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$\mathcal{H}_2$ performance analysis of reset control systems

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***Abstract*— To overcome fundamental limitations of linear controllers, reset controllers were proposed in literature. Since the closed loop system including such a reset controller is of a hybrid nature, it is difficult to determine its performance. The focus in this paper is to determine the performance of a SISO reset control system in $\mathcal{H}_2$ sense. The method is generally applicable in the sense that it is valid for any proper LTI plant and linear-based reset controller. We derive convex optimization problems in terms of LMIs to compute an upperbound on the $\mathcal{H}_2$ norm, using dissipativity theory with piecewise quadratic Lyapunov functions. Finally, by means of a simple multiobjective tracking example, we show that reset control can outperform a linear controller obtained via a standard multiobjective control design method.**

***Index Terms*— Reset control, $\mathcal{H}_2$, stability, hybrid systems, linear matrix inequality, step response.**

I. INTRODUCTION

In order to overcome the fundamental performance limitations that linear controllers are known to be subject to [1]–[3], various nonlinear feedback controllers for linear time-invariant (LTI) plants were proposed in literature. An example of such a nonlinear feedback is the reset controller, which is basically a linear controller whose states (or subset of states) are reset to zero whenever its input and output satisfy certain conditions.

The concept of reset control was first introduced in 1958 by means of the resetting integrator of Clegg [4], but it was not until 1974 that it was first used in a control design procedure [5]. Subsequently, in [6], a *first order reset element (FORE)* was introduced, together with a controller design procedure based on frequency domain techniques. An overview of these results is given in [7].

At the end of the '90s there has been renewed interest in reset control systems, resulting in various stability analysis techniques. The first results were reported in [8] and [9], stating stability criterions for zero-input closed loops with a second order plant and a Clegg integrator or a FORE, respectively. However, these criterions involve explicit computation of reset times and closed loop solutions, and are hard to generalize for higher order systems.

In following publications stability conditions were formulated using Lyapunov based conditions. This was first done in [10], in which only second order closed loops with constant inputs were considered. These results have been extended in [11] to a sufficient criterion for BIBO (bounded input bounded output) stability, and later to the so-called H_β-condition [12]. The possible advantages of reset controllers over linear ones have been shown both in simulations [9], [13] and experiments [7], [11], [14]. A clear overview of this work for general reset systems is provided in [15].

The H_β-condition, which is a reformulation of Lyapunov based stability linear matrix inequalities (LMIs), is however conservative. In more recent publications [16], [17] the authors were able to remove part of this conservatism by introducing two adjustments. First, a slightly different resetting condition was suggested, i.e. resetting when the controller input and output have opposite sign instead of when the input is zero. Second, *piecewise quadratic* Lyapunov functions were used, to capture a broader class of stability problems. Additionally, the analysis has been extended such that the closed loop $\mathcal{L}_2$ gain from input to output of a reset control system can be approximated by an upperbound. However, the results in [16] and [17] are not universally applicable, since only Clegg integrators and FOREs are considered.

Still, the proposed calculation of the $\mathcal{L}_2$ gain is very useful, since it expresses the performance of certain reset control systems in a quantitative measure. However, the $\mathcal{L}_2$ gain is typically a steady state measure, whereas the advantage of reset control over linear control is especially apparent during the transient behavior of constrained problems [13]. In particular, it has been shown that reset controllers are able to reduce the overshoot of step responses, decreasing the total energy of the error signal. This observation shows similarities with one of the interpretations of the $\mathcal{H}_2$ norm, which can be seen as the total output energy (of e.g. the tracking error) of a closed loop system to either an impulse input or non-zero initial values. Because of this transient interpretation, we believe that the $\mathcal{H}_2$ norm is a very helpful measure, to objectively show that reset controllers can outperform linear ones, especially in (input) constrained problems.

For this reason this paper derives an LMI-based analysis method to calculate upperbounds on the $\mathcal{H}_2$ norm of a closed loop reset control system. The results can be used to approximate the energy content of the output resulting from specific input signals. We will use the same reset condition as in [17] and also adopt piecewise quadratic Lyapunov functions to reduce conservatism of the analysis. However, in contrast to [17], our results are not only useful for FOREs, but for any reset controller with linear flow dynamics and they use $\mathcal{H}_2$ instead of $\mathcal{L}_2$ gain performance. Moreover, we provide a simple though convincing example to illustrate the accuracy of our proposed $\mathcal{H}_2$ norm calculation and show that, for an input constrained $\mathcal{H}_2$ problem, reset control can indeed outperform a linear controller designed by a common multiobjective design method.

This paper is organized as follows. Section II provides some background on the $\mathcal{H}_2$ norm for linear systems. In Section III we introduce the closed loop layout under consideration, and give mathematical descriptions of the dynamics

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of the plant and the reset controller. Our main results on the $\mathcal{H}_2$ norm for reset control systems are derived in Section IV. Finally, the advantage of reset control in $\mathcal{H}_2$ sense is shown by an example in Section V.

Notation. The set of real numbers is denoted by $\mathbb{R}$, the set of positive real numbers is denoted by $\mathbb{R}_+$. The set of real symmetric matrices with nonnegative elements is denoted by $\mathbb{S}_+$. The identity matrix of dimension $n\times n$ is denoted by $I_n \in \mathbb{R}^{n\times n}$. Given two vectors x_1, x_2 we write (x_1, x_2) to denote $[x_1^T, x_2^T]^T$. A vector $x \in \mathbb{R}^n$ is nonnegative, denoted by $x \geq 0$, if its elements $x_i \geq 0$ for $i = 1, \ldots, n$. A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite, denoted by $A \succ 0$, if $x^T A x > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$. A sequence of scalars $(u^1, u^2, \ldots, u^k)$ is called lexicographically nonnegative, written as $(u^1, u^2, \ldots, u^k) \geq_\ell 0$, if $(u^1, u^2, \ldots, u^k) = (0, 0, \ldots, 0)$ or $u^j > 0$ where $j = \min\{p = 1, \ldots, k : u^p \neq 0\}$. For a sequence of vectors $(x^1, x^2, \ldots, x^k)$ with $x^j \in \mathbb{R}^n$, we write $(x^1, x^2, \ldots, x^k) \geq_\ell 0$ when $(x_i^1, x_i^2, \ldots, x_i^k) \geq_\ell 0$ for all $i = 1, \ldots, n$. Likewise, we write $(x^1, x^2, \ldots, x^k) \leq_\ell 0$ to denote a lexicographically nonpositive sequence of vectors, meaning that $-(x^1, x^2, \ldots, x^k) \geq_\ell 0$.

II. LINEAR $\mathcal{H}_2$ THEORY

Our main results on the $\mathcal{H}_2$ analysis of reset control systems use some common $\mathcal{H}_2$ results for linear systems

$$\Sigma : \begin{cases} \dot{x} &= \mathcal{A}x + \mathcal{B}w \\ z &= \mathcal{C}x. \end{cases} \quad (1)$$

where $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times m}$, and $\mathcal{C} \in \mathbb{R}^{l\times n}$ are the system matrices, $x \in \mathbb{R}^n$ is the state, and $w \in \mathbb{R}^m$ and $z \in \mathbb{R}^l$ denote the input and output, respectively. We shortly summarize some of these results here (see [18], [19] for details).

A. $\mathcal{H}_2$ norm for linear systems

It is well-known that one of the possible interpretations of the $\mathcal{H}_2$ norm is the total energy content of the output z due to an impulsive input w. The response of system (1) to such an impulsive input is equivalent to the response when the system is subjected to the initial condition $x_0 = \mathcal{B}_j$ or e.g. $x_0 = \mathcal{B}_i + \mathcal{B}_j$ (with $i \neq j$), where $\mathcal{B}_j$ denotes the j-th column of $\mathcal{B}$ corresponding to the j-th input. In this paper we use the latter interpretation that corresponds to $w = 0$.

Definition 1 *Consider the linear system* (1)*, with initial state $x_0 \in \mathbb{R}^n$ and no input ($w = 0$). The total output energy in z is then defined by:*

$$\int_0^\infty z^T z\, dt = \int_0^\infty x_0^T \mathrm{e}^{\mathcal{A}^T t} \mathcal{C}^T \mathcal{C} \mathrm{e}^{\mathcal{A}t} x_0\, dt. \quad (2)$$

The square root of this integral is called the $\mathcal{H}_2$ norm for x_0 and is denoted by $||\Sigma||_{2,x_0}$.

To calculate (2) we introduce the observability gramian

$$M := \int_0^\infty \mathrm{e}^{\mathcal{A}^T t} \mathcal{C}^T \mathcal{C} \mathrm{e}^{\mathcal{A}t} dt, \quad (3)$$

so that the squared $\mathcal{H}_2$ norm is equal to

$$||\Sigma||_{2,x_0}^2 = \int_0^\infty z^T z\, dt = x_0^T M x_0. \quad (4)$$

It is well known that for Hurwitz matrices $\mathcal{A}$ the observability gramian M is the solution to the Lyapunov equality

$$\mathcal{A}^T M + M\mathcal{A} + \mathcal{C}^T \mathcal{C} = 0. \quad (5)$$

B. An LMI-approach to $\mathcal{H}_2$

The $\mathcal{H}_2$ norm of a linear system can also be obtained using the concept of dissipativity [18], [20].

Definition 2 *The linear system* (1) *with state $x \in \mathbb{R}^n$, input $w \in \mathbb{R}^m$ and output $z \in \mathbb{R}^l$ is dissipative w.r.t. a supply function $s : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ if there exists a storage function $V : \mathbb{R}^n \to \mathbb{R}$ such that*

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(w(t), z(t)) dt \quad (6)$$

for all $t_1 \geq t_0$ and $V(0) = 0$.

For the initial state $\mathcal{H}_2$ problem there is no input and we are only interested in the energy content of the output, hence we select $s(w, z) = -z^T z \leq 0$. Using quadratic storage functions $V(x) = x^T P x$ the differential form of (6) yields

$$\begin{aligned} \tfrac{dV}{dx}\dot{x} &\leq s(w,z) && \forall x \\ \dot{x}^T P x + x^T P \dot{x} &\leq -z^T z && \forall x \\ x^T(\mathcal{A}^T P + P\mathcal{A})x + x^T \mathcal{C}^T \mathcal{C} x &\leq 0 && \forall x \\ \mathcal{A}^T P + P\mathcal{A} + \mathcal{C}^T \mathcal{C} &\preceq 0, && (7) \end{aligned}$$

which is an LMI in the design variable P. Additionally, global asymptotic stability can be taken into account by choosing V a positive definite Lyapunov function (i.e. we add $P \succ 0$) and making (7) strict. The actual $\mathcal{H}_2$ norm $||\Sigma||_{2,x_0}$ can be upperbounded by using (6), where we set $t_0 = 0$, $s(w, z) = -z^T z$ and let $t_1 \to \infty$ and $\lim_{t_1\to\infty} V(x(t_1)) = 0$. Indeed, then we obtain:

$$||\Sigma||_{2,x_0}^2 = \int_0^\infty z^T z\, dt \leq V(x(t_0)) = x_0^T P x_0. \quad (8)$$

We can now examine asymptotic stability and approximate the actual $\mathcal{H}_2$ norm by minimizing γ^2 subject to the LMIs

$$\mathcal{A}^T P + P\mathcal{A} + \mathcal{C}^T \mathcal{C} \prec 0 \quad (9a)$$

$$x_0^T P x_0 < \gamma^2 \quad (9b)$$

which should be solved for $P \succ 0$ and γ^2. The infimum of this optimization problem retrieves $||\Sigma||_{2,x_0}^2$. These LMIs are, of course, closely related to (4) and (5).

III. SYSTEM DESCRIPTION

In this section we introduce the closed loop layout and reset control system for which the $\mathcal{H}_2$ norm will be calculated.

A. Closed loop layout

In this paper we focus on the general reset control system layout depicted in Figure 1, consisting of various input filters, a linear plant P and a reset controller K. Our goal is to calculate the total output energy of the unfiltered output z, consisting of the signals e (tracking error), u_p (control) and y_p (plant output) or a subset of these signals, subject to certain *specific* inputs, i.e. the signals r (reference), d

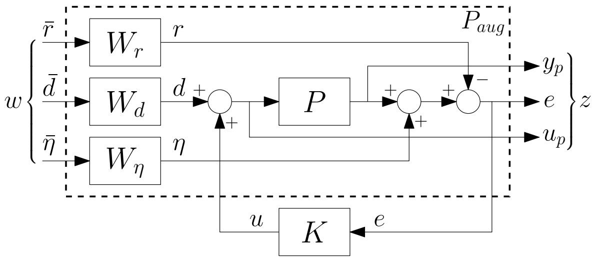

Fig. 1. Closed loop layout with input w and output z

(disturbance) and η (measurement noise) or a subset of these inputs. For ease of exposition, all these signals are assumed to take values in $\mathbb{R}$, i.e. we only consider SISO (single input single output) plants and controllers. The inputs are assumed to be known a priori, and this knowledge is captured in input filters W_r, W_d and W_η, whose inputs are bundled in the exogenous input w. Possible input filters include:

- unit step: $W(s) = \frac{1}{(s+\varepsilon)}$;
- unit ramp: $W(s) = \frac{1}{(s+\varepsilon)^2}$;
- sine wave with frequency ω: $W(s) = \frac{\omega}{(s+\varepsilon)^2+\omega^2}$;

where s is the Laplace variable and $\varepsilon > 0$ is a small offset. This offset is standard in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems as it is needed to ensure closed loop stability. Note that when an impulse input is applied to these filters, their outputs indeed approximate a step, ramp or sine wave respectively. Hence, the total energy in z as a result of such specific signals in r, d or η gets arbitrarily close to the total output energy of the impulse response from w to z, which is an interpretation of the $\mathcal{H}_2$ norm. As discussed in Section II, to compute this $\mathcal{H}_2$ norm we will use the initial condition setting and assume $w = 0$. However, in order to be able to select appropriate initial states $x_0 = \mathcal{B}_j$ later on, we will elaborate on the plant and controller dynamics without this assumption.

B. Plant dynamics

The plant P and all input filters are LTI systems. Together they form the LTI augmented plant P_{aug}, depicted by the dashed box in Figure 1. We can describe its dynamics by

$$\begin{array}{rcl} \dot{x}_p &=& Ax_p + B_w w + Bu \\ z &=& C_z x_p + D_{zw} w + D_z u \\ e &=& Cx_p + D_w w, \end{array} \tag{10}$$

where $A, B_w, B, C_z, D_{zw}, D_z, C, D_w$ are matrices of appropriate dimension, $x_p \in \mathbb{R}^{n_p}$ is the augmented plant state, and $w \in \mathbb{R}^{n_w}$ and $z \in \mathbb{R}^{n_z}$ are the exogenous input and the to be controlled output, respectively. The tracking or stabilization error is available for feedback and we assume that there is no direct feedthrough from u to e.

C. Reset controller

The reset controller K is modeled as a linear controller which resets whenever its input e and output u satisfy a certain condition. This controller is described by

$$\begin{array}{rcll} \dot{x}_k &=& A_K x_k + B_K e & \text{if } (e,u) \in \mathfrak{C}' \\ x_k^+ &=& A_r x_k & \text{if } (e,u) \in \mathfrak{D}' \\ u &=& C_K x_k + D_K e, & \end{array} \tag{11}$$

where $x_k \in \mathbb{R}^{n_k}$ is the controller state. The closed loop state then becomes $x = (x_p, x_k)$ where $x \in \mathbb{R}^n$ and $n = n_p + n_k$. The reset controller can be considered as a hybrid system with a *flow set* $\mathfrak{C}'$ and a *reset set* $\mathfrak{D}'$ [16]. Indeed, as long as $(e, u) \in \mathfrak{C}'$ the controller behaves linearly and its output flows conform (A_K, B_K, C_K, D_K). When $(e, u) \in \mathfrak{D}'$ the state is changed instantaneously from x_k to x_k^+ by the discrete map corresponding to $A_r \in \mathbb{R}^{n_k \times n_k}$.

For analysis purposes the sets $\mathfrak{C}'$ and $\mathfrak{D}'$ should be closed and such that $\mathfrak{C}' \cup \mathfrak{D}' = \mathbb{R}^2$ [16]. The sets are defined by a resetting condition, for which many choices are possible, but here we follow [16], [17] where resets occur whenever input and output have opposite sign, i.e. $eu \leq 0$. Hence, the controller flows when $e \geq 0, u \geq 0$ or $e \leq 0, u \leq 0$, yielding

$$\mathfrak{C}' := \left\{ \begin{bmatrix} e \\ u \end{bmatrix} \in \mathbb{R}^2 : E_f \begin{bmatrix} e \\ u \end{bmatrix} \geq 0 \text{ or } E_f \begin{bmatrix} e \\ u \end{bmatrix} \leq 0 \right\} \tag{12a}$$

$$\mathfrak{D}' := \left\{ \begin{bmatrix} e \\ u \end{bmatrix} \in \mathbb{R}^2 : E_R \begin{bmatrix} e \\ u \end{bmatrix} \geq 0 \text{ or } E_R \begin{bmatrix} e \\ u \end{bmatrix} \leq 0 \right\} \tag{12b}$$

where

$$E_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E_R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The flow set (12a) and reset set (12b) can also be expressed in terms of x and w. Therefore we introduce a transformation matrix $T = \left[\begin{array}{c|c} T_x & T_w \end{array}\right]$:

$$\begin{bmatrix} e \\ u \end{bmatrix} = T \begin{bmatrix} x \\ w \end{bmatrix} = \left[\begin{array}{cc|c} C & 0 & D_w \\ D_K C & C_K & D_K D_w \end{array}\right] \begin{bmatrix} x_p \\ x_k \\ w \end{bmatrix},$$

such that

$$\tilde{\mathfrak{C}} := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{R}^{n+n_w} : E_f T \begin{bmatrix} x \\ w \end{bmatrix} \geq 0 \text{ or } E_f T \begin{bmatrix} x \\ w \end{bmatrix} \leq 0 \right\} \tag{13a}$$

$$\tilde{\mathfrak{D}} := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{R}^{n+n_w} : E_R T \begin{bmatrix} x \\ w \end{bmatrix} \geq 0 \text{ or } E_R T \begin{bmatrix} x \\ w \end{bmatrix} \leq 0 \right\} \tag{13b}$$

The flow and reset regions $\tilde{\mathfrak{C}}$ and $\tilde{\mathfrak{D}}$ depend on both the closed loop state x and the input w, due to the presence of D_w. To overcome this, we adopt the following assumption.

Assumption 3 *The input filters W_r, W_d and W_η are all strictly proper.*

Note that strict properness of these filters is quite natural in practice, as is also indicated by the three examples in Section III-A. This strict properness implies that there is no direct feedthrough between the input w and the controller input signal e (i.e. $D_w = 0$) or between w and z (i.e. $D_{zw} = 0$). Also note that in the impulsive interpretation of the $\mathcal{H}_2$ norm this is a necessary requirement to ensure a bounded $\mathcal{H}_2$ norm. This simplifies the definitions of $\tilde{\mathfrak{C}}$ and $\tilde{\mathfrak{D}}$ to

$$\mathfrak{C} := \{x \in \mathbb{R}^n : E_f T_x x \geq 0 \quad \text{or} \quad E_f T_x x \leq 0\} \tag{14a}$$

$$\mathfrak{D} := \{x \in \mathbb{R}^n : E_R T_x x \geq 0 \quad \text{or} \quad E_R T_x x \leq 0\} \tag{14b}$$

which now only depend on x.

The reset action is defined by the reset matrix A_r, for which various choices are possible. A common and appropriate choice in most cases, is

$$A_r = \begin{bmatrix} I_{n_k - n_r} & 0 \\ 0 & 0_{n_r} \end{bmatrix},$$

stating that only the last n_r of the n_k controller states are reset to zero, while the other states remain unchanged.

D. Closed loop dynamics

We can now combine the augmented plant and the reset controller into one closed loop system, described by Σ:

$$\Sigma : \begin{cases} \dot{x} &= \mathcal{A}x + \mathcal{B}w \quad \text{if } x \in \mathfrak{C} \\ x^+ &= A_R x \quad \text{if } x \in \mathfrak{D} \\ z &= \mathcal{C}x + \mathcal{D}w, \end{cases} \tag{15}$$

where, using $D_w = 0$ and $D_{zw} = 0$,

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{cc|c} A + BD_K C & BC_K & B_w \\ B_K C & A_K & 0 \\ \hline C_z + D_z D_K C & D_z C_K & 0 \end{array}\right], \; A_R = \begin{bmatrix} I_{n_p} & 0 \\ 0 & A_r \end{bmatrix}.$$

The linear closed loop system, which results when resetting is omitted (i.e. $\mathfrak{C} = \mathbb{R}^n$ and $\mathfrak{D} = \emptyset$), is called the *base linear system*. At this point we return to the assumption that $w = 0$ and consider non-zero initial values of the input filters. Hence we take $w = 0$ and $x_0 = \mathcal{B}_j$ (see Section II-A). Note that $x_0 = \mathcal{B}_i + \mathcal{B}_j$ where $i \neq j$ is also a valid initial condition.

To guarantee that after each reset the system can flow on a non-trivial time interval (local existence of solutions), we adopt the following assumption.

Assumption 4 *The plant and reset controller dynamics and the regions $\mathfrak{C}$ and $\mathfrak{D}$ are such that*

$$x \in \mathfrak{D} \quad \Rightarrow \quad x^+ = A_R x \in \mathcal{F}_{\mathfrak{C}}. \tag{16}$$

where $\mathcal{F}_{\mathfrak{C}}$ is given by

$$\mathcal{F}_{\mathfrak{C}} := \{x_0 \in \mathfrak{C} : \exists \epsilon > 0 \quad \forall \tau \in [0, \epsilon) \quad x_{x_0}(\tau) \in \mathfrak{C}\} \tag{17}$$

defining all initial conditions x_0 for which the closed loop solution x_{x_0} remains inside $\mathfrak{C}$ at least on the interval $[0, \epsilon)$.

This assumption implies that the state after a reset x^+ (which forms an initial condition in (17)) should either lie in the interior of $\mathfrak{C}$, or the flow dynamics should not drive x directly outside $\mathfrak{C}$ when it is on the boundary of $\mathfrak{C}$. The following lemma characterizes $\mathcal{F}_{\mathfrak{C}}$, which can be used to check (16).

Lemma 5 *Given the system (15), the set $\mathcal{F}_{\mathfrak{C}}$ in Assumption 4 can be characterized with two lexicographic orderings, i.e.*

$$\begin{aligned} \mathcal{F}_{\mathfrak{C}} = \{x_0 \in \mathfrak{C} : \big(&E_f T_x x_0, E_f T_x \mathcal{A} x_0, E_f T_x \mathcal{A}^2 x_0, \\ &\ldots, E_f T_x \mathcal{A}^{n-1} x_0\big) \geq_\ell 0 \\ \textit{or} \; \big(&E_f T_x x_0, E_f T_x \mathcal{A} x_0, E_f T_x \mathcal{A}^2 x_0, \\ &\ldots, E_f T_x \mathcal{A}^{n-1} x_0\big) \leq_\ell 0\} \end{aligned} \tag{18}$$

Proof: The proof is based on inspecting the values of $x_{x_0}(\tau)$ and its derivatives $\dot{x}_{x_0}(\tau), \ddot{x}_{x_0}(\tau), \ldots$ at time $\tau = 0$, and combining them with (14a). The Cayley-Hamilton theorem is used to obtain a finite characterization involving only the first $(n-1)$ derivatives (see [21] for more details). ■

Under Assumption 4 local existence of solutions is guaranteed. However, we have to be careful as the reset times may accumulate (so called Zeno behavior). At this point we assume that either Zenoness is absent or we can continue beyond the accumulation point in such a way that global existence of solutions is guaranteed.

IV. Main results

We now present the main results on the LMI-based calculation of an upperbound on the $\mathcal{H}_2$ norm of reset control systems. For this $\mathcal{H}_2$ analysis we use the concept of dissipativity as in Section II-B, resulting in a set of computable LMIs. Moreover, we adopt the following definition.

Definition 6 *The squared $\mathcal{H}_2$ norm of reset control system (15), or the squared total energy in its output $z \in \mathbb{R}^{n_z}$ corresponding to a non-zero initial value $x_0 \in \mathbb{R}^n$, is defined as*

$$||\Sigma||_{2,x_0}^2 = \int_0^\infty z^T z \, dt \tag{19}$$

A. Common Lyapunov function

Although a reset control system behaves in a hybrid manner, the mathematical description of its dynamics (15) shows that both the flow and the reset part can be described in a linear fashion. This motivates our choice to use a common quadratic Lyapunov function $V(x) = x^T P x$.

Theorem 7 *Consider the reset control system (15) with $\mathfrak{C}$ and $\mathfrak{D}$ as defined in (14). This system is asymptotically stable and its $\mathcal{H}_2$ norm $||\Sigma||_{2,x_0} \leq \gamma$ if there exist $P \succ 0$ and $U_f, U_R \in \mathbb{S}_+^{2\times 2}$ such that*

$$\mathcal{A}^T P + P\mathcal{A} + \mathcal{C}^T \mathcal{C} + T_x^T E_f^T U_f E_f T_x \prec 0 \tag{20a}$$

$$A_R^T P A_R - P + T_x^T E_R^T U_R E_R T_x \preceq 0 \tag{20b}$$

$$\gamma^2 - x_0^T P x_0 \geq 0 \tag{20c}$$

Proof: The proof that $||\Sigma||_{2,x_0} \leq \gamma$ is based on showing that hypotheses (20) imply, for $V(x) = x^T P x$ and $s(w, z) = -z^T z$, that

$$\frac{d}{dt} V(x) < s(w, z) \quad \text{when } x \in \mathfrak{C} \backslash \{0\} \tag{21a}$$

$$V(x^+) \leq V(x) \quad \text{when } x \in \mathfrak{D}. \tag{21b}$$

Indeed, if (21) holds then for all $t_1 \geq t_0$

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(w(t), z(t)) dt \tag{22}$$

showing that (15) is dissipative w.r.t. the supply function $s(w, z) = -z^T z$. Moreover, note that (20a) also implies that

$$\frac{d}{dt} V(x(t)) \leq -\epsilon V(x(t)) \qquad \text{when } x(t) \in \mathfrak{C} \tag{23}$$

for some $\epsilon > 0$. Together with (21b) and since V is positive definite, this yields global asymptotic stability [22]. Then by letting $t_1 \to \infty$ and using $\lim_{t_1 \to \infty} V(x(t_1)) = 0$, this yields

$$||\Sigma||_{2,x_0}^2 = \int_0^\infty z^T z \, dt = -\int_0^\infty s(w, z) dt \leq V(x(0)) \leq \gamma^2. \tag{24}$$

To show (21a) and (21b), note that

$$x \in \mathfrak{C} \quad \Rightarrow \quad x^T T_x^T E_f^T U_f E_f T_x x \geq 0 \tag{25a}$$

$$x \in \mathfrak{D} \quad \Rightarrow \quad x^T T_x^T E_R^T U_R E_R T_x x \geq 0, \tag{25b}$$

since $U_f, U_R \in \mathbb{S}_+^{2\times 2}$ only have non-negative elements. Hence, combining (25a) with (20a) yields that

$$x^T(\mathcal{A}^T P + P\mathcal{A} + \mathcal{C}^T \mathcal{C})x < 0 \qquad \text{if } x \in \mathfrak{C} \backslash \{0\} \tag{26}$$

and combining (25b) with (20b) gives

$$x^T(A_R^T P A_R - P)x \leq 0 \qquad \text{if } x \in \mathfrak{D}, \tag{27}$$

which are just reformulations of (21a) and (21b). Hence, the proof is complete. ■

B. Piecewise quadratic Lyapunov functions

Although Theorem 7 provides an easy way to determine an upperbound on the $\mathcal{H}_2$ norm for a specific x_0, it can be conservative. Indeed, quadratic storage functions might not be flexible enough to approximate the actual $\mathcal{H}_2$ norm, since this could require a very complicated function, given the hybrid nature of the closed loop.

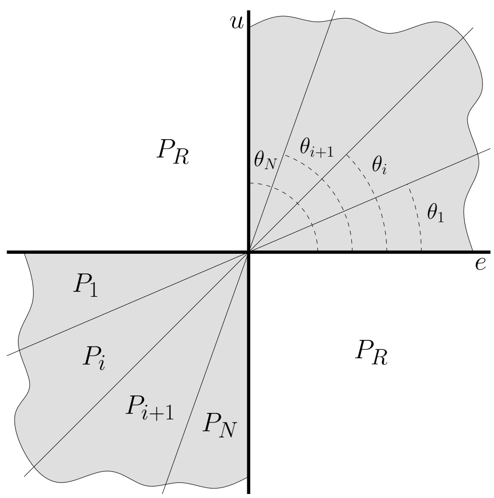

Fig. 2. Partitioning of the (e, u)-space

To reduce this conservatism, we will use *piecewise quadratic Lyapunov functions* [23]. These piecewise Lyapunov functions are obtained by partitioning the flow set $\mathfrak{C}'$ into smaller regions $\mathfrak{C}'_i$ and assigning a different quadratic Lyapunov function $V_i(x) = x^T P_i x$ to each of them, see Figure 2. The angles θ_i and θ_{i-1} uniquely define two lines

$$\begin{aligned} u\cos(\theta_i) &= e\sin(\theta_i) \\ u\cos(\theta_{i-1}) &= e\sin(\theta_{i-1}) \end{aligned}$$

which bound each region $\mathfrak{C}'_i$. These angles should be chosen such that $0 < \theta_0 < \theta_1 < \ldots < \theta_N = \frac{\pi}{2}$. Here we choose to distribute θ_i equidistantly, so $\theta_i = \frac{i}{N}\frac{\pi}{2}$, where $i = 0, \ldots, N$ and N is the number of desired subregions. Using the coordinate transformation matrix T_x, we can now define regions $\mathfrak{C}_i$ and $\mathfrak{D}$ as

$$\mathfrak{C}_i := \{x \in \mathbb{R}^n : E_i T_x x \geq 0 \quad \text{or} \quad E_i T_x x \leq 0\} \tag{28a}$$
$$\mathfrak{D} := \{x \in \mathbb{R}^n : E_R T_x x \geq 0 \quad \text{or} \quad E_R T_x x \leq 0\} \tag{28b}$$

where

$$E_i = \begin{bmatrix} -\sin(\theta_{i-1}) & \cos(\theta_{i-1}) \\ \sin(\theta_i) & -\cos(\theta_i) \end{bmatrix}, \quad E_R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Furthermore, the boundaries of the regions are defined by

$$\begin{bmatrix} -\sin(\theta_i) & \cos(\theta_i) \end{bmatrix} T_x x = \Phi_i x = 0, \tag{29}$$

whose solutions are in the kernel of Φ_i. We can also use an image representation for these boundaries, yielding $\mathrm{im}(W_{\Phi i}) = \ker(\Phi_i)$, where $W_{\Phi i} \in \mathbb{R}^{n\times(n-1)}$ is a matrix with full column rank, and $\mathrm{im}(W_{\Phi i})$ denotes its image.

Using this partitioning we can now formulate our main result on the calculation of an $\mathcal{H}_2$ upperbound.

Theorem 8 *The reset control system* (15) *is asymptotically stable with an* $\mathcal{H}_2$ *norm* $||\Sigma||_{2,x_0} \leq \gamma$ *if, for a given* N*, there exists* $P_i, P_R \succ 0$ *and* $U_i, U_R, V_i, V_R \in \mathbb{S}_+^{2\times 2}$ *such that*

$$\mathcal{A}^T P_i + P_i\mathcal{A} + \mathcal{C}^T\mathcal{C} + T_x^T E_i^T U_i E_i T_x \prec 0, \qquad i = 1,\ldots,N \tag{30a}$$
$$A_R^T P_R A_R - P_R + T_x^T E_R^T U_R E_R T_x \preceq 0 \tag{30b}$$
$$P_i - T_x^T E_i^T V_i E_i T_x \succ 0 \qquad i = 1,\ldots,N \tag{30c}$$
$$P_R - T_x^T E_R^T V_R E_R T_x \succ 0 \tag{30d}$$
$$W_{\Phi i}^T (P_i - P_{i+1}) W_{\Phi i} = 0, \qquad i = 1,\ldots,N-1 \tag{30e}$$
$$W_{\Phi 0}^T (P_R - P_1) W_{\Phi 0} = 0 \tag{30f}$$
$$W_{\Phi N}^T (P_N - P_R) W_{\Phi N} = 0 \tag{30g}$$
$$\gamma^2 - x_0^T P_j x_0 \geq 0, \qquad j \in I(x_0) \tag{30h}$$

where $I(x_0) := \{i : x_0 \in \mathfrak{C}_i\}$ *denotes the indices of the regions that contain* x_0.

Proof: Note that V, defined as $V(x) = V_i(x) := x^T P_i x$ when $x \in \mathfrak{C}_i$ and $V(x) = x^T P_R x$ when $x \in \mathfrak{D}$, is a continuous piecewise quadratic function due to continuity constraints (30e), (30f) and (30g). Furthermore, V is positive definite. To show this, note that for $x \in \mathfrak{C}_i\backslash\{0\}$ (hence not necessarily for all x) it holds that

$$V(x) = x^T P_i x \overset{(30c)}{>} x^T T_x^T E_i^T V_i E_i T_x x \geq 0 \tag{31}$$

as $x \in \mathfrak{C}_i \Rightarrow x^T T_x^T E_i^T V_i E_i T_x x \geq 0$ due to the fact that $V_i \in \mathbb{S}_+^{2\times 2}$ has non-negative elements. The same applies for $x \in \mathfrak{D}$ using (30d). Using (30a) and (30b) similarly as in the proof of Theorem 7 again shows global asymptotic stability, as V is then a Lyapunov function. Furthermore, applying (30a) and (30b) similarly as in the dissipativity part of the proof of Theorem 7 now gives

$$\frac{\partial V_i}{\partial x}\mathcal{A}x < -z^T z \qquad \text{if } x \in \mathfrak{C}_\mathrm{i},\ x \neq 0. \tag{32a}$$
$$V(x^+) - V(x) \leq 0 \qquad \text{if } x \in \mathfrak{D} \tag{32b}$$

Hence, (32a) and (32b) guarantee that

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(w(t), z(t))dt \tag{33}$$

which again yields that $||\Sigma||_{2,x_0}^2 \leq V(x(0)) \leq \gamma^2$. Now applying further analogous arguments as in the proof of Theorem 7, completes this proof. ■

Remark 9 The LMI formulations to assess the $\mathcal{H}_2$ norm of reset systems were originally intended for setting up efficient controller synthesis techniques. Unfortunately, it seems hard to obtain LMI-type of conditions for such a controller design. Nonlinear combinations of the design variables (i.e. P and the controller parameters), which are not linearizable using [19], force the design problem into a (non-convex) *bilinear* matrix inequality, which is generally hard to solve. □

V. Example

As we mentioned before, performance improvement by using reset control is especially apparent in the transient closed loop behavior of constrained problems, which motivates our choice to consider the $\mathcal{H}_2$ norm of reset control systems. Indeed, in this section we show, by means of an input constrained problem, that reset control can perform better than the linear controller obtained via the usual mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design approximation for this type of problem.

Example 1 (Tracking performance of an integrator plant) In this example we consider a closed loop system with an integrator plant, $G(s) = \frac{1}{s}$, which should track a unit step reference $r(t) = 1(t)$. Our goal is to minimize the energy in the error e for this specific reference, subject to a maximum allowed control signal u to the plant, as is usually the case in practical situations. Hence, our design problem is

$$\min_K \quad \sqrt{\int_0^\infty e^2\,dt} \tag{34a}$$

$$\text{subject to} \quad |u(t)| \leq 1. \tag{34b}$$

The theoretical best non-linear controller for our integrator plant is described by the discontinuous feedback

$$u = \operatorname{sign}(e). \tag{35}$$

This controller produces the maximum control signal as long as possible, and vanishes as soon as the plant output reaches the desired value. This way the plant reacts as fast as possible, without any overshoot, thus realizing a minimal amount of energy in e, i.e. $\sqrt{1/3} \approx 0.577$. The closed loop response resulting from this discontinuous feedback is depicted in Figure 4 in grey.

Common multiobjective controller design methods rely on norm based optimization functionals and constraints [19]. Problem (34), however, is given in terms of time domain signals. A common attempt to capture the essence of time domain specifications such as (34b) is the reformulation into the frequency domain, which is in general only an approximation and might be conservative. However, in [24] the authors propose to use a static output filter $W_u = \frac{m}{U_0}$ for a standard $\mathcal{H}_\infty$ optimization problem in order to obtain that $|u(t)| \leq U_0$ for the specific step reference $r(t) = m \cdot 1(t)$. Furthermore, the step reference can be accurately approximated by including an input filter $W_r(s) = \frac{1}{s+\varepsilon}$ and considering an impulsive input to this filter. These considerations result in the multiobjective problem as depicted in Figure 3, where, using $U_0 = m = 1$, $\varepsilon = 10^{-4}$, the matrices defining the augmented plant P_{aug} are given by

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -\varepsilon \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$
$$C_z = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix} \quad D_z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

According to [24] (34) can then be approximated via the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem

$$\min_K \quad \gamma_2 = ||T_2||_2 \tag{36a}$$

$$\text{subject to} \quad ||T_\infty||_\infty < \gamma_\infty \tag{36b}$$

where T_2 denotes the transfer function from w_1 to z_1, T_∞ is the transfer between w_2 and z_2, and $\gamma_\infty = 1$. The standard method to approximate the optimum for (36) is by requiring a common Lyapunov function for both specifications, and applying a linearizing change of variables [19]. However, this approach is to some extend conservative. It is therefore necessary to iteratively increase γ_∞ until the $\mathcal{H}_\infty$ norm of the control sensitivity function $R(s) = \frac{u(s)}{r(s)} = \frac{K}{1+KG}$ is just below 1, which in this case happens when $\gamma_\infty = 1.25$. Afterwards, the actual $\mathcal{H}_2$ norm with the obtained controller should be recalculated with a separate linear $\mathcal{H}_2$ analysis. Using this method, we obtain $\gamma_2 = 1$ and the static linear controller $K = 1$. The closed loop response using this controller is depicted in Figure 4 by the dashed line. The actual $\mathcal{H}_2$ norm from w_1 to z_1 is $\sqrt{1/2} \approx 0.707$ for this linear controller, which can be validated by either an LMI optimization or numerical integration of the output energy in e.

However, a better performance, i.e. a smaller minimum for (34a) while maintaining (34b), can easily be obtained by using e.g. the reset controller satisfying (11) where

$$A_K = \begin{bmatrix} -0.01 & 7.8125 \\ 0 & -62.5 \end{bmatrix}, \quad B_K = \begin{bmatrix} 0 \\ 8 \end{bmatrix},$$
$$C_K = \begin{bmatrix} 1.32 & 7.0313 \end{bmatrix}, \quad D_K = 0, \quad A_r = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

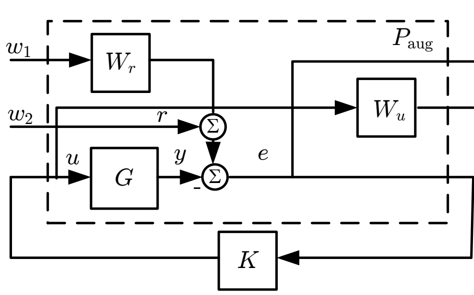

Fig. 3. Augmented plant description for the multiobjective problem

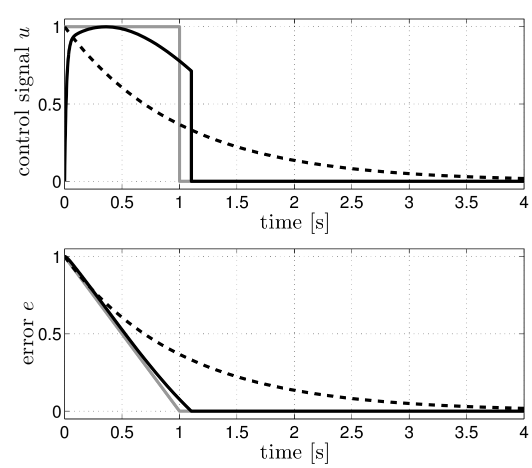

Fig. 4. Closed loop responses u and e using hybrid feedback (grey), the linear controller (dashed), and a reset controller (solid)

TABLE I

$\mathcal{H}_2$ NORMS FOR VARIOUS CONTROLLERS

	hybrid	linear	reset control N=2	N=5	N=10	N=50
Numerical int.	0.577	0.707	0.601	0.601	0.601	0.601
LMI approx.	0.577	0.707	0.721	0.618	0.604	0.601

The solid lines in Figure 4 show its closed loop response. Theorem 8 can now be applied to approximate the $\mathcal{H}_2$ norm from w_1 to z_1 for this controller by selecting the appropriate rows and columns of the augmented plant matrices. With $N = 5$ subregions to divide the state space, we obtain $\gamma_2 = 0.618$, which is much smaller than the linear $\mathcal{H}_2$ norm. It can be seen in Figure 4 that our reset controller approximates the discontinuous controller (35) fairly well, resulting in an almost as fast response and an $\mathcal{H}_2$ norm that is only slightly larger. The actual energy content in e obtained by the reset controller and calculated by numerical integration of the energy in e equals 0.601. To check the approximation power of Theorem 8, we divide the state space into more regions N. All results are summarized in Table I. Note that indeed our LMI-based $\mathcal{H}_2$ analysis converges to the correct value as N increases. □

We would like to emphasize that it is not guaranteed that the constructed linear controller is the optimal one for problem (34). There might still be (higher order) linear controllers with a better performance. However, the above example shows that it is easy to find a reset controller that outperforms the linear controller that is designed via a common design routine for these type of problems.

VI. Conclusion

Motivated by recent publications on the potential advantages of reset control, we have developed an LMI-based framework which can be used to calculate the $\mathcal{H}_2$ norm of a general reset control system. We removed much conservatism by introducing piecewise quadratic Lyapunov functions, which are much more flexible than quadratic ones. Furthermore, we have presented an example which shows that reset control can be close to the performance of the optimal (discontinuous) controller for a constrained $\mathcal{H}_2$ problem, while a common method to design a good linear controller provides a worse $\mathcal{H}_2$ performance. The example also shows the accuracy of our LMI-based calculation of the $\mathcal{H}_2$ norm in the sense that by increasing the number of regions in the piecewise quadratic Lyapunov function, we recover the actual $\mathcal{H}_2$ performance of the reset control system. Finally, we remark that it is still of interest to find a systematic synthesis method for reset controllers.

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