \in \mathbb{C}, \tag{9}$$

where $\Gamma = \text{diag}(\eta_1, \dots \eta_p) \in \mathbb{R}^{p \times p}$, with $\eta_i > 0$ for $i = 1, \dots, p$. A cascade that represents system (8) together with the multiplier $M(s)$ is shown in Figure 1(a). Using the dynamic multiplier $M(s)$ we aim to transform the original system into a feedback interconnection of two passive systems (with the perturbation input e), as is done in [4] and [2] for systems with Lipschitz continuous nonlinearities.

In state-space formulation, the interconnected system Σ_1, Σ_2 takes the following form:

$$\Sigma_1 = \begin{cases} \dot{x} = (A+BK)x + Gw - BKe(t) \\ \tilde{z} = \tilde{H}x + \tilde{D}w + \tilde{Z}e(t) \end{cases} \tag{10a}$$

$$\Sigma_2 = \begin{cases} \dot{z} = -\Gamma^{-1}z + \Gamma^{-1}\tilde{z} \\ w \in -\varphi(z), \end{cases} \tag{10b}$$

see also Figure 1. Herein, $\tilde{z} \in \mathbb{R}^p$ and the matrices $\tilde{H} \in \mathbb{R}^{p \times n}$, $\tilde{D} \in \mathbb{R}^{p \times p}$ and $\tilde{Z} \in \mathbb{R}^{p \times n}$ can be derived from the fact that $\tilde{z} = z + \Gamma \dot{z}$ (due to the choice of the multiplier $M(s)$ as in (9)) and, hence,

$$\begin{aligned} &\tilde{H} = H + \Gamma H(A+BK), \\ &\tilde{D} = \Gamma HG, \qquad \tilde{Z} = -\Gamma HBK. \end{aligned} \tag{11}$$

The following theorem states sufficient conditions under which system (8) is ISS with respect to input e for any $\varphi \in [0, \infty]$, i.e. the system (8) is absolutely ISS.

Theorem 1
Consider system (8) *and suppose there exists a diagonal matrix* $\Gamma = diag(\eta_1, ..., \eta_p) \in \mathbb{R}^{p \times p}$ *with* $\eta_i > 0, i = 1, ..., p,$ *such that* $(A+BK, G, \tilde{H}, \tilde{D})$*, with* $\tilde{D}$ *and* $\tilde{H}$ *as in* (11)*, is strictly passive. Then system* (8) *is absolute ISS with respect to input* e *for any* φ *satisfying Assumption 1.*

Clearly, if the input e is zero, the origin $x = 0$ of system (8) is absolutely stable under the conditions of Theorem 1. An advantage of achieving absolute stability is the robustness to uncertainties in the nonlinearity φ in the feedback loop. We also note that, in [5], frequency-domain conditions (including Popov-type conditions) guaranteeing a property close to GAS are stated for Lur'e-type systems with discontinuous nonlinearities. Here, we provide a Popov-like criterion for systems with set-valued nonlinearities that guarantees ISS with respect to input e (instead of only global asymptotic stability).

B. Observer design

Following [19], we propose the following observer for the system (5):

$$\dot{\hat{x}} = (A - LC)\hat{x} + G\hat{w} + Bu + Ly \tag{12a}$$

$$\hat{w} \in -\varphi(\hat{z}) \tag{12b}$$

$$\hat{z} = (H - NC)\hat{x} + Ny \tag{12c}$$

$$\hat{y} = C\hat{x}. \tag{12d}$$

with the observer gains $N \in \mathbb{R}^{p \times \kappa}$ and $L \in \mathbb{R}^{n \times \kappa}$.
At this point, we state an additional assumption on the set-valued nonlinearity $\varphi(\cdot)$ of system (5):

Assumption 2
The set-valued nonlinearity $\varphi : \mathbb{R}^p \to \mathbb{R}^p$ *is such that* φ *is monotone, i.e. for all* $z_1 \in \mathbb{R}^p$ *and* $z_2 \in \mathbb{R}^p$ *with* $w_1 \in -\varphi(z_1)$ *and* $w_2 \in -\varphi(z_2)$*, it holds that* $\langle w_1 - w_2, z_1 - z_2 \rangle \geq 0$*, where* $\langle \cdot, \cdot \rangle$ *denotes the inner product in* $\mathbb{R}^p$*.*

Since the right-hand side of (12a) is again upper semi-continuous in (t, x) due to continuity of y and piecewise continuity of u, using Assumptions 1 and 2 on φ it can be shown that there exist global solutions of (12), see [17], [18]. Knowing that both the plant and the observer have global solutions, the dynamics for the observer error $e := x - \hat{x}$ is given by

$$\dot{e} = (A - LC)e + G(w - \hat{w}) \tag{13a}$$

$$w \in -\varphi(Hx) \tag{13b}$$

$$\hat{w} \in -\varphi(H\hat{x} + N(y(t) - C\hat{x})). \tag{13c}$$

The problem of the observer design is finding the gains L and N such that all solutions to the observer error dynamics converge exponentially to the origin, which implies that $\lim_{t\to\infty} |\hat{x}(t) - x(t)| = 0$.

Theorem 2 ([19])
Consider system (5) *and the observer* (12) *with* $(A - LC, G, H - NC, 0)$ *strictly passive and the matrix* G *being*

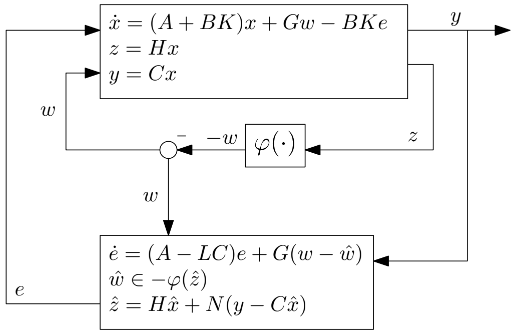


Fig. 2: Combination of the observer design and the controller design.

of full column rank. Then, the point $e = 0$ is a globally exponentially stable equilibrium point of the observer error dynamics (13) *for any $\varphi(\cdot)$ satisfying Assumptions 1 and 2.*

C. Output-feedback control

In this section, an observer-based output-feedback controller is presented, where we use the observer design, presented in the previous section, to provide an estimate $\hat{x}$ for the system state. Next, the estimated state $\hat{x}$ is fed back to the system (5) via $u = K\hat{x} = K(x - e)$ as in (7), where e now represent the estimation error. Application of the observer-based output-feedback controller results in an interconnection of system (8) and system (13), which is depicted in Figure 2. We aim to prove global asymptotic stability (GAS) of the equilibrium (x, e)=(0,0) of the interconnected system (8), (13).

Theorem 3
Consider system (8) *and observer* (12). *Suppose there exists a matrix $\Gamma = diag(\eta_1, ..., \eta_p) \in \mathbb{R}^{p\times p}$ with $\eta_i > 0, i = 1, ..., p$, such that $(A + BK, G, \tilde{H}, \tilde{D})$ is strictly passive with $\tilde{D}$ and $\tilde{H}$ as in* (11). *Moreover, suppose, $(A - LC, G, H - NC, 0)$ is strictly passive and G being full column rank. Then, $(x, e) = (0, 0)$ is a globally asymptotically stable equilibrium point of the interconnected system* (8), (13) *for any $\varphi(\cdot)$ satisfying Assumptions 1 and 2 (i.e. system* (8), (13) *is absolutely stable).*

IV. Application to a rotor dynamic system with non-collocated friction and actuation

In this section, we apply the controller design proposed in the previous section to tackle a motion control problem for mechanical systems with non-collocation of friction and actuation. As a typical example, we consider a rotor dynamic system as depicted schematically in Figure 3, which is a model of an experimental setup as presented in [20]. The system consists of an upper disc actuated by a drive part (power amplifier, motor) and a lower disc. The upper disc is connected to the lower disc by a steel string, which is a low-stiffness connection between the discs. In the experimental setup, a (lubricated) brake mechanism applies a friction torque T_{fl} to the lower disc. Moreover, the friction torque T_{fu} acting on the upper disc is due to friction in the bearings at the upper disc and electromagnetic effects in the drive part. The respective angular positions θ_l, θ_u of the lower and the upper discs can be measured and u is the input voltage to the drive part. The configuration of this setup can be recognized in the structure of drilling systems and other rotor dynamic motion systems.

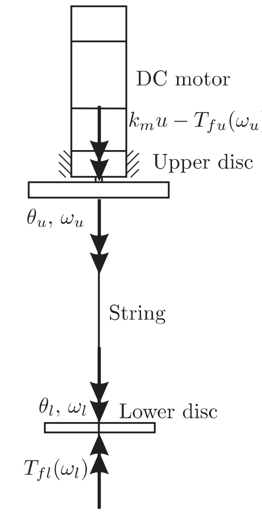


Fig. 3: The rotor dynamic system.

In order to describe the dynamics of the system in first-order Lur'e-type form, we define the state vector x as $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top = \begin{bmatrix} \alpha & \omega_u & \omega_l \end{bmatrix}^\top = \begin{bmatrix} \theta_u - \theta_l & \dot{\theta}_u & \dot{\theta}_l \end{bmatrix}^\top$.

Note that the state $x_1 = \alpha = \theta_u - \theta_l$ represents the relative angular displacement of the lower disc with respect to the upper disc, which can be obtained via the encoder measurements of θ_u and θ_l ($y = x_1$). The desired solution for the rotor dynamic system is a constant (and identical) velocity for both discs, which corresponds to an equilibrium (note that in drilling systems such constant velocity solution corresponds to nominal operating conditions). The state-space equations of the rotor dynamic system in Lur'e-type form are given by (5), with state $x \in \mathbb{R}^3$, $w, z \in \mathbb{R}^2$, input $u \in \mathbb{R}$, measured output $y \in \mathbb{R}$, and $\varphi(z) = [\varphi_1(z_1) \quad \varphi_2(z_2)]^\top = [T_{fu}(z_1) \quad T_{fl}(z_2)]^\top$ with $\varphi_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$. The matrices and the nonlinearity $\varphi(z)$ in (5) are given by

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -\frac{k_\theta}{J_u} & -b & b \\ \frac{k_\theta}{J_l} & b & -b \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{k_m}{J_u} \\ 0 \end{bmatrix}, \tag{14}$$

$$G = \begin{bmatrix} 0 & 0 \\ \frac{1}{J_u} & 0 \\ 0 & \frac{1}{J_l} \end{bmatrix}, H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{15}$$

$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $\varphi(z) = \begin{bmatrix} T_{fu}(z_1) & T_{fl}(z_2) \end{bmatrix}^\top$.

Herein, J_u and J_l represent the moments of inertia of the upper and lower disc, respectively, k_θ is the torsional stiffness of the steel string, b represents the material damping in the string and k_m is a motor constant. Set-valued force laws are needed to model the friction acting on the upper and lower disc to account for the pronounced sticking effect in both characteristics, see [20], [21]:

$$T_{fu}(\dot{\theta}_u) \in \begin{cases} T_{cu}(\dot{\theta}_u)\mathrm{sgn}(\dot{\theta}_u) & \text{for } \dot{\theta}_u \neq 0 \\ [-T_{su} + \Delta T_{su}\,, & \\ \qquad T_{su} + \Delta T_{su}] & \text{for } \dot{\theta}_u = 0, \end{cases} \tag{16}$$

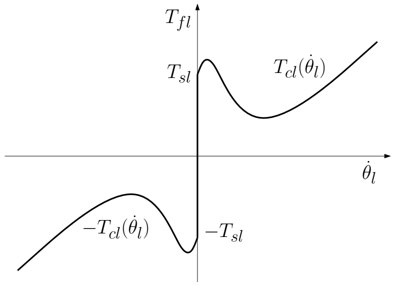


Fig. 4: Lower friction model T_{fl}.

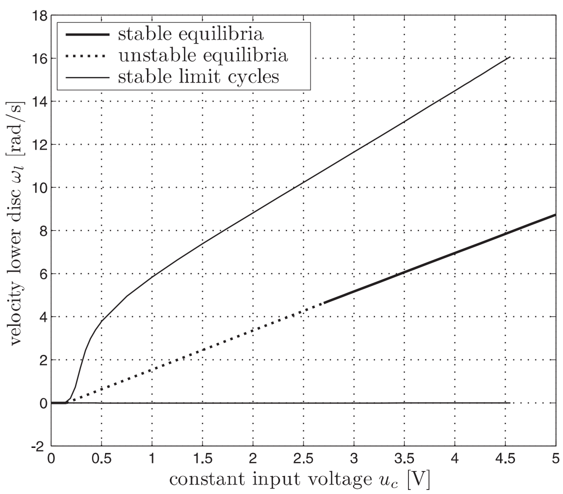


Fig. 5: Bifurcation diagram for the open-loop rotor dynamic system.

$$T_{cu}(\dot{\theta}_u) = T_{su} + \Delta T_{su}\mathrm{sgn}(\dot{\theta}_u) + b_u|\dot{\theta}_u| + \Delta b_u\dot{\theta}_u. \quad (17)$$

which accounts for an asymmetric combined Coulomb and viscous friction characteristic, and

$$T_{fl}(\dot{\theta}_l) \in \begin{cases} T_{cl}(\dot{\theta}_l)\mathrm{sgn}(\dot{\theta}_l) & \text{for } \dot{\theta}_l \neq 0 \\ [-T_{sl}, T_{sl}] & \text{for } \dot{\theta}_l = 0, \end{cases} \quad (18)$$

$$T_{cl}(\dot{\theta}_l) = T_{cl} + (T_{sl} - T_{cl})e^{-|\frac{\dot{\theta}_l}{\omega_{sl}}|^{\delta_{sl}}} + b_l|\dot{\theta}_l|. \quad (19)$$

Figure 4 depicts a schematic representation of the friction law (18), (19) at the lower disc and reveals the presence of a Stribeck effect.

The parameters of the rotor dynamic model are taken from [20] and are estimated by dedicated parameter identification experiments. The parameter values are given by: $k_m = 4.3228$ Nm/V, $J_u = 0.4765$ kg m^2, $T_{su} = 0.37975$ Nm, $\Delta T_{su} = -0.00575$ Nm, $b_u = 2.4245$ kg m^2/rad s, $\Delta b_u = -0.0084$ kg m^2/rad s, $k_\theta = 0.075$ Nm/rad, $b = 0$ kg m^2/rad s, $J_l = 0.035$ kg m^2, $T_{sl} = 0.26$ Nm, $T_{cl}0.05$ Nm, $\omega_{sl} = 2.2$ rad/s, $\delta_{sl} = 1.5$ and $b_l = 0.009$ kg m^2/rad s.

For varying constant inputs u_c (i.e. $u = u_c$ in (5)), different (co-existing) steady-state solutions of the rotor dynamic system exist and are depicted in a bifurcation diagram in Figure 5, with u_c the bifurcation parameter. For the case where the steady-state response is a periodic solution (stick-slip limit cycle), we plot the maximum and minimum value of the state variable ω_l (velocity of the lower disc) in the bifurcation diagram. For the region with constant input voltages up to approximately $u_c = 2.7$ V, we observe only

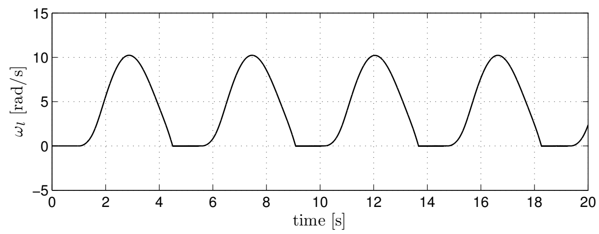


Fig. 6: Limit cycle response of the rotor dynamic system for $u_c = 2.5$ V.

stable limit cycles. Figure 6 shows such a limit cycle for $u_c = 2.5$ V. In the region from approximately 2.7 V - 4.5 V, two stable steady-state solutions co-exist: an equilibrium point and a stick-slip limit cycle. For constant input voltages higher than 4.5 V, only a stable equilibrium point occurs. As we remarked earlier, the equilibria of the rotor dynamic system correspond to both discs rotating with the same constant velocity, which are the desired operating conditions. As such, the control goal is to stabilize the unstable equilibria up to $u_c = 2.7$ V and to eliminate the limit cycles up to $u_c = 4.5$ V.

A. Output-feedback controller

One could opt to design an output-feedback controller by using the circle criterion, instead of the more involved Popov-criterion-inspired approach with the dynamic multiplier $M(s)$. However, the controller design based on the circle criterion is not feasible for the rotor dynamic system (5) according the presented feasibility conditions in [1]. In order to satisfy the feasibility conditions in [1], the damping coefficient b should satisfy $\{b > \min \frac{\partial}{\partial \omega_l} T_{fl}(\omega_l) | \omega_l > 0\}$. This would imply that the negative damping in the friction model T_{fl}, which is, basically, the cause of the friction-induced stick-slip vibrations is dominated by such viscous damping b in the string. However, the damping coefficient b reflects only material damping in the string which is generally very low and will not satisfy the above condition ($b = 0$ as introduced earlier). As many mechanical motion systems consist of inertias coupled by a low-damping connection, there exists a large class of systems for which a circle-criterion based controller design is not feasible. For these systems, the output-feedback controller design, presented in Section III-C, can be a solution since the use of the dynamic multiplier relaxes the circle-criterion conditions.

The control strategy presented in Section III-C is applied to the rotor dynamic system (5) and the output-feedback control law is given by

$$u = u_c + u_{comp} + K(\hat{x} - x_{eq}), \quad (20)$$

with $x_{eq} = [\alpha_{eq} \quad \omega_{eq} \quad \omega_{eq}]^\top$ the desired equilibrium of the rotor dynamic system (5), the control gain $K \in \mathbb{R}^{1\times 3}$ and $u_{comp} = (T_{fu}(\hat{x}_2) - b_u\hat{x}_2)/k_m$.

The part u_{comp} of the control law compensates partly the friction acting at the upper disc of the rotor dynamic system. The 'effective' friction after compensation acting at the upper disc is purely viscous. Note that such a friction compensation can not be employed to compensate for the friction at the lower disc (which is responsible for the stick-slip limit cycling), due to the non-collocation of actuation and friction. We can easily transform the closed-loop rotor

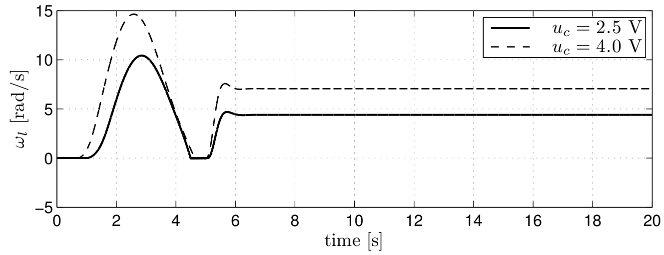


Fig. 7: Closed-loop response of the rotor dynamic system for $u_c =$ 2.5 V and 4.0 V, respectively.

dynamic system (5), (20) to a system in Lur'e-type form with the origin as equilibrium by choosing, for example, the new state $\xi = x - x_{eq}$. For the sake of brevity, we will omit this transformation here (see [20] for more details).

Assumption 2 requires that the set-valued nonlinearity φ is monotone. If we consider the friction model T_{fl}, see Figure 4, which is contained in φ, then it is clear that T_{fl} is not monotone. We will render φ monotone by applying a loop transformation, which will add 'viscous' damping to T_{fl} and subtract it from the linear part of (5), see [20]. The following feedback and observer gains satisfy Theorem 3: $K = \begin{bmatrix} 15.9 & 1.57 & 27.6 \end{bmatrix}, L^\top = \begin{bmatrix} 195 & -312 & -9080 \end{bmatrix}, N^\top = \begin{bmatrix} -2.22 & -37.8 \end{bmatrix}$, with the multiplier matrix $\Gamma = 10I$. A solution for P that satisfies the strict passivity condition for $(A + BK, G, \tilde{H}, \tilde{D})$, i.e. satisfies the corresponding LMI condition (4), is

$$P = \begin{bmatrix} 3.639 & 0.431 & 6.382 \\ 0.431 & 0.070 & 0.740 \\ 6.382 & 0.740 & 11.627 \end{bmatrix}. \tag{21}$$

B. Closed-loop Simulations

The presented output-feedback controller is applied to the rotor dynamic system to stabilize the equilibria of the rotor dynamic setup for any bounded constant input u_c. Simulations are performed and we show the closed-loop transient response for the constant input voltages $u_c = 2.5$ V and $u_c = 4.0$ V in Figure 7, where the initial open-loop solutions are stick-slip limit cycles. The output-feedback controller is switched on at $t = 5$ s for $u_c = 2.5$ V and the closed-loop system converges to the equilibrium state ($\omega_{eq} = 4.40$ rad/s). Also for $u_c = 4.0$ V, the closed-loop system converges to the equilibrium state ($\omega_{eq} = 7.06$ rad/s).

V. Conclusions

In this paper, we considered the output-feedback control of Lur'e-type systems with set-valued nonlinearities. The concept of absolute input-to-state stability (ISS) was presented, together with a generalization of a Popov-like criterion that guarantees absolute ISS for systems with set-valued nonlinearities. The latter concept is used to design a state-feedback controller and the related absolute stability conditions are less restrictive than those related to a controller design based on the circle criterion. Since the presented controller induces absolute ISS, the closed-loop system is robust to uncertainties in the nonlinearities and measurement errors. Furthermore, an output-feedback controller design is constructed by exploiting the ISS property, where a model-based observer, for which stability of the error dynamics is proven, is used to estimate the system state. We provided a separation principle for the considered class of Lur'e-type systems.

The control strategy is applied to a mechanical motion system with non-collocation of actuation and dry friction for which well-known strategies such as direct friction compensation techniques fail. The effectiveness of the proposed output-feedback control strategy is shown in simulations.

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