

Periodic Event-Triggered Control for Nonlinear Networked Control Systems

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I. INTRODUCTION

Abstract—Periodic event-triggered control (PETC) is an appealing paradigm for the implementation of controllers on platforms with limited communication resources, a typical example being networked control systems. In PETC, transmissions over the communication channel are triggered by an event generator, which depends solely on the available plant and controller data and is only evaluated at given sampling instants to enable its digital implementation. In this paper, we consider the general scenario, where the controller communicates with the plant via multiple decoupled networks. Each network may contain multiple nodes, in which case a dedicated protocol is used to schedule transmissions among these nodes. The transmission instants over the networks are asynchronous and generated by local event generators. At given sampling instants, the local event generator evaluates a rule, which only involves the measurements and the control inputs available locally, to decide whether a transmission is needed over the considered network. Following the emulation approach, we show how to design local triggering generators to ensure input-to-state stability and \mathcal{L}_p stability for the overall system based on a continuous-time output-feedback controller that robustly stabilizes the network-free system. The method is applied to a class of Lipschitz nonlinear systems, for which we formulate the design conditions as linear matrix inequalities. The effectiveness of the scheme is illustrated via simulations of a nonlinear example.

Index Terms—Lyapunov methods, networked control systems, nonlinear control systems, robust stability.

NETWORKED control systems (NCSs) refer to systems, in which the plant and the controller communicate via networks. Integrating networks into control systems, compared with the traditional dedicated point-to-point (wired) links, has major advantages, such as lower cost, reduced weight and power, simpler installation and maintenance, and higher reliability [1]. Moreover, the NCS configuration is essential when the plant consists of many subsystems, which are physically distributed and interconnected to coordinate their tasks and achieve an overall objective; see their applications in smart grids, wide-area systems, or for systems with distributed sensors, actuators, and controllers. A major challenge in NCSs is to design control strategies, which do not “overuse” the network, to limit the transmission delays and the occurrence of packet losses, which may destroy the desired closed-loop system properties. An attractive approach in this context is event-triggered control (ETC), which adapts the transmission instants based on the current state, input, and/or output measurement of the plant (see [2] and the references therein). The idea of ETC is to use the network only when this is needed by generating transmissions whenever a state- or output-dependent condition is satisfied. Most literature on ETC focuses on *continuous event-triggered control* (CETC), in the sense that the triggering condition is evaluated at all times (see, for instance, [3]–[7]). Although CETC may significantly reduce the number of transmissions compared with traditional periodic sampling, the continuous evaluation of the triggering condition causes issues when sensors are battery powered for mobility and/or flexibility reasons. Moreover, it is not even possible to evaluate triggering rules continuously when the implementation platform is digital. In this case, it is more natural to evaluate the triggering criterion at some discrete sampling instants, leading to *periodic event-triggered control* (PETC) (see [8] and [9]).

Hybrid systems are commonly used to model CETC systems (e.g., [4], [7], [10], [11]), as the plant and the controller are often described by continuous-time systems, and transmissions are discrete events, which can be modeled by jumps. The generic results in [12] about the sampled-and-hold implementations of hybrid controllers ensure that the emulation of a continuous event-triggered controller as a *periodic* event-triggered controller still “works,” if the sampling period is sufficiently small. To be more precise, the uniform global asymptotic stability of a compact set ensured by CETC is semiglobally and practically preserved for *fast* sampling by PETC. Unfortunately, these results do not provide exploitable explicit bounds on the sampling period. Furthermore, it is of interest to preserve global asymptotic

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stability properties in PETC, instead of semiglobal practical asymptotic stability. Works addressing these points have mostly been developed for systems with linear dynamics (see [8] and [13]–[16]). On the other hand, PETC results for nonlinear systems are scarce. In [9, Ch. 6.5] and [17], it is explained how to convert general continuous state-feedback event-triggered controllers to periodic event-triggered ones, while (approximately) preserving the properties of the former. The work in [18] develops observer-based output-feedback controllers for a class of nonlinear Lipschitz systems, and a practical stability property is ensured at the end. Another work is [19], where the output-feedback PETC scheme is studied to ensure global asymptotic stability for a class of polynomial nonlinear systems. Obviously, PETC for nonlinear systems is at its early stage, and a lot remains to be done. In particular, there is a need for systematic design frameworks, which are flexible enough to cope with output feedbacks, as well as exogenous disturbances. The primary aim of this paper is to address this challenge.

We study plants modeled by a continuous-time nonlinear system affected by exogenous disturbances and for which only some output is available for control. We proceed by emulation to design the periodic event-triggered controller. Thus, we first assume that we know an output-feedback controller, which robustly stabilizes the plant in the absence of communication constraints, in the sense that it ensures either an input-to-state stability or an \mathcal{L}_p stability property for the closed-loop system with respect to the exogenous disturbances, as well as output and input noises. At this stage, any continuous-time design technique can be applied. We then implement the controller over networks. We investigate the scenario where multiple asynchronously operating networks are used to connect the controller to the plant: this is an additional novelty of this paper. This setup is relevant, for instance, when one network ensures the communication from the sensors to the controller, and another one is used to connect the controller to the actuators. The sensors and the actuators are grouped into nodes, which are connected to a given network. The transmissions over each network are generated by a local triggering generator. The latter collects measurements and control inputs, which are locally available; at some sampling instants specific to the considered network (and not necessarily periodic), it evaluates a criterion and then decides whether a node needs to transmit its packet over this network. The transmitting node is selected according to the local scheduling rule, such as the round-robin (RR) or try-once-discard (TOD) protocol considered in [1] and [20]. To design local triggering generators, we, therefore, have to define three elements: the criterion, the sampling instants at which the criterion is evaluated, and the scheduling rule. Regarding the scheduling rules, we require that they are *uniformly globally asymptotically stable* (UGAS) as characterized in [20], which cover the RR, TOD, and the sampled-data protocols. We also make assumptions on the robust stability of the original closed-loop system in the absence of a network, which can be checked *a priori*. Note that imposing robust stability properties is required for any nonlinear control systems to be implementable in practice. Based on these assumptions, we provide the expression of the local triggering conditions, as well as an explicit bound on the maximum allowable sampling periods (MASP), which are used to characterize

the sampling instants. We actually show that there is a tradeoff between the MASP of each triggering generator and a parameter used to define the corresponding triggering condition.

The overall system is modeled as a hybrid system using the formalism of [21] and [22], for which a jump corresponds to a sampling instant of one local triggering generator. We then ensure an input-to-state stability or an \mathcal{L}_p stability with respect to the exogenous disturbances, depending on the assumptions. These results lead to a uniform global asymptotic stability property in the absence of disturbances. The analysis relies on a novel hybrid Lyapunov function. We apply the results to a class of globally Lipschitz nonlinear systems and formulate the assumptions as linear matrix inequalities (LMIs). The obtained LMIs are always verified in special cases when the nonlinearity only involves the measured output or for any stabilizable and detectable LTI systems. The latter case appears to be a contribution in its own right, as it extends the centralized and state-feedback PETC for linear systems in [13] and [8] and the output-feedback PETC in [14] to decentralized implementations. Simulation results on a nonlinear system, which is not globally Lipschitz, are also provided.

The decentralized setup we investigate is similar to the one in [4], where *continuous event-triggered controllers* are synthesized. The fact that we consider PETC, as opposed to CETC, benefits for digital implementations, which leads to additional difficulties. Because the triggering rules are continuously evaluated in CETC, properties, which are essential to guarantee stability, are ensured at all times. This is no longer the case in PETC, as the triggering criteria are only checked at some sampling instants and may, therefore, be violated between two successive sampling instants. As a result, our approach requires a new hybrid model, a different set of assumptions, as well as a novel hybrid Lyapunov function compared to [4]. Note that CETC as proposed in [4] relies on time regularization, as the triggering criterion is evaluated continuously after a fixed waiting time has elapsed since the previous event. This is different from PETC as done in this paper, as the triggering conditions here are evaluated *only* at some sampling instants, which facilitates digital implementations. Compared to [9, Ch. 6.5] and [17], the results are applicable for decentralized output-feedback control, tolerate the presence of exogenous disturbances, and explicitly reveal a link between the triggering conditions and the sampling instants. Compared to [18], we consider exogenous disturbances, a decentralized scenario, we do not restrict our attention to nonlinear systems with a specific structure, and we ensure asymptotic stability in the absence of perturbations. Conference versions of this paper can be found in [23] and [24]. In particular, a centralized full-state-feedback PETC is provided for disturbance-free systems in [23], and a centralized output-feedback control for systems implemented on a single network is studied in [24], where only input-to-state stability results are provided.

To summarize, our work leads to the following contributions on PETC:

- 1) a generic design framework of triggering generators for nonlinear systems, which is applicable for output-feedback control, as well as in the presence of exogenous disturbances;

- 2) decentralized PETC strategies over multiple asynchronously operating networks, for the first time to the best of our knowledge;
- 3) a novel hybrid Lyapunov function is constructed to investigate stability properties of the system;
- 4) even in the particular case of linear systems, the results extend those in [8], [13], and [14] to decentralized output-feedback control.

The rest of this paper is organized as follows. The notation and preliminaries on hybrid systems are given in Section II. We state the problem and present the hybrid model in Section III. The main results are provided in Section IV and applied to a class of globally Lipschitz nonlinear systems in Section V. Simulation results for a nonlinear system are given in Section VI, and conclusions are provided in Section VII. The proofs are postponed to the Appendixes, where technical lemmas are also provided.

II. PRELIMINARIES

Let $\mathbb{Z}_{>0} := \{1, 2, \dots\}$, $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$, $\mathbb{R} := (-\infty, \infty)$ and $\mathbb{R}_{\geq 0} := [0, \infty)$. For $k_0 \in \mathbb{Z}_{\geq 0}$ and $\Gamma \subset \mathbb{Z}_{\geq 0}$, $k_0 + \Gamma := \{k_0 + k : k \in \Gamma\}$. For sets A and B in a universe U , $A \setminus B := \{x \in U : x \in A \text{ and } x \notin B\}$. Let $\mathbf{0}_n$ and $\mathbf{1}_n$, $n \in \mathbb{Z}_{>0}$, be the n -dimensional vector, for which elements are all zeros and ones, respectively. Let $\mathbf{0}_{n \times n}$ and $\mathbf{I}_{n \times n}$ be the square zero matrix and the identity matrix of dimension n , respectively. Let $|x|$ denote the Euclidean norm of the vector $x \in \mathbb{R}^n$. Let $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ stand for the minimum and maximum eigenvalues of real symmetric matrix P , respectively. For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, (x, y) stands for $[x^T, y^T]^T$. Given a set $\mathcal{A} \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define the distance of x to \mathcal{A} as $|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|$. A set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is outer semicontinuous when its graph $\{(y, z) \in \mathbb{R}^m \times \mathbb{R}^n : z \in M(y)\}$ is closed (see [22, Lemma 5.10]). A function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class- \mathcal{K} , if it is continuous, zero at zero, and strictly increasing, and it is of class- \mathcal{K}_{∞} if, in addition, it is unbounded. A function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class- \mathcal{KL} , if it is continuous, $\gamma(\cdot, r)$ is of class- \mathcal{K} for each $r \in \mathbb{R}_{\geq 0}$, and for each $s \in \mathbb{R}_{\geq 0}$, $\gamma(s, \cdot)$ is decreasing to zero. The notation $(t', j') \leq (t, j)$ refers to $t' \leq t$ and $j' \leq j$ with $t, t' \in \mathbb{R}_{\geq 0}$ and $j, j' \in \mathbb{Z}_{\geq 0}$. For $x, v \in \mathbb{R}^n$ and locally Lipschitz $U : \mathbb{R}^n \rightarrow \mathbb{R}$, $U^\circ(x; v)$ is the Clarke derivative of the function U at x in the direction v , i.e., $U^\circ(x; v) := \limsup_{y \rightarrow x, \lambda \downarrow 0} \frac{U(y + \lambda v) - U(y)}{\lambda}$. This notion will be useful as we will be working with locally Lipschitz Lyapunov functions, which are not differentiable everywhere. We omit the definitions on hybrid systems and refer the reader to [21] and [22].

III. PETC SETUP AND HYBRID MODEL

In this section, we introduce the setup and model the overall system as a hybrid system. We then formally state the problem.

A. PETC Setup

We consider the plant model

$$\begin{aligned} \dot{x}_p &= f_p(x_p, u, w) \\ y &= g_p(x_p) \end{aligned} \quad (1)$$

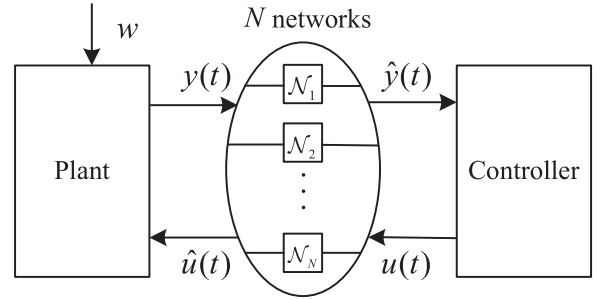


Fig. 1. Block diagram of the setup.

where $x_p \in \mathbb{R}^{n_p}$ is the state, $w \in \mathbb{R}^{n_w}$ is the exogenous disturbance, $u \in \mathbb{R}^{n_u}$ is the control input, and $y \in \mathbb{R}^{n_y}$ is the plant output. As already mentioned in Section I, we use an emulation-based design approach. We, therefore, assume that we know an output-feedback controller

$$\begin{aligned} \dot{x}_c &= f_c(x_c, y) \\ u &= g_c(x_c) \end{aligned} \quad (2)$$

with state $x_c \in \mathbb{R}^{n_c}$, which robustly stabilizes the origin of (1) in a sense made precise in Section IV-A. The functions f_p and f_c are assumed to be continuous, and g_p and g_c are assumed to be continuously differentiable and zero at zero. Any controller design method can be used to obtain controller (2), such as backstepping, forwarding, feedback linearization, high-gain techniques etc.

We consider the scenario, where plant (1) and controller (2) communicate with each other via multiple networks, as illustrated in Fig. 1. In particular, sensors and actuators are connected by $N \in \mathbb{Z}_{>0}$ independently and asynchronously operating networks $\mathcal{N}_1, \dots, \mathcal{N}_N$. Let $\bar{N} := \{1, 2, \dots, N\}$ and $v := (y, u) \in \mathbb{R}^{n_y + n_u}$. For simplicity of exposition, we assume $v = (v_1, \dots, v_N)$ (after reordering, if necessary), where v_i , $i \in \bar{N}$, corresponds to the sensors and the actuators whose signals are transmitted through network \mathcal{N}_i .

A local event-triggering generator generates the sequence of transmission instants for each network \mathcal{N}_i , $i \in \bar{N}$, in the following manner. A triggering condition is evaluated at each sampling instant s_j^i , $i \in \bar{N}$, $j \in \mathbb{Z}_{\geq 0}$, where

$$\varepsilon_i \leq s_{j+1}^i - s_j^i \leq T_i \quad (3)$$

with $T_i > 0$ the upper bound on the intersampling times and $\varepsilon_i \in (0, T_i]$ the minimum time between two successive evaluations of the triggering condition. Note that each network has its own sequence of sampling instants, which is not necessarily periodic or synchronized with the other networks. Consequently, the sequence of transmission instants of network \mathcal{N}_i , which we denote $\{t_{\kappa_i}^i\}_{\kappa_i \in \mathbb{Z}_{\geq 0}}$, is a subsequence of $\{s_j^i\}_{j \in \mathbb{Z}_{\geq 0}}$, and two successive transmissions are spaced by at least ε_i units of time in view of (3), thereby avoiding the Zeno phenomenon. Parameter ε_i reflects the minimum achievable transmission interval given by the hardware constraints. Note that ε_i can be chosen arbitrarily in the set $(0, T_i]$. In fact, the stability and performance results below apply for any $\varepsilon_i \in (0, T_i]$. In practical, $\varepsilon_i > 0$ is determined by the hardware constraint. We assume that transmission delays and quantization effects are negligible. Each

transmission generator consists of a triggering law and a scheduling rule. We need to introduce some variables before presenting those.

We denote by \hat{u} the networked version of u available to plant (1). Similarly, controller (2) has access to \hat{y} , the networked version of y . We let \hat{v} be the networked version of v and we partition it as $(\hat{v}_1, \dots, \hat{v}_N)$ in the same way as v is. Thus, \hat{v}_i , $i \in \overline{N}$, is related to the network \mathcal{N}_i . Between two successive transmission instants, \hat{v}_i is governed by

$$\dot{\hat{v}}_i = \hat{f}_{v_i}(\hat{v}, g_p(x_p), g_c(x_c)), t \in (s_j^i, s_{j+1}^i), j \in \mathbb{Z}_{\geq 0}, i \in \overline{N} \quad (4)$$

where \hat{f}_{v_i} is the holding function corresponding to network \mathcal{N}_i , and we define $\hat{f}_v := (\hat{f}_{v_1}, \dots, \hat{f}_{v_N})$. Zero-order-hold devices correspond to $\hat{f}_v = 0$ for instance. Other holding functions can also be envisioned, like model-based ones (see, for example, [5]). Before modeling the dynamics of \hat{v}_i , $i \in \overline{N}$, at each sampling instant s_j^i , we introduce the vector of network-induced errors $e_i := \hat{v}_i - v_i \in \mathbb{R}^{n_{e_i}}$, where $n_{e_i} \in \mathbb{Z}_{>0}$ satisfies $\sum_{i=1}^N n_{e_i} = n_y + n_u$. Hence, n_{e_i} is the number of sensor/actuator signals associated with network \mathcal{N}_i .

At each sampling instant s_j^i , $j \in \mathbb{Z}_{\geq 0}$ and $i \in \overline{N}$, a function $\Upsilon_i : \mathbb{R}^{n_{e_i}} \times \mathbb{R}^{n_{e_i}} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$ is evaluated, which depends on v_i , \hat{v}_i , and an auxiliary variable κ_i , which counts the number of transmissions over network \mathcal{N}_i . The expression of Υ_i will be given in Section IV-B. A transmission is triggered depending on the sign of Υ_i , which leads to the update law for \hat{v}_i given by

$$\hat{v}_i(s_j^{i+}) \in \begin{cases} \{v_i(s_j^i) + \chi_i(e_i(s_j^i), \kappa_i(s_j^i))\} \\ \text{when } \Upsilon_i(e_i(s_j^i), v_i(s_j^i), \kappa_i(s_j^i)) > 0 \\ \{\hat{v}_i(s_j^i)\} \\ \text{when } \Upsilon_i(e_i(s_j^i), v_i(s_j^i), \kappa_i(s_j^i)) < 0 \\ \{v_i(s_j^i), v_i(s_j^i) + \chi_i(e_i(s_j^i), \kappa_i(s_j^i))\} \\ \text{when } \Upsilon_i(e_i(s_j^i), v_i(s_j^i), \kappa_i(s_j^i)) = 0 \end{cases} \quad (5)$$

where χ_i models the scheduling protocol corresponding to network \mathcal{N}_i , such as the RR¹ or TOD protocol,² or the so-called sampled-data protocol for which $\chi_i = 0$ when the network is composed of a single node. Expressions of χ_i for various protocols are available in [20] and [25], and the cases of RR and TOD are provided next for completeness.

Example 1 (TOD protocol): Let $i \in \overline{N}$ and $\ell_i \in \mathbb{Z}_{>0}$ denote the number of nodes of \mathcal{N}_i network. TOD protocol is modeled as $\chi_i(e_i) := (I - \tilde{\Psi}_i(e_i))e_i$, where $\tilde{\Psi}_i(e_i) := \text{diag}\{\tilde{\psi}_1^i(e_i)\mathbf{I}_{n_1^i \times n_1^i}, \tilde{\psi}_2^i(e_i)\mathbf{I}_{n_2^i \times n_2^i}, \dots, \tilde{\psi}_{\ell_i}^i(e_i)\mathbf{I}_{n_{\ell_i}^i \times n_{\ell_i}^i}\}$. The functions $\tilde{\psi}_s^i$ satisfy $\tilde{\psi}_s^i(e_i) = 1$ when $s = \min(\arg \max_{j \in \{1, \dots, \ell_i\}} |e_j^i|)$ and $\tilde{\psi}_s^i(e_i) = 0$ otherwise, for $s \in \{1, 2, \dots, \ell_i\}$. ■

Example 2 (RR protocol): The RR protocol has the form of $\chi_i(e_i, \kappa_i) := (I - \Delta_i(\kappa_i))e_i$, where $\Delta_i(\kappa_i) = \text{diag}\{\delta_1^i(\kappa_i)\mathbf{I}_{n_1^i \times n_1^i}, \delta_2^i(\kappa_i)\mathbf{I}_{n_2^i \times n_2^i}, \dots, \delta_{\ell_i}^i(\kappa_i)\mathbf{I}_{n_{\ell_i}^i \times n_{\ell_i}^i}\}$, $\delta_s^i(\kappa_i) = 1$ for $s \in \{1, 2, \dots, \ell_i\}$ when $s - 1 = \kappa_i \bmod \ell_i$ and $\delta_s^i(\kappa_i) = 0$ otherwise. ■

¹The RR protocol assigns access to network in a predetermined and cyclic manner.

²The TOD protocol gives access to the node with the largest mismatch between the current signal value and the last transmitted one.

Let $\ell_i \in \mathbb{Z}_{>0}$ be the number of nodes of network \mathcal{N}_i , and v_i and \hat{v}_i are, respectively, partitioned as $(v_{i,1}, \dots, v_{i,\ell_i})$, and $(\hat{v}_{i,1}, \dots, \hat{v}_{i,\ell_i})$ (after reordering these, if needed), where $v_{i,j}$ and $\hat{v}_{i,j}$, $j \in \{1, \dots, \ell_i\}$, denote a group of sensors or/and actuators associated with the same node on \mathcal{N}_i . In view of (5), when $\Upsilon_i(e_i(s_j^i), v_i(s_j^i), \kappa_i(s_j^i)) > 0$, a transmission occurs over network \mathcal{N}_i at time s_j^i and the scheduling protocol grants access to the network to a single node, say the k th node with $k \in \{1, 2, \dots, \ell_i\}$. Then, $\hat{v}_{i,k}(s_j^{i+}) = v_{i,k}(s_j^i)$ and $\hat{v}_{i,m}(s_j^{i+}) = \hat{v}_{i,m}(s_j^i)$ for all $m \in \{1, 2, \dots, \ell_i\} \setminus \{k\}$. When $\Upsilon_i(e_i(s_j^i), v_i(s_j^i), \kappa_i(s_j^i)) < 0$, no transmission occurs and κ_i and the complete vector \hat{v}_i remain unchanged. When $\Upsilon_i(e_i(s_j^i), v_i(s_j^i), \kappa_i(s_j^i)) = 0$, the model allows two possibilities: either a transmission occurs or not. This construction ensures that the jump map in (5) is outer semicontinuous, which is essential for the hybrid model presented below to be (nominally) well-posed (see [22, Ch. 6] for more details). Note that the transmissions over the N networks are independently generated; as a result, several transmissions can occur at the same time, but over distinct networks.

We are almost ready to model the overall system. Before that, we need to write the dynamics of the network-induced errors. Let $x := (x_p, x_c) \in \mathbb{R}^{n_x}$ and $n_x := n_p + n_c$. We deduce from (5) that the variable e_i has the following dynamics at jumps:

$$e_i(s_j^{i+}) \in h_i(x(s_j^i), e_i(s_j^i), \kappa_i(s_j^i)) \quad (6)$$

where

$$h_i(x, e_i, \kappa_i) := (1 - \Gamma_i(e_i, v_i, \kappa_i))e_i + \Gamma_i(e_i, v_i, \kappa_i)\chi_i(e_i, \kappa_i) \quad (7)$$

and $\Gamma_i : \mathbb{R}^{n_{e_i}} \times \mathbb{R}^{n_{e_i}} \times \mathbb{Z}_{\geq 0} \Rightarrow \{0, 1\}$ in (7) indicates whether a transmission occurs. Based on the discussion above (5), $\Gamma_i(e_i, v_i, \kappa_i) = \{1\}$ when $\Upsilon_i(e_i, v_i, \kappa_i) > 0$, which corresponds to a transmission and $h_i(x, e_i, \kappa_i) = \chi_i(e_i, \kappa_i)$ in this case. When $\Upsilon_i(e_i, v_i, \kappa_i) < 0$, $\Gamma_i(e_i, v_i, \kappa_i) = \{0\}$, and this corresponds to no transmission and $h_i(x, e_i, \kappa_i) = e_i$. When $\Upsilon_i(e_i, v_i, \kappa_i) = 0$, $\Gamma_i(e_i, v_i, \kappa_i) = \{0, 1\}$ covers the above two possibilities. In agreement with [20], we call (6) the *protocol map*. We see from the right-hand side of (7) that h_i depends on v_i and not on the complete vector of x . Writing

$$v_i = g_{v_i}(x) \quad (8)$$

since v_i is composed of components of y and u , which depend on x in view of (1) and (2), we make h_i depend on x and not v_i in (7), for the sake of convenience. We note that h_i depends on the state x contrary to [20], [26], and [27], which will have important consequences on the stability property and analysis of the protocols compared to the latter references (see Remark 3 in Section IV-B).

B. Hybrid Model

We model the overall system as a hybrid system using the formalism of [21] and [22] so that we can resort to the analytical tools of [21] and [22] to study the stability properties of the system. We introduce for this purpose clock variables $\tau_i \in \mathbb{R}_{\geq 0}$

for $i \in \bar{N}$ to keep track of the time elapsed since the last evaluation of the triggering criterion of network \mathcal{N}_i . Thus, τ_i and κ_i , the transmission counter of network \mathcal{N}_i , have the dynamics

$$\begin{cases} \dot{\tau}_i = 1 \\ \dot{\kappa}_i = 0 \end{cases} \quad \text{when } \tau_i \in [0, T_i] \\ \begin{cases} \tau_i^+ = 0 \\ \kappa_i^+ \in \kappa_i + \Gamma_i(e_i, v_i, \kappa_i) \end{cases} \quad \text{when } \tau_i \in [\varepsilon_i, T_i]$$

where Γ_i is introduced after (7). Let $\tau := (\tau_1, \dots, \tau_N)$ and $\kappa := (\kappa_1, \dots, \kappa_N)$. We model the overall closed-loop system as

$$\begin{aligned} \dot{q} &= F(q, w), & q \in C \\ q^+ &\in G(q), & q \in D \end{aligned} \quad (9)$$

where $q := (x, e, \kappa, \tau) \in \mathcal{X} := \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{Z}^N \times \mathbb{R}^N$

$$\begin{aligned} C &:= \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{Z}_{\geq 0}^N \times \mathbb{T}_1 \times \dots \times \mathbb{T}_N \\ D &:= \bigcup_{i=1}^N D_i \end{aligned} \quad (10)$$

$$\begin{aligned} D_i &:= \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{Z}_{\geq 0}^N \times \mathbb{T}_1 \times \dots \times \mathbb{T}_{i-1} \times [\varepsilon_i, T_i] \\ &\quad \times \mathbb{T}_{i+1} \times \dots \times \mathbb{T}_N \end{aligned}$$

and $\mathbb{T}_i := [0, T_i]$. The mapping F in (9) is defined as: for $q \in C$, $F(q, w) := (f(x, e, w), g(x, e, w), \mathbf{0}_N, \mathbf{1}_N)$, where $f(x, e, w) := (f_p(x_p, g_c(x_c) + e_u, w), f_c(x_c, g_p(x_p) + e_y))$, $g(x, e, w) := (g_1(x, e, w), \dots, g_N(x, e, w))$, and for $i \in \bar{N}$,

$$\begin{aligned} g_i(x, e, w) &:= \hat{f}_{v_i}(g_{v_i}(x) + e_i, g_{v_i}(x)) - f_{v_i}(x, e, w) \\ f_{v_i}(x, e, q) &:= \frac{\partial g_{v_i}}{\partial x} f(x, e, w) \end{aligned} \quad (11)$$

with $f_p, g_p, f_c, g_c, \hat{f}_{v_i}$, and g_{v_i} coming from (1), (2), (4), and (8), respectively. The set-valued mapping G is defined, for $q \in \mathcal{X}$, as $G(q) := \bigcup_{i=1}^N \mathcal{G}_i(q)$ with

$$\mathcal{G}_i(q) := \begin{cases} \left\{ \left(\begin{array}{c} x \\ \mathcal{H}_i(x, e, \kappa) \\ \kappa + \Gamma_i(e_i, x, \kappa_i) \bar{\Lambda}_i \\ \Lambda_i \tau \end{array} \right) \right\} & \text{when } q \in D_i \\ \emptyset & \text{when } q \notin D_i \end{cases} \quad (12)$$

where $\Lambda_i \in \mathbb{R}^{N \times N}$ and $\bar{\Lambda}_i \in \mathbb{R}^{N \times N}$ are diagonal matrices; for the former, the diagonal elements are 1 except the i th one, which is 0, and for the latter, the diagonal elements are 0 except the i th one, which is 1. Hence, $\Lambda_i + \bar{\Lambda}_i = \mathbf{I}_{N \times N}$. The function $\mathcal{H}_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{Z}_{\geq 0}^N \rightarrow \mathbb{R}^{n_e}$ is defined as $\mathcal{H}_i(x, e, \kappa) := (e_1, e_2, \dots, e_{i-1}, h_i(x, e_i, \kappa_i), e_{i+1}, \dots, e_N)$, where h_i comes from (7). The map \mathcal{G}_i describes how e jumps when a transmission occurs over network \mathcal{N}_i : e_i is updated to $h_i(x, e_i, \kappa_i)$, κ_i is incremented to $\kappa_i + 1$ when the local generator triggers a transmission; otherwise, it keeps the same value, and τ_i is always reset to 0 after a jump. The function \mathcal{G}_i keeps x, e_j, κ_j, τ_j unchanged

for all $j \in \bar{N} \setminus \{i\}$. In model (9), simultaneous transmissions over different networks are modeled by successive jumps with no flow in between.

C. Problem Statement

Our objective is to design the local triggering generators, namely Υ_i and $T_i, i \in \bar{N}$, to ensure either input-to-state stability or \mathcal{L}_p stability properties for system (9), as defined next.

Definition 1: Set $\mathcal{S} \subset \mathcal{X}$ is *input-to-state stable* (ISS) for system (9) if there exist $\beta \in \mathcal{KL}$ and $\psi \in \mathcal{K}_\infty$ such that any solution pair (φ, w) satisfies³ $|\varphi(t, j)|_{\mathcal{S}} \leq \beta(|\varphi(0, 0)|_{\mathcal{S}}, t + j) + \psi(\|w\|_\infty)$ for all $(t, j) \in \text{dom } \varphi$. We say that \mathcal{S} is *exponentially ISS with a linear gain* when $\beta(s_1, s_2) = k s_1 \exp(-c s_2)$ and $\psi(s) = \gamma s$ for some $k, c, \gamma > 0$ and for $s_1, s_2, s \geq 0$. ■

We define \mathcal{L}_p stability with respect to output $z := \eta(x, w)$, which may correspond to y or not.

Definition 2: System (9) is \mathcal{L}_p stable from w to z with respect to set $\mathcal{S} \subset \mathcal{X}$ with gain less than or equal to θ , if there exists $\tilde{\gamma} \in \mathcal{K}_\infty$ such that any solution pair (φ, w) satisfies⁴ $\|z\|_{\mathcal{L}_p} \leq \tilde{\gamma}(|\varphi(0, 0)|_{\mathcal{S}}) + \theta \|w\|_{\mathcal{L}_p}$. ■

IV. MAIN RESULTS

In this section, we first state the assumption we make on the closed-loop system (1), (2) and the scheduling rule, based on which we construct the triggering condition Υ_i and the bound on T_i , for $i \in \bar{N}$. We then present the stability guarantees.

A. Assumptions

We assume that each e_i system in (9) satisfies the following properties.

Assumption 1: For each $i \in \bar{N}$, there exist a locally Lipschitz function $W_i : \mathbb{R}^{n_{e_i}} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, a continuous function $\tilde{H}_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_w} \rightarrow \mathbb{R}_{\geq 0}$, $\underline{\alpha}_{W_i}, \bar{\alpha}_{W_i} \in \mathcal{K}_\infty$, $\rho_i \in [0, 1]$, and $L_{W_i} \geq 0$ such that the following hold.

- i) For any $e_i \in \mathbb{R}^{n_{e_i}}$ and $\kappa_i \in \mathbb{Z}_{\geq 0}$, $\underline{\alpha}_{W_i}(|e_i|) \leq W_i(e_i, \kappa_i) \leq \bar{\alpha}_{W_i}(|e_i|)$.
- ii) For any $(e_i, x, \kappa_i) \in \mathbb{R}^{n_{e_i}} \times \mathbb{R}^{n_x} \times \mathbb{Z}_{\geq 0}$, $W_i(\chi_i(e_i, \kappa_i), \kappa_i + 1) \leq \rho_i W_i(e_i, \kappa_i)$.
- iii) For almost all $e_i \in \mathbb{R}^{n_{e_i}}$, all $\kappa_i \in \mathbb{Z}_{\geq 0}$ and $(x, w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_w}$, $\langle \frac{\partial W_i(e_i, \kappa_i)}{\partial e_i}, g_i(x, e, w) \rangle \leq L_{W_i} W_i(e_i, \kappa_i) + \tilde{H}_i(x, e, w)$ with g_i coming from (11). ■

Items (i) and (ii) are exclusively related to the scheduling protocol implemented on network \mathcal{N}_i . Indeed, these items state that the protocol is UGAS (see [25, Definition 1]). These conditions are always satisfied for the sampled-data case and RR and TOD protocols for which expressions of W_i are available (see [20]). Then, given W_i , item (iii) of Assumption 1 essentially requires that W_i exponentially grows on flows. Such a property is natural, as the e_i system is typically unstable between two transmission instants. Item (iii) of Assumption 1 is always feasible when W_i

³See the definition of $\|w\|_\infty$ in [21].

⁴See the definition of $\|w\|_{\mathcal{L}_p}$ in [4].

is globally Lipschitz in e_i uniformly in κ_i , and g_i satisfies a linear growth condition for instance (see [20, Remark 11]).

We assume that controller (2) has been designed to robustly stabilize system (1) in the following sense.

Assumption 2: There exist a locally Lipschitz function $V : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$, $\underline{\alpha}_V, \bar{\alpha}_V, \alpha_W \in \mathcal{K}_\infty$, locally Lipschitz functions $\delta_i : \mathbb{R}^{n_{e_i}} \rightarrow \mathbb{R}_{\geq 0}$ satisfying $\delta_i(0) = 0$, and continuous functions $\tilde{\alpha}_V : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_w} \rightarrow \mathbb{R}$ and $J_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_w} \rightarrow \mathbb{R}_{\geq 0}$, $\gamma_i > 0$, $L_{\delta_i} \in \mathbb{R}$, $i \in \bar{N}$, such that the following hold.

- i) For all $x \in \mathbb{R}^{n_x}$, $\underline{\alpha}_V(|x|) \leq V(x) \leq \bar{\alpha}_V(|x|)$.
- ii) For almost all $x \in \mathbb{R}^{n_x}$ and all $(e, w) \in \mathbb{R}^{n_e} \times \mathbb{R}^{n_w}$, $\langle \nabla V(x), f(x, e, w) \rangle \leq -\tilde{\alpha}_V(x, e, w) + \sum_{i=1}^N (\gamma_i^2 W_i^2(e_i, \kappa_i) - \tilde{H}_i^2(x, e, w) - J_i(x, e, w) - \delta_i(v_i))$, where W_i and \tilde{H}_i come from Assumption 1.
- iii) For almost all $x \in \mathbb{R}^{n_x}$ and all $(e, w) \in \mathbb{R}^{n_e} \times \mathbb{R}^{n_w}$, $\langle \nabla \delta_i(v_i), f_{v_i}(x, e, w) \rangle \leq L_{\delta_i} \delta_i(v_i) + \tilde{H}_i^2(x, e, w) + J_i(x, e, w)$ with f_{v_i} coming from (11). ■

Assumption 2 states properties of the closed-loop system (1), (2), and it neither requires any knowledge on the network nor implies the stability of (9). Indeed, variable e is here understood as a generic perturbation affecting (y, u) . To verify whether Assumption 2 holds, we simply have to take the Lyapunov function V used to ensure the stability of (1) and (2) in the absence of network and study whether the required conditions are verified.

The function $\tilde{\alpha}_V$ in Assumption 2 will be taken in the following as $\tilde{\alpha}_V(x, e, w) = \alpha_V(|x|) + \alpha_W(|e|) - \sum_{i=1}^N \varrho_{V,i}(|w|)$ for some $\alpha_V, \alpha_W, \varrho_{V,i} \in \mathcal{K}_\infty$, $i \in \bar{N}$, when investigating input-to-state stability, and as $\tilde{\alpha}_V(x, e, w) = -\mu(\theta^p |w|^p - |z|^p)$ with $\mu > 0$, $\theta \geq 0$, when studying \mathcal{L}_p stability. Item (ii) of Assumption 2 means that either the origin of (1) and (2) is ISS with respect to input (e, w) or system (1), (2) is \mathcal{L}_p stable from w to z . These types of conditions are natural as we approach the problem by emulation, that is, the original closed-loop system needs to satisfy some robustness properties to cope with the errors induced by the network, as does any nonlinear controller, which is implemented in practice. But again, this does not mean that (9) satisfies desired stability properties because the e system is typically unstable. Similar assumptions as item (ii) of Assumption 2 are often made in the literature on NCSs; see, e.g., [4], [10], and [28], where examples of systems satisfying these conditions are provided. The functions δ_i in Assumption 2 will be used to define the local event triggering condition. Item (iii) is an exponential growth condition of δ_i on flows, where the function J_i is used to collect the redundant terms when we bound the norm of the derivative of $\delta_i(v_i)$ with $L_{\delta_i} \delta_i(v_i) + \tilde{H}_i^2(x, e, w)$.

We show in Section V how to satisfy Assumptions 1 and 2 for a class of globally Lipschitz systems. A nonlinear example, which is not globally Lipschitz and satisfies all the required conditions, is provided in Section VI.

Remark 1: Assumptions 1 and 2 may be verified by systems subject to model uncertainties. Indeed, these Lyapunov-like conditions do not necessarily require a precise model of the plant to be checked, as will be illustrated in Section VI. ■

Remark 2: Assumptions 1 and 2 impose conditions on the class of systems to which the results apply. It is possible to

relax these assumptions to only hold in a given compact set. In this case, the forthcoming results can be adapted to derive local stability properties, at the price of more technicalities, which we do not present in order not to blur the main message of this paper. ■

B. Local Triggering Generators

We exploit Assumptions 1 and 2 to design the triggering generators and T_i , $i \in \bar{N}$. We define Υ_i in (5) as, for $v_i, e_i \in \mathbb{R}^{n_{e_i}}$ and $\kappa_i \in \mathbb{Z}_{\geq 0}$,

$$\Upsilon_i(e_i, v_i, \kappa_i) = \gamma_i W_i^2(e_i, \kappa_i) - \lambda_i \bar{\rho}_i \delta_i(v_i) \quad (13)$$

where $\bar{\rho}_i := \max\{\rho_i, \frac{\gamma_i \lambda_i}{1 - \lambda_i L_{\delta_i}}\} \geq 0$ with ρ_i and W_i coming from Assumption 1, $L_{\delta_i} \in \mathbb{R}$, $\gamma_i > 0$, δ_i coming from Assumption 2, and $\lambda_i \geq 0$ is a design parameter. The triggering condition (13) is similar to those proposed in [3], [4], [10], and [11] for CETC in different contexts. Note that $\bar{\rho}_i$ in (13) depends on ρ_i and thus on the scheduling protocol. The mapping Υ_i only depends on the local variables e_i, v_i , and κ_i , and not the whole state q , which is essential for the envisioned setup and for the decentralized implementation of the triggering rule.

We select λ_i in (13) such that $\lambda_i < \lambda_i^*$, where λ_i^* is defined as

$$\lambda_i^* := \begin{cases} 1, & \text{when } L_{\delta_i} \leq -\gamma_i \\ \min\left\{1, \frac{1}{L_{\delta_i} + \gamma_i}\right\}, & \text{when } L_{\delta_i} > -\gamma_i. \end{cases} \quad (14)$$

Given $\lambda_i \in [0, \lambda_i^*)$, we select T_i defined after (9) such that $T_i < T_{\text{MASP},i}(\lambda_i)$, where $T_{\text{MASP},i}(\lambda_i)$ is the MASP of network \mathcal{N}_i and is defined as

$$T_{\text{MASP},i}(\lambda_i) := \begin{cases} \frac{1}{L_{W_i} r_i} \arctan(\vartheta_i), & \text{when } \gamma_i > L_{W_i} \\ \frac{1}{L_{W_i}} \frac{1 - \bar{\rho}_i}{1 + \bar{\rho}_i}, & \text{when } \gamma_i = L_{W_i} \\ \frac{1}{L_{W_i} r_i} \operatorname{arctanh}(\vartheta_i), & \text{when } \gamma_i < L_{W_i} \end{cases} \quad (15)$$

where $\bar{\rho}_i$ is defined above, $r_i := \sqrt{|(\frac{\gamma_i}{L_{W_i}})^2 - 1|}$, $\vartheta_i := \frac{r_i(1 - \bar{\rho}_i)}{2 \frac{\bar{\rho}_i}{1 + \bar{\rho}_i} (\frac{\gamma_i}{L_{W_i}} - 1) + 1 + \bar{\rho}_i}$, and $L_{W_i} \geq 0$ and $\gamma_i > 0$ come, respectively, from Assumptions 1 and 2.

The bound in (15) depends on the triggering parameter λ_i . More precisely, the bound is decreasing in λ_i . In other words, the larger the λ_i , the smaller $T_{\text{MASP},i}(\lambda_i)$ and vice versa.

Remark 3: The local event-triggering condition in (13) ensures that the protocol equation (6) is ISS with respect to v_i (see [29, Definition 5.3]). In particular, in view of the definition of Υ_i in (13) and item (ii) of Assumption 1, a transmission is triggered when $W_i(e_i, \kappa_i) \geq \sqrt{\lambda_i \bar{\rho}_i \delta_i(v_i) / \gamma_i}$, which ensures that $W_i(\chi_i(e_i, \kappa_i), \kappa_i + 1) \leq \rho_i W_i(e_i, \kappa_i)$, where $\rho_i \in [0, 1)$, for any $v_i, e_i \in \mathbb{R}^{n_{e_i}}$. Although the actual protocol equation (6), which is implemented, is ISS, the scheduling rule itself, which is modeled by χ_i and decides which nodes gets access to the network, is UGAS in view of items (i) and

(ii) of Assumption 1; see the definition of UGAS protocols in [20, Remark 7]. ■

Remark 4: When $\lambda_i = 0, i \in \bar{N}$, the triggering function Υ_i is always nonnegative. Consequently, transmissions occur at every sampling instant according to (5). We then recover the time-triggered results of [28], in particular the bound on the maximal allowable transmission interval is the same when a single network is used, and there are no disturbances, i.e., $w = 0$, as well as those in [27] when the network consists of a single node. ■

C. Input-to-State Stability

We are ready to state the next result about the input-to-state stability of system (9).

Theorem 1: Consider system (9) and suppose the following hold.

- 1) Assumption 1 holds with $\tilde{H}_i(x, e, w) = H_i(x, e) + \varrho_{W_i}(|w|)$ for some continuous functions $H_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \rightarrow \mathbb{R}_{\geq 0}$ and $\varrho_{W_i} \in \mathcal{K}_{\infty}, i \in \bar{N}$.
- 2) Assumption 2 holds with $\tilde{\alpha}_V(x, e, w) = \alpha_V(|x|) + \alpha_W(|e|) - \sum_{i=1}^N \varrho_{V,i}(|w|)$ for some $\alpha_V, \alpha_W, \varrho_{V,i} \in \mathcal{K}_{\infty}, i \in \bar{N}$.
- 3) For each $i \in \bar{N}$, let $\lambda_i \in [0, \lambda_i^*]$ and $T_i \in [\varepsilon_i, T_{\text{MASP},i}(\lambda_i)]$, where λ_i^* and $T_{\text{MASP},i}(\lambda_i)$ are defined in (14) and (15), respectively.

Then, the set $\mathcal{A} := \{q \in C \cup D : x = 0, e = 0, \kappa_i \in \mathbb{Z}_{\geq 0}, \tau_i \in [0, T_i], i \in \bar{N}\}$ is ISS for system (9). ■

Theorem 1, whose proof is given in Appendix A, shows that set \mathcal{A} is ISS for system (9). This implies that: 1) x and e globally converge to a neighborhood of the origin, whose “size” depends on the norm of the disturbance w ; and 2) the set \mathcal{A} is UGAS [22, Definition 3.6] when $w = 0$.

Remark 5: Theorem 1 relies on small-gain techniques. The general idea is that the x -system is assumed to satisfy an ISS property with respect to $(w, W_1(e_1, \kappa_1), \dots, W_N(e_N, \kappa_N))$ on flows according to items (i) and (ii) of Assumption 2, and remains constant at jumps. On the other hand, Assumption 1 leads to an ISS property of the e -system with respect to (x, w) as well, as shown in [20, Proposition 6], thanks to the definition of the event generators. Then, by carefully selecting the triggering conditions and T_i , the small-gain condition applies, and the desired result is obtained. While the connection with small-gain techniques is easier to see in the case where the controller is a state-feedback law and there is only one network as in [24], the fact that output-feedback control is addressed and the decentralized scenario we investigate prevent us to directly apply existing hybrid small-gain results. That is the reason why we propose a completely novel hybrid Lyapunov construction in the proof of Theorem 1. ■

Tailored results can be derived from Theorem 1 either under stronger conditions or for more specific implementation setups. Thus, an exponentially ISS property is obtained by strengthening the conditions of Theorem 1 as stated next, whose proof follows directly from the proof of Theorem 1 and is, therefore, omitted.

Corollary 1: Consider system (9). Suppose that items 1) and 2) of Theorem 1 are satisfied, and there exist $\underline{a}_{W_i}, \bar{a}_{W_i}, \underline{a}_V, \bar{a}_V, a_V, a_W > 0, i \in \bar{N}$, such that Assumptions 1 and 2, respectively, hold with $\underline{\alpha}_{W_i}(s) = \underline{a}_{W_i}s, \bar{\alpha}_{W_i}(s) = \bar{a}_{W_i}s, \underline{\alpha}_V(s) = \underline{a}_V s^2, \bar{\alpha}_V(s) = \bar{a}_V s^2, \alpha_V(s) = a_V s^2,$ and $\alpha_W(s) = a_W s^2$ for $s \geq 0$. Then, the set \mathcal{A} defined in Theorem 1 is exponentially ISS with a linear gain. ■

When a single network is used and the state of plant x_p is available for control, i.e., $y = x_p$ in (1), we can relax Assumption 2 and modify the triggering condition. Since we consider only one network here, only one triggering generator is needed. We, therefore, use the notation Υ to define the triggering condition, and χ to denote the scheduling rule. As we need to specify the expressions of \tilde{H}_i and J_i in Assumptions 1 and 2 for this special case, we rewrite those conditions here as follows.

Assumption 3: There exist locally Lipschitz functions $V : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$ and $W : \mathbb{R}^{n_e} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with V and W positive definite, $\varrho_W \in \mathcal{K}_{\infty}$, and continuous functions $\tilde{\alpha}_V : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_w} \rightarrow \mathbb{R}_{\geq 0}, \rho \in [0, 1), a_V, \gamma > 0,$ and $L_W, L_V \geq 0$ such that the following hold.

- i) For any $(e, x, \kappa) \in \mathbb{R}^{n_e} \times \mathbb{R}^{n_x} \times \mathbb{Z}_{\geq 0}, W(\chi(e, \kappa), \kappa + 1) \leq \rho W(e, \kappa)$.
- ii) For almost all $e \in \mathbb{R}^{n_e}$, all $\kappa \in \mathbb{Z}_{\geq 0}$ and $(x, w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_w}, \langle \frac{\partial W(e, \kappa)}{\partial e}, g(x, e, w) \rangle \leq L_W W(e, \kappa) + L_V \sqrt{V(x)} + \varrho_W(|w|)$.
- iii) For almost all $x \in \mathbb{R}^{n_x}$ and all $(e, w) \in \mathbb{R}^{n_e} \times \mathbb{R}^{n_w}, \langle \nabla V(x), f(x, e, w) \rangle \leq -a_V V(x) - \tilde{\alpha}_V(x, e, w) + \gamma^2 W^2(e, \kappa)$. ■

We define the single triggering condition Υ , as, for $(x, e) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_e}$ and $\kappa \in \mathbb{Z}_{\geq 0}$,

$$\Upsilon(e, x, \kappa) = \gamma W^2(e, \kappa) - \lambda \bar{\rho} V(x) \quad (16)$$

where $\bar{\rho} := \max\{\rho, \frac{\gamma \lambda}{a_V}\}$. We select λ such that $\lambda < \lambda^*$ with

$$\lambda^* := \min\left\{1, \frac{a_V}{\gamma}\right\}. \quad (17)$$

For each $\lambda \in [0, \lambda^*)$, the MASP $T_{\text{MASP}}(\lambda)$ is defined as

$$T_{\text{MASP}}(\lambda) := \begin{cases} \frac{1}{L_W r} \arctan(\vartheta), & \text{when } \gamma L_V > L_W \\ \frac{1}{L_W} \frac{1 - \bar{\rho}}{1 + \bar{\rho}}, & \text{when } \gamma L_V = L_W \\ \frac{1}{L_W r} \operatorname{arctanh}(\vartheta), & \text{when } \gamma L_V < L_W \end{cases} \quad (18)$$

where $\bar{\rho}$ is defined below (16), $r := \sqrt{|(\frac{\gamma L_V}{L_W \sqrt{a_V}})^2 - 1|}, \vartheta := \frac{r(1 - \bar{\rho})}{2 \frac{\bar{\rho}}{1 + \bar{\rho}} (\frac{\gamma}{L_W} \frac{L_V^2 + a_V}{2 a_V} - 1) + 1 + \bar{\rho}}, L_W, L_V \geq 0,$ and $a_V, \gamma > 0$ come from Assumption 3. The MASP in (18) is different to (15), since extra parameters $L_V \geq 0$ and $a_V > 0$ are introduced in Assumption 3, and it is consistent with (15) when $L_V^2/a_V = 1$.

We can state the next theorem.

Theorem 2: Consider system (9). Suppose that Assumption 3 holds with $\tilde{\alpha}_V(x, e, w) = \alpha_W(|e|) - \varrho_V(|w|)$ for some $\alpha_W, \varrho_V \in \mathcal{K}_{\infty}$ from Assumption 3. Let $\lambda \in [0, \lambda^*)$ and $T <$

$T_{\text{MASP}}(\lambda)$, where λ^* and $T_{\text{MASP}}(\lambda)$ are defined in (17) and (18), respectively. Then, set \mathcal{A} defined in Theorem 1 is ISS. ■

Theorem 2 extends the main result of [23] to scheduling and allows considering exogenous disturbances acting on the plant and [24] to scheduling. Note that the conditions in [24], which are parallel to Assumption 3, are slightly different, where $\rho = 0$ and \tilde{H}_i, J_i are not specified as in Assumption 3.

D. \mathcal{L}_p Stability

We now consider the input-to-output stability of system (9) from w to the output $z = \eta(x, w)$.

Theorem 3: Consider system (9) and suppose the following hold.

- 1) Assumptions 1 and 2 are verified with $\tilde{\alpha}_V(x, e, w) = -\mu(\theta^p |w|^p - |\eta(x, w)|^p)$ for some $\mu, \theta > 0$.
- 2) For each $i \in \bar{N}$, let $\lambda_i \in [0, \lambda_i^*)$ and $T_i < T_{\text{MASP},i}(\lambda_i)$, where λ_i^* and $T_{\text{MASP},i}(\lambda_i)$ are defined in (14) and (15), respectively.

Then, system (9) is \mathcal{L}_p stable from w and to z with respect to the set \mathcal{A} defined in Theorem 1 with gain less than or equal to θ . ■

Like in Section IV-C, we derive a tailored result for the case, where a single network is used and $y = x_p$. Its proof follows from the ones to Theorems 2 and 3 and is, therefore, again omitted.

Theorem 4: Consider system (9). Suppose that Assumption 3 holds with $\tilde{\alpha}_V(x, e, w) = -\mu(\theta^p |w|^p - |\eta(x, w)|^p)$ for some $\mu, \theta > 0$. Let $\lambda \in [0, \lambda^*)$ and $T < T_{\text{MASP}}(\lambda)$, where λ^* and $T_{\text{MASP}}(\lambda)$ are defined in (17) and (18), respectively. Then, system (9) is \mathcal{L}_p stable from w and to z with respect to the set \mathcal{A} defined in Theorem 1 with gain less than or equal to θ . ■

V. CASE STUDY

A. Setup and Hybrid Model

We consider nonlinear systems of the form

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u + D_p \psi(x_p) + E_p w \\ y &= C_p x_p \end{aligned} \quad (19)$$

where $x_p \in \mathbb{R}^{n_p}$ is the state; $u \in \mathbb{R}^{n_u}$ is the control input; $w \in \mathbb{R}^{n_w}$ is the external disturbance; $y \in \mathbb{R}^{n_y}$ is the measured output; A_p, B_p, C_p, D_p , and E_p are matrices of appropriate dimensions; (A_p, B_p) and (A_p, C_p) are assumed to be stabilizable and detectable, respectively; and $\psi: \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_r}$ satisfies $\psi(0) = 0$ and $|\psi(x_1) - \psi(x_2)| \leq L|x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}^{n_p}$ with constant $L > 0$, where $n_r \in \mathbb{Z}_{>0}$.

We focus on observer-based controllers of the form

$$\begin{aligned} \dot{x}_c &= A_p x_c + B_p u + D_p \psi(x_c) - M(C_p x_c - y) \\ u &= K x_c \end{aligned} \quad (20)$$

where $x_c \in \mathbb{R}^{n_p}$ is the state estimate, and M and K are matrices of appropriate dimensions such that $\mathcal{A}_1 := \begin{bmatrix} A_p & B_p K \\ M C_p & A_p + B_p K - M C_p \end{bmatrix}$ is Hurwitz, which is always possible, since (A_p, B_p) and (A_p, C_p) are stabilizable and detectable, respectively.

We consider the scenario where the plant and the controller communicate via N independently operating networks, as described in Section III. Zero-order-hold devices are used so that $\hat{f}_v = 0$ as defined after (4). Each network is scheduled by an arbitrary *uniformly globally exponentially stable* (UGES) protocol, whenever the local triggering rule is satisfied. Hence, items (i) and (ii) of Assumption 1 hold with $W_i: \mathbb{R}^{n_{e_i}} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, $\rho_i \in [0, 1)$, $\bar{\alpha}_{W_i}(s) = \bar{a}_{W_i} s$, $\underline{\alpha}_{W_i}(s) = \underline{a}_{W_i} s$, $i \in \bar{N}$, for some $\bar{a}_{W_i} \geq \underline{a}_{W_i} > 0$ and all $s \geq 0$, which depend on the considered protocol. We further assume that there exists $\varpi_i \geq 0$ such that $|\frac{\partial W_i(e_i, \kappa_i)}{\partial e_i}| \leq \varpi_i$ for almost all $e_i \in \mathbb{R}^{n_{e_i}}$ and $\kappa_i \in \mathbb{Z}_{\geq 0}$, which is the case for the sampled-data case and RR and TOD protocols according to [20, Section VI].

Let $x = (x_p, x_c) \in \mathbb{R}^{n_x}$, $v = (y, u) \in \mathbb{R}^{n_e}$, $e = (\hat{v} - v)$, $n_x = n_p + n_c$, and $n_e = n_y + n_u$. We write $v := \bar{C}x$, $\bar{C}: \begin{bmatrix} C_p & 0 \\ 0 & K \end{bmatrix} \in \mathbb{R}^{n_e \times n_x}$, and

$$v_i := \bar{C}_i x \in \mathbb{R}^{n_{e_i}}, \quad i \in \bar{N} \quad (21)$$

be the i th element of v associated with network \mathcal{N}_i with $\bar{C}_i \in \mathbb{R}^{n_{e_i} \times n_x}$. Define $e_y := \hat{y} - y = C_y e$, $e_u := \hat{u} - u = C_u e$, with appropriate matrices $C_y \in \mathbb{R}^{n_y \times n_e}$ and $C_u \in \mathbb{R}^{n_u \times n_e}$.

In this case, the hybrid model (9) is given by

$$\dot{q} = \begin{pmatrix} \mathcal{A}_1 x + \mathcal{B}_1 e + \mathcal{D}_1 \bar{\psi}(x) + \mathcal{E}_1 w \\ \mathcal{A}_2 x + \mathcal{B}_2 e + \mathcal{D}_2 \bar{\psi}(x) + \mathcal{E}_2 w \\ \mathbf{0}_N \\ \mathbf{1}_N \end{pmatrix}, \quad q \in C \quad (22)$$

$$q^+ \in G(q), \quad q \in D$$

where

$$q = (x, e, \kappa, \tau), \mathcal{B}_1 := \begin{bmatrix} B_p C_u \\ M C_y \end{bmatrix}, \mathcal{D}_1 := \begin{bmatrix} D_p & 0 \\ 0 & D_p \end{bmatrix},$$

$$\mathcal{E}_1 := \begin{bmatrix} E_p \\ 0 \end{bmatrix}, \bar{\psi}(x) := (\psi(x_p), \psi(x_c)), \mathcal{A}_2 := - \begin{bmatrix} \bar{C}_1 \mathcal{A}_1 \\ \vdots \\ \bar{C}_N \mathcal{A}_1 \end{bmatrix},$$

$$\mathcal{B}_2 := - \begin{bmatrix} \bar{C}_1 \mathcal{B}_1 \\ \vdots \\ \bar{C}_N \mathcal{B}_1 \end{bmatrix}, \mathcal{D}_2 := - \begin{bmatrix} \bar{C}_1 \mathcal{D}_1 \\ \vdots \\ \bar{C}_N \mathcal{D}_1 \end{bmatrix},$$

and

$$\mathcal{E}_2 := - \begin{bmatrix} \bar{C}_1 \mathcal{E}_1 \\ \vdots \\ \bar{C}_N \mathcal{E}_1 \end{bmatrix}.$$

The flow and jump sets C and D are defined after (9), and the jump map G is given in (12).

B. Input-to-State Stability

Before stating the main results of this section, we need to introduce some notation. For any $\pi = (\pi_1, \dots, \pi_N) \in \mathbb{R}_{\geq 0}^N$, we

define

$$\Psi(\pi) := \text{diag}\{\pi_1 I_{n_{e_1}}, \dots, \pi_N I_{n_{e_N}}\} \in \mathbb{R}^{n_e \times n_e}. \quad (23)$$

Let

$$\begin{aligned} \bar{\Psi}_i &:= \text{diag}\{\mathbf{0}_{n_{e_1} \times n_{e_1}}, \dots, \mathbf{0}_{n_{e_{i-1}} \times n_{e_{i-1}}}, \mathbf{I}_{n_{e_i} \times n_{e_i}}, \\ &\quad \mathbf{0}_{n_{e_{i+1}} \times n_{e_{i+1}}}, \dots, \mathbf{0}_{n_{e_N} \times n_{e_N}}\} \in \mathbb{R}^{n_e \times n_e}, \\ \bar{\mathcal{B}}_{2,i} &:= \bar{C}_i \mathcal{B}_1(I_{n_e} - \bar{\Psi}_i) \in \mathbb{R}^{n_{e_i} \times n_e}, i \in \bar{N}, \end{aligned}$$

and

$$\bar{\mathcal{B}}_2 := \begin{bmatrix} \bar{\mathcal{B}}_{2,1} \\ \vdots \\ \bar{\mathcal{B}}_{2,N} \end{bmatrix} \in \mathbb{R}^{n_e \times n_e},$$

where \bar{C}_i comes from (21).

The next proposition states that all conditions of Corollary 1 hold if the LMI in (24) is satisfied.

Proposition 1: If there exist a positive-definite symmetric matrix P , $a_V, a_W, \tilde{\theta}, \underline{a}_{W_i}, \epsilon_i > 0$, $\varpi_i \geq \underline{a}_{W_i}$ and $\nu_i \geq a_W + \varpi_i^2 |\bar{\mathcal{B}}_{2,i}|^2$, $i \in \bar{N}$, such that the following LMI holds:

$$\begin{bmatrix} \Sigma_{11} & \star & \star \\ \Sigma_{21} & \Sigma_{22} & \star \\ \Sigma_{31} & \mathcal{E}_2^T \Psi^T(\varpi) \Psi(\varpi) \bar{\mathcal{B}}_2 & \Sigma_{33} \end{bmatrix} < 0 \quad (24)$$

where Ψ is defined in (23), $\varpi := (\varpi_1, \dots, \varpi_N)$, $\epsilon := (\epsilon_1, \dots, \epsilon_N)$, Σ_{11} , Σ_{21} , Σ_{22} , Σ_{31} and Σ_{33} are given in Table I. Then, Assumptions 1 and 2 hold with the data given in Table I. ■

An immediate consequence of Proposition 1 is that the set $\mathcal{A} = \{q \in C \cup D : x=0, e=0, \kappa_i \in \mathbb{Z}_{\geq 0}, \tau_i \in [0, T_i], i \in \bar{N}\}$ is exponentially ISS with a linear gain for system (22) according to Corollary 1 by suitably defining λ_i and T_i , as Proposition 1 ensures the satisfaction of all the conditions of Corollary 1.

C. \mathcal{L}_2 Stability

We now consider \mathcal{L}_2 stability for system (22) using Theorem 3, with respect to the performance output $z := C_z x + D_z w$.

We can follow the proof of Proposition 1, provided in Appendix A, to show that the conditions of Theorem 3 hold.

Proposition 2: If there exist a positive-definite symmetric matrix P , $\mu, \theta, \epsilon_i, \underline{a}_{W_i} > 0$, $\varpi_i \geq \underline{a}_{W_i}$, and $\nu_i \geq \varpi_i^2 |\bar{\mathcal{B}}_{2,i}|^2$, $i \in \bar{N}$, such that (24) holds. Then, Assumptions 1 and 2 hold with the data given in Table I. ■

Based on Proposition 2, system (22) is \mathcal{L}_2 stable from w to z with respect to the set \mathcal{A} with gain less than or equal to θ according to Theorem 3, when $\lambda_i < \lambda_i^*$ and $T_i < T_{\text{MASP},i}(\lambda_i)$ with λ_i^* in (14) and $T_{\text{MASP},i}(\lambda_i)$ in (15).

D. Special Cases

When ψ in (19) only depends on the output y , not the state x_p , and $D_p = B_p$, condition (24) slightly differs and can be shown to always hold as formalized next.

Lemma 1: Consider system (19) with $\psi(y)$ instead of $\psi(x_p)$, and $D_p = B_p$. Let $\mathcal{D}_1 := \begin{bmatrix} B_p & 0 \\ 0 & 0 \end{bmatrix}$ and replace $\bar{\psi}(x)$ in (22) by $\bar{\psi}(y, e) := \begin{bmatrix} \psi(y) - \psi(y + e_y) \\ 0 \end{bmatrix}$. Then, there exist a positive-definite symmetric matrix P , $a_V, a_W, \tilde{\theta}, \underline{a}_{W_i}, \epsilon_i > 0$, $\varpi_i \geq \underline{a}_{W_i}$, and $\nu_i \geq a_W + L^2 |\mathcal{D}_1|^2 |P|^2 / a_V + \varpi_i^2 |\bar{\mathcal{B}}_{2,i}|^2 + \epsilon_i^2 L^2 |\bar{C}_i \mathcal{D}_1|^2 + 2 \varpi_i^2 (L^2 |\bar{C}_i \mathcal{D}_1|^2 + L |\bar{C}_i \mathcal{D}_1| |\bar{\mathcal{B}}_{2,i}|)$, $i \in \bar{N}$, such that (24) holds with Σ_{21} , Σ_{31} , and Σ_{33} from Table I and $\Sigma_{11} := \mathcal{A}_1^T P + P \mathcal{A}_1 + \mathcal{A}_2^T (\Psi^T(\sqrt{2}\varpi) \Psi(\sqrt{2}\varpi) + \Psi^T(\epsilon) \Psi(\epsilon)) \mathcal{A}_2 + 2a_V I_{n_x} + \bar{C}^T \Psi^T(\epsilon) \Psi(\epsilon) \bar{C}$, $\Sigma_{22} := -\Psi(\nu) + (a_W + L^2 |\mathcal{D}_1|^2 |P|^2 / a_V) I_{n_e} + \bar{\mathcal{B}}_2^T \Psi^T(\varpi) \Psi(\varpi) \bar{\mathcal{B}}_2 + \Psi^T(\epsilon L |\mathcal{D}_2|) \Psi(\epsilon L |\mathcal{D}_2|) + \Psi^T(k\varpi) \Psi^T(k\varpi)$ with $k^2 = 2L^2 |\mathcal{D}_2|^2 + 2L |\mathcal{D}_2| |\bar{\mathcal{B}}_2|$. As a result, Assumptions 1 and 2 hold with the data from Table I except that $H_i(x, e) := \varpi_i (|\bar{C}_i \mathcal{A}_1 x + \bar{\mathcal{B}}_{2,i} e| + L |\bar{C}_i \mathcal{D}_2| |e|)$, $\gamma_i^2 = (\nu_i - (a_W + L^2 |\mathcal{D}_1|^2 |P|^2 / a_V + \varpi_i^2 |\bar{\mathcal{B}}_{2,i}|^2 + \epsilon_i^2 L^2 |\bar{C}_i \mathcal{D}_1|^2 + 2 \varpi_i^2 (L^2 |\bar{C}_i \mathcal{D}_1|^2 + L |\bar{C}_i \mathcal{D}_1| |\bar{\mathcal{B}}_{2,i}|)) / \underline{a}_{W_i}^2)$, $J_i(x, e, w) := \epsilon_i^2 ((|\mathcal{A}_1^T \bar{C}_i^T \bar{C}_i \mathcal{A}_1| + |\bar{C}_i|^2) |x|^2 + L^2 |\bar{C}_i \mathcal{D}_1|^2 |e|^2) + 2\epsilon_i^2 |x^T \bar{C}_i^T \bar{C}_i (\mathcal{B}_1 e + \mathcal{E}_1 w)|$. ■

It is important to note that Lemma 1 covers linear time-invariant systems as in this case $\psi(y) = 0$. In other words, the proposed approach can always be applied to stabilizable and detectable linear time-invariant systems.

VI. ILLUSTRATIVE EXAMPLE

In this section, we provide an example of a nonlinear system, which is not globally Lipschitz contrary to the systems addressed in Section V, to which our results apply. The control system consists of two coupled plants P_1 and P_2 , whose origin is unstable, as in [4, Section VII.B]. The plants P_i , $i \in \{1, 2\}$, are modeled as

$$P_1 : \begin{cases} \dot{x}_1 = d_1 x_1^2 - x_1^3 + x_2 + u_1 + w \\ y_1 = x_1 \end{cases} \quad (25)$$

$$P_2 : \begin{cases} \dot{x}_2 = d_2 x_2^2 - x_2^3 + x_1 + u_2 + w \\ y_2 = x_2 \end{cases} \quad (26)$$

where $x_i \in \mathbb{R}$, $i \in \{1, 2\}$, is the state of subsystem P_i , $y_i = x_i$ is its output, $d_1, d_2 \in \mathbb{R}$ are unknown uncertain parameters (potentially time-varying) verifying $|d_1| \leq 1$ and $|d_2| \leq 1$, and $w \in \mathbb{R}$ is the exogenous disturbance. For each subsystem, its own controller is collocated with the actuator and is given by $u_i = -2y_i$.

We consider the case where the output measurements of y_1 and y_2 are, respectively, transmitted via two independently operating networks, \mathcal{N}_1 and \mathcal{N}_2 , and received by the controller as \hat{y}_1 and \hat{y}_2 , as illustrated in Fig. 2. Zero-order-hold devices are used to implement the controller, and this gives $\hat{f}_v = 0$, as defined after (4). Let $e_1 = \hat{y}_1 - y_1$ and $e_2 = \hat{y}_2 - y_2$ be the networked-induced errors (there is no need to introduce $\hat{u} - u$ since the controller is static), $x = (x_1, x_2)$, and $e = (e_1, e_2)$. Let $\tau = (\tau_1, \tau_2)$ with $\tau_1, \tau_2 \in \mathbb{R}_{\geq 0}$. Note that $\kappa \in \mathbb{Z}_{\geq 0}$ in (9) is irrelevant here, since both networks have only one node. We obtain system (9) with $q = (x, e, \tau)$, $F(q, w) = (f(e, x, w), g(e, x, w), 1, 1)$, $f(x, e, w) = (d_1 x_1^2 - x_1^3 + x_2 - 2(x_1 + e_1) + w, d_2 x_2^2 - x_2^3$

TABLE I
EXPRESSION OF FUNCTIONS AND PARAMETERS IN SECTION V

	Proposition 1/Lemma 1	Proposition 2
$\Sigma_{11} :=$	$\mathcal{A}_1^T P + P \mathcal{A}_1 + a_V I_{n_x} + \mathcal{D}_2^T \Psi^T(\sqrt{2\varpi}L)\Psi(\sqrt{2\varpi}L)\mathcal{D}_2$ $+ 2L \mathcal{D}_1 P + \mathcal{A}_2^T(\Psi^T(\varpi)\Psi(\varpi) + \Psi^T(\epsilon)\Psi(\epsilon))\mathcal{A}_2$ $+ \mathcal{D}_2^T \Psi^T(\sqrt{2L\varpi})\Psi(\sqrt{2L\varpi})\mathcal{A}_2$ $+ \overline{C}^T \Psi^T(\epsilon\sqrt{2L \mathcal{D}_1 +1})\Psi(\epsilon\sqrt{2L \mathcal{D}_1 +1})\overline{C}$	$\mathcal{A}_1^T P + P \mathcal{A}_1 + \mathcal{D}_2^T \Psi^T(\varpi L)\Psi(\varpi L)\mathcal{D}_2$ $+ 2L \mathcal{D}_1 P + \mathcal{A}_2^T(\Psi^T(\varpi)\Psi(\varpi) + \Psi^T(\epsilon)\Psi(\epsilon))\mathcal{A}_2^T$ $+ \mathcal{D}_2^T \Psi^T(\sqrt{2L\varpi})\Psi(\sqrt{2L\varpi})\mathcal{A}_2$ $+ \overline{C}^T \Psi^T(\epsilon\sqrt{2L \mathcal{D}_1 +1})\Psi(\epsilon\sqrt{2L \mathcal{D}_1 +1})\overline{C} + \mu C_z^T C_z$
$\Sigma_{21} :=$	$\mathcal{B}_1^T P + \overline{\mathcal{B}}_2^T \Psi^T(\varpi)\Psi(\varpi)\mathcal{A}_2 + \mathcal{B}_2^T \Psi^T(\epsilon)\Psi(\epsilon)\overline{C}$	$\mathcal{B}_1^T P + \overline{\mathcal{B}}_2^T \Psi^T(\varpi)\Psi(\varpi)\mathcal{A}_2 + \mathcal{B}_2^T \Psi^T(\epsilon)\Psi(\epsilon)\overline{C}$
$\Sigma_{22} :=$	$-(\Psi(\nu) - a_W I_{n_e}) + \overline{\mathcal{B}}_2^T \Psi^T(\varpi)\Psi(\varpi)\overline{\mathcal{B}}_2$	$-\Psi(\nu) + \overline{\mathcal{B}}_2^T \Psi^T(\varpi)\Psi(\varpi)\overline{\mathcal{B}}_2$
$\Sigma_{31} :=$	$\mathcal{E}_1^T P + \mathcal{E}_2^T \Psi^T(\varpi)\Psi(\varpi)\mathcal{A}_2 + \mathcal{E}_2^T \Psi^T(\epsilon)\Psi(\epsilon)\overline{C}$	$\mathcal{E}_1^T P + \mathcal{E}_2^T \Psi^T(\varpi)\Psi(\varpi)\mathcal{A}_2 + \mathcal{E}_2^T \Psi^T(\epsilon)\Psi(\epsilon)\overline{C} + \mu D_z^T C_z$
$\Sigma_{33} :=$	$-\theta I_{n_w} + \mathcal{E}_2^T \Psi^T(\varpi)\Psi(\varpi)\mathcal{E}_2^T$	$-\mu\theta^2 I_{n_w} + \mu D_z^T D_z + \mathcal{E}_2^T \Psi^T(\varpi)\Psi(\varpi)\mathcal{E}_2$
$\underline{\alpha}_{W_i}(s) :=$	$\underline{a}_{W_i} s$	$\underline{a}_{W_i} s$
$\overline{\alpha}_{W_i}(s) :=$	$\overline{a}_{W_i} s$	$\overline{a}_{W_i} s$
$L_{W_i} :=$	$\varpi_i / \underline{a}_{W_i} \overline{C}_i \mathcal{B}_1 \overline{\Psi}_i $	$\varpi_i / \underline{a}_{W_i} \overline{C}_i \mathcal{B}_1 \overline{\Psi}_i $
$H_i(x, e) :=$	$\varpi_i (\overline{C}_i \mathcal{A}_1 x + \overline{\mathcal{B}}_{2,i} e + L \overline{C}_i \mathcal{D}_1 x)$	$\varpi_i (\overline{C}_i \mathcal{A}_1 x + \overline{\mathcal{B}}_{2,i} e + L \overline{C}_i \mathcal{D}_1 x)$
$\varrho_{W_i}(w) :=$	$\varpi_i \overline{C}_i \mathcal{E}_1 w $	$\varpi_i \overline{C}_i \mathcal{E}_1 w $
$V(x) :=$	$x^T P x$	$x^T P x$
$\delta_i(v_i) :=$	$\epsilon_i^2 v_i ^2$	$\delta_i(v_i) = \epsilon_i^2 v_i ^2$
$\underline{\alpha}_V(s) :=$	$\lambda_{\min}(P) s^2$	$\lambda_{\min}(P) s^2$
$\overline{\alpha}_V(s) :=$	$\lambda_{\max}(P) s^2$	$\lambda_{\max}(P) s^2$
$\tilde{\alpha}_V(x, e, w) :=$	$a_V x ^2 + a_W e ^2 - \sum_{i=1}^N \frac{1}{N} \tilde{\theta} w ^2$	$\mu (C_z x + D_z w ^2 - \theta^2 w ^2)$
$\gamma_i :=$	$\sqrt{\frac{\nu_i - a_W - \varpi_i^2 \overline{\mathcal{B}}_{2,i} ^2}{\underline{a}_{W_i}^2}}$	$\sqrt{\frac{\nu_i - \varpi_i^2 \overline{\mathcal{B}}_{2,i} ^2}{\underline{a}_{W_i}^2}}$
$J_i(x, e, w) :=$	$\epsilon_i^2 \left(\mathcal{A}_1^T \overline{C}_i^T \overline{C}_i \mathcal{A}_1 + 2L \overline{C}_i^T \overline{C}_i \mathcal{D}_1 \right) x ^2$ $+ 2\epsilon_i^2 x^T \overline{C}_i^T \overline{C}_i (\mathcal{B}_1 e + \mathcal{E}_1 w) $	$\epsilon_i^2 \left(\mathcal{A}_1^T \overline{C}_i^T \overline{C}_i \mathcal{A}_1 + 2L \overline{C}_i^T \overline{C}_i \mathcal{D}_1 \right) x ^2$ $+ 2\epsilon_i^2 x^T \overline{C}_i^T \overline{C}_i (\mathcal{B}_1 e + \mathcal{E}_1 w) $
$L_{\delta_i} :=$	1	1

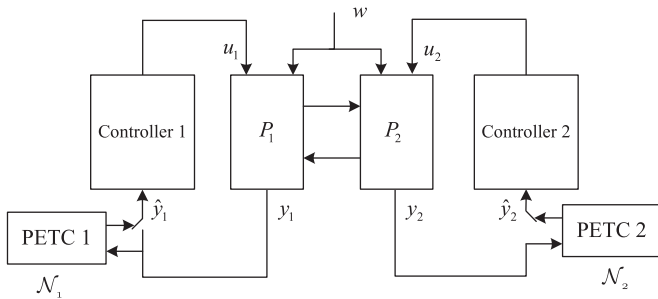


Fig. 2. Control setup of the two coupled systems.

$+ x_1 - 2(x_2 + e_2) + w$, $g(x, e, w) = -f(x, e, w)$, C, D defined as after (9), and $G(q)$ in (12), with $h_i(e_i, x)$ being defined as $h_i(e_i, x) = (1 - \Gamma_i(e_i, x))e_i$, $i \in \{1, 2\}$.

We now verify Assumptions 1 and 2. Let $i \in \{1, 2\}$, we take $W_i(e_i) = |e_i|$. Assumption 1 holds with $\underline{\alpha}_{W_i}(s) = \overline{\alpha}_{W_i}(s) = s$ for all $s \geq 0$, $\rho_i = 0$ (since the networks consist of one node), $L_{W_i} = 2$, and $\tilde{H}_i(x, e, w) = |-d_i x_i^2 + x_i^3 - x_{3-i} + 2x_i| + |w|$, $i \in \{1, 2\}$.

To verify Assumption 2, we take $\delta_i(v_i) = 0.5v_i^2$ and consider the candidate Lyapunov function $V(x) = a^2 \sum_{i=1}^2 (b \frac{x_i^2}{2} + c \frac{x_i^4}{4})$ for any $x_i \in \mathbb{R}$, some $a, b, c > 0$ and $i \in \{1, 2\}$. Since $\langle \nabla \delta_i(v_i), f_{v_i}(x, e, w) \rangle \leq -2x_i^2 + |d_i x_i^3 + x_i x_{3-i} - 2x_i e_i + x_i w| \leq -4\delta_i(v_i) + |d_i x_i^3 - x_i^4 + x_i x_{3-i}| + |2x_i^2| + |e_i|^2 + |w|^2 \leq L_{\delta_i} \delta_i(v_i) + J_i(x, e, w)$, item (iii) of Assumption 2 holds with $L_{\delta_i} = -4$, $J_i(x, e, w) = |x_i^3 + x_i x_{3-i}| + |2d_i x_i^2| + |e_i^2| + |w|^2$, $i \in \{1, 2\}$. We now verify item (ii) of

Assumption 2. Since $2s_1 s_2 \leq s_1^2 + s_2^2$ for any $s_1, s_2 \in \mathbb{R}$, we have that

$$\langle \nabla V(x), f(x, e, w) \rangle \leq a^2 \left(\sum_{i=1}^2 [(b^2 + c^2)e_i^2 + 0.5(b^2 + c^2)w^2 + (-c + 1.5)x_i^6 + cd_i x_i^5 + (-b - 2c)x_i^4 + bd_i x_i^3 + (-2b + 1.5)x_i^2] + 2bx_1 x_2 + cx_1^3 x_2 + cx_1 x_2^3 \right).$$

Noting that $(s_1 + s_2)^2 \leq 2(s_1^2 + s_2^2)$ for any $s_1, s_2 \in \mathbb{R}$, $\tilde{H}_i^2(x, e, w) \leq 2(-d_i x_i^2 + x_i^3 - x_{3-i} + 2x_i)^2 + 2w^2 \leq 4(-d_i x_i^2 + x_i^3 + 2x_i)^2 + 4x_{3-i}^2 + 2w^2 \leq 4(x_i^6 - 2d_i x_i^5 + 5x_i^4 - 4d_i x_i^3 + 4x_i^2) + 4x_{3-i}^2 + 2w^2$ and $J_i(x, e, w) \leq 0.5x_i^2 + 0.5(x_i^2 + x_{3-i})^2 + 2d_i x_i^2 + e_i^2 + w^2 \leq x_i^4 + x_{3-i}^2 + (2d_i + 0.5)x_i^2 + e_i^2 + w^2$, $i \in \{1, 2\}$. We then subtract and add $\tilde{\alpha}(x, e, w) = \nu x^2 + \nu e^2$, $\delta_i(v_i) = 0.5x_i^2$, $\tilde{H}_i^2(x, e, w)$, and $J_i(x, e, w)$ to get

$$\langle \nabla V(x), f(x, e, w) \rangle \leq -\alpha_V(x) - \alpha_W(e) + a^2 p(x) + \sum_{i=1}^2 [\varrho_{V_i}^2(|w|) - \delta_i(v_i) - \tilde{H}_i^2(x, e, w)] + \sum_{i=1}^2 [-J_i(x, e, w) + a^2(b^2 + c^2 + a^{-2}(1 + \nu))e_i^2]$$

where $p(x) := \sum_{i=1}^2 x_i^2 [(-2b + 1.5 + a^{-2}(21 + 2d_i + \nu)) + d_i(b - 16a^{-2})x_i + (-b - 2c + 21a^{-2})x_i^2 + d_i(c - 8a^{-2})x_i^3]$

TABLE II
AVERAGE INTERTRANSMISSION TIME FOR \mathcal{N}_1 AND \mathcal{N}_2 NETWORKS

	Average inter-transmission time					
	$\lambda_1 = 0.06$ $\lambda_2 = 0.02$		$\lambda_1 = 0.1$ $\lambda_2 = 0.09$		$\lambda_1 = 0.15$ $\lambda_2 = 0.12$	
	\mathcal{N}_1	\mathcal{N}_2	\mathcal{N}_1	\mathcal{N}_2	\mathcal{N}_1	\mathcal{N}_2
$T_1 = 0.01$ $T_2 = 0.02$	0.068	0.0403	0.0793	0.0781	0.0952	0.08
$T_1 = 0.04$ $T_2 = 0.05$	0.0971	0.0556	0.101	0.098	×	×
$T_1 = 0.08$ $T_2 = 0.1$	0.1103	0.101	×	×	×	×

$+(-c + 1.5 + 4a^{-2})x_i^4] + 2bx_1x_2 + cx_1^3x_2 + cx_1x_2^3$, $\varrho_{V_i}(s) := (0.5(b^2 + c^2) + 3a^{-2})s^2$ for all $s \geq 0$. This implies that item (ii) of Assumption 2 holds with $\gamma_1 = \gamma_2 = a\sqrt{b^2 + c^2 + a^{-2}(1 + \nu)}$ provided parameters a, b, c, ν are such that $p(x) \leq 0$ for all $x \in \mathbb{R}^2$. We take $(a, b, c, \nu) = (1.7, 3.93, 2.9, 0.01)$ to ensure $p(x) \leq 0$ for all $x \in \mathbb{R}^2$, which yields $\gamma_i = 8.36$ and determines the expression of $V(x)$; hence, item (i) of Assumption 2 holds.

Note that $\tilde{H}_i(x, e, w) = H_i(x, e) + \varrho_{W_i}(|w|)$ with $H_i(x, e) := |-d_i x_i^2 + x_i^3 - x_{3-i} + 2x_i|$ and $\varrho_{W_i}(|w|) := |w|$ for $i \in \{1, 2\}$, and $\tilde{\alpha}_V(x, e, w) = \alpha_V(|x|) + \alpha_W(|e|)$ with $\alpha_V(|x|) := \nu|x|^2$, $\alpha_W(|e|) := \nu|e|^2$. Items 1) and 2) of Theorem 1 are, therefore, verified. We have that $\lambda_i^* = 0.2289$ according to (14), from which we derive $T_{\text{MASP},i}(\lambda_i)$ for any $\lambda_i \in [0, \lambda_i^*]$. Indeed, $T_{\text{MASP},1}$ can be taken as a function of λ_1 , which tends to zero as λ_1 tends to its maximal value λ_1^* , and the maximal value for $T_{\text{MASP},1}$ is 0.1634, which arises when $\lambda_1 \rightarrow 0$. As a result, the set \mathcal{A} is ISS according to Theorem 1.

To illustrate the impact of λ_i and sampling period T_i , $i \in \{1, 2\}$, on the number of transmissions over the networks, we have considered different values of λ_i and T_i with $T_i < T_{\text{MASP},i}(\lambda_i)$ being satisfied, where $T_{\text{MASP},i}(\lambda_i)$ is the MASP determined by the given $\lambda_i \in (0, \lambda_i^*)$. We have set $\varepsilon_i = T_i$ for all $i \in \{1, 2\}$, and run 50 simulations over 10 s with parameters $d_1 = d_2 = 0.8$ and initial conditions randomly selected in $[-20, 20]$ for both systems. Parameter ε_i was selected as T_i , so that the triggering generators periodically evaluate their triggering condition. We have taken $w(t) = 2 \sin(20\pi t)$. The obtained average intertransmission times over the 50 simulations are reported in Table II.

Empty boxes in Table II mean that the condition $T_i < T_{\text{MASP},i}(\lambda_i)$ is violated. In view of the lines of Table II, we see that the average intertransmission times increase when λ_i grows for the same sampling period T_i . Also, when we keep the same triggering parameter λ_i and vary the sampling period T_i , the average intertransmission times increase with T_i . This suggests that, for this example and this set of simulations, setting sampling periods close to $T_{\text{MASP},i}(\lambda_i)$ uses less network bandwidth and ensures system stability. Interestingly, selecting T_i large and λ_i small, or T_i small and λ_i large, lead to similar average intertransmission times in view of Table II.

VII. CONCLUSION

We considered PETC of nonlinear systems subject to exogenous disturbances, where the controller communicates with the plant via multiple asynchronously operating networks. An emulation-based systematic design procedure was proposed,

which is applicable for output-feedback control. The starting point of the design is the availability of a controller, which robustly stabilizes the system in the absence of communication constraints. In the next step, the implementation of the controller over the networks was considered. Each network consists of multiple nodes, in which case a protocol is used to schedule transmissions. Moreover, a transmission over each network is triggered when a criterion, which only depends on the local measurements and the local control signals, is violated at given discrete sampling instants. We derived a hybrid system model to describe the resulting dynamics of the NCS and constructed a novel hybrid Lyapunov function for stability analysis. We provided conditions on the controller and scheduling protocols in order to design the local event-triggering criteria and explicit bounds on the MASPs, to ensure input-to-state stability and \mathcal{L}_p stability of the NCS. We showed that our design framework is applicable to a class of globally Lipschitz nonlinear systems and formulated the required conditions as LMIs. We also showed explicitly that our results are applicable to any stabilizable and detectable linear time-invariant system. The effectiveness of the scheme was illustrated via simulations for a nonlinear example, which is not globally Lipschitz and suffered from parametric uncertainties.

Several extensions can be envisioned based on the framework laid down in this paper. Refined results could be developed for more specific classes of nonlinear systems. The results on LTI systems in Section V may also serve as a basis to derive codesign techniques, where both the triggering generator and the controller are designed simultaneously, similarly to [30], where CETC is studied.

APPENDIX A PROOFS

A. Proof of Theorem 1

We define, for any $q \in C \cup D$, the Lyapunov function as

$$U(q) := V(x) + \sum_{i=1}^N S_i(q) \quad (27)$$

$$S_i(q) := \max \{ \gamma_i \phi_i(\tau_i) W_i^2(e_i, \kappa_i), \lambda_i \delta_i(v_i) \}$$

where W_i , δ_i , and V come from Assumptions 1 and 2, and $\phi_i : [0, T_i] \rightarrow [\underline{\mu}_i, \bar{\mu}_i]$ with $\bar{\mu}_i > \underline{\mu}_i > 0$ is defined as in Lemma 3 in Appendix B.

We first show that the following properties hold for system (9). There exist $\bar{\alpha}_U, \underline{\alpha}_U, \alpha_U, \theta_F \in \mathcal{K}_\infty$ such that:

- U is locally Lipschitz in x , e , and τ , and, for all $q \in C \cup D$, $\underline{\alpha}_U(|q|_{\mathcal{A}}) \leq U(q) \leq \bar{\alpha}_U(|q|_{\mathcal{A}})$;

b) for all $q \in C$ and $w \in \mathbb{R}^{n_w}$, $U^\circ(q; F(q, w)) \leq -\alpha_U U(q) + \theta_F(|w|)$;

c) for all $q \in D$, $w \in \mathbb{R}^{n_w}$ and $g \in G(q)$, $U(g) \leq U(q)$.

Proof of item a): It follows from Assumptions 1 and 2, the definition of ϕ_i in Lemma 3, that the Lipschitz property of U in item a) is satisfied. Since δ_i is continuous and positive semidefinite, $v_i = g_{v_i}(x)$ with g_{v_i} in (8) is continuous and $g_{v_i}(0) = 0$, as $g = (g_p, g_c)$ and $g_p(0) = g_c(0) = 0$ with g_p and g_c in (1) and (2); there exists $\bar{\alpha}_{\delta_i} \in \mathcal{K}_\infty$ such that $\delta_i(v_i) \leq \bar{\alpha}_{\delta_i}(|x|)$ (see [31, Lemma 4.3]). In view of Lemma 3, $\phi_i(\tau_i) \in [\underline{\mu}_i, \bar{\mu}_i]$ for all $\tau_i \in [0, T_i]$ with $0 < \underline{\mu}_i^* < \underline{\mu}_i < \bar{\mu}_i < \bar{\mu}_i^*$, $i \in \bar{N}$. Consequently, in view of items (i) of Assumptions 1 and 2 and (27), $\alpha_U(|x|) + \sum_{i=1}^N \gamma_i \underline{\mu}_i \alpha_{W_i}^2(|e|) \leq U(q) \leq \bar{\alpha}_U(|x|) + \sum_{i=1}^N (\gamma_i \bar{\mu}_i \bar{\alpha}_{W_i}^2(|e|) + \lambda_i \bar{\alpha}_{\delta_i}(|x|))$. By applying Lemma 4 in Appendix B, we derive that there exist $\underline{\alpha}_U$, $\bar{\alpha}_U \in \mathcal{K}_\infty$ such that $\underline{\alpha}_U(|q|_A) \leq U(q) \leq \bar{\alpha}_U(|q|_A)$; hence, item a) holds.

Proof of item b): Let $q \in C$, $w \in \mathbb{R}^{n_w}$, and $i \in \bar{N}$. We distinguish three cases according to Lemma 2 in Appendix B: *Case I:* $\gamma_i \phi_i(\tau_i) W_i^2(e_i, \kappa_i) < \lambda_i \delta_i(v_i)$; *Case II:* $\gamma_i \phi_i(\tau_i) W_i^2(e_i, \kappa_i) > \lambda_i \delta_i(v_i)$; and *Case III:* $\gamma_i \phi_i(\tau_i) W_i^2(e_i, \kappa_i) = \lambda_i \delta_i(v_i)$. Suppose that Cases I–III, respectively, hold for $i \in N_I$, $i \in N_{II}$, and $i \in N_{III}$, where $N_I, N_{II}, N_{III} \subseteq \bar{N}$ and $N_I \cup N_{II} \cup N_{III} = \bar{N}$. Then, in view of item (ii) of Assumption 2, and items 1) and 2) of Theorem 1, we have

$$U^\circ(q; F(q, w)) \leq -\alpha_V(|x|) - \alpha_W(|e|) + \sum_{i \in \bar{N}} (Z_i(q, w) + S_i^\circ(q; F(q, w))) \quad (28)$$

where $Z_i(q, w) = \gamma_i^2 W_i^2(e_i) - \tilde{H}_i^2(x, e, w) - J_i(x, e, w) - \delta_i(v_i) + \varrho_{V,i}(|w|)$. We next consider $S_i^\circ(q; F(q, w))$ for $i \in N_I$, $i \in N_{II}$, and $i \in N_{III}$, respectively.

Case I. $i \in N_I$: We have that $S_i(q) = \lambda_i \delta_i(v_i)$ in (27) in this case. It then follows from item (iii) of Assumption 2 and [32] that

$$S_i^\circ(q; F(q, w)) \leq \lambda_i (L_{\delta_i} \delta_i(v_i) + \tilde{H}_i^2(x, e, w) + J_i(x, e, w)).$$

Since $\phi_i(\tau_i) \geq \underline{\mu}_i > \underline{\mu}_i^* = \bar{\rho}_i \geq \frac{\gamma_i \lambda_i}{1 - \lambda_i L_{\delta_i}}$ according to Lemma 3, $1 - \lambda_i L_{\delta_i} > 0$ as ensured by $\lambda_i < \lambda_i^*$ with λ_i^* defined in (14); $\gamma_i \phi_i(\tau_i) W_i^2(e_i, \kappa_i) < \lambda_i \delta_i(v_i)$ implies that $\gamma_i^2 W_i^2(e_i, \kappa_i) < \frac{\gamma_i \lambda_i \delta_i(v_i)}{\phi_i(\tau_i)} < \frac{\gamma_i \lambda_i \delta_i(v_i)}{\bar{\rho}_i} \leq \frac{\gamma_i \lambda_i \delta_i(v_i)}{1 - \lambda_i L_{\delta_i}} = (1 - \lambda_i L_{\delta_i}) \delta_i(v_i)$, and

$$\begin{aligned} Z_i(q, w) + S_i^\circ(q; F(q, w)) &\leq \gamma_i^2 W_i^2(e_i) - \tilde{H}_i^2(x, e, w) \\ &\quad - J_i(x, e, w) - \delta_i(v_i) + \varrho_{V,i}(|w|) \\ &\quad + \lambda_i (L_{\delta_i} \delta_i(v_i) + \tilde{H}_i^2(x, e, w) + J_i(x, e, w)) \\ &\leq -(1 - \lambda_i) \left(\tilde{H}_i^2(x, e, w) + J_i(x, e, w) \right) + \varrho_{V,i}(|w|) \\ &\leq \varrho_{V,i}(|w|) \end{aligned} \quad (29)$$

since $\lambda_i < \lambda_i^* \leq 1$ in view of (14).

Case II. $i \in N_{II}$: We have that $S_i(q) = \gamma_i \phi_i(\tau_i) W_i^2(e_i, \kappa_i)$ in this case. We omit below the dependence of ϕ_i on τ_i for the sake of convenience. In view of item (ii) in

Assumption 1 and the facts that $\phi_i(\tau_i) \leq \bar{\mu}_i$ according to Lemma 3, $\tilde{H}_i(x, e, w) = H_i(x, e) + \varrho_{W_i}(|w|)$ according to item 1) of Theorem 1, $\sqrt{s_1^2 + s_2^2} \leq s_1 + s_2$ and $2s_3 s_4 \leq \nu s_3^2 + \frac{1}{\nu} s_4^2$, for $s_1, s_2, s_3, s_4 \geq 0$ and $\nu > 0$, we have

$$\begin{aligned} S_i^\circ(q; F(q, w)) &= \gamma_i (-2(L_{W_i} + \tilde{v}_i) \phi_i - \gamma_i (\phi_i^2 + 1)) W_i^2(e_i, \kappa_i) \\ &\quad + 2\gamma_i \phi_i W_i(e_i, \kappa_i) (L_{W_i} W_i(e_i, \kappa_i) + H_i(x, e) + \varrho_{W_i}(|w|)) \\ &\leq \gamma_i (-2L_{W_i} + \tilde{v}_i) \phi_i - \gamma_i (\phi_i^2 + 1) W_i^2(e_i, \kappa_i) \\ &\quad + 2\gamma_i L_{W_i} \phi_i W_i^2(e_i, \kappa_i) + \gamma_i^2 \phi_i^2 W_i^2(e_i, \kappa_i) + H_i^2(x, e) \\ &\quad + \gamma_i \phi_i \left(\tilde{v}_i W_i^2(e_i, \kappa_i) + \frac{1}{\tilde{v}_i} \varrho_{W_i}^2(|w|) \right) \\ &\leq -\gamma_i^2 W_i^2(e_i, \kappa_i) + H_i^2(x, e) + \frac{\bar{\mu}_i \gamma_i}{\tilde{v}_i} \varrho_{W_i}^2(|w|) \end{aligned}$$

where $\tilde{v}_i > 0$ is given in Lemma 3. Then, since $\bar{\mu}_i < \bar{\mu}_i^*$ and $\tilde{H}_i^2(x, e, w) \geq H_i^2(x, e)$, we have

$$\begin{aligned} Z_i(q, w) + S_i^\circ(q; F(q, w)) &\leq \gamma_i^2 W_i^2(e_i) - \tilde{H}_i^2(x, e, w) \\ &\quad - J_i(x, e, w) - \delta_i(v_i) + \varrho_{V,i}(|w|) \\ &\quad - \gamma_i^2 W_i^2(e_i, \kappa_i) + H_i^2(x, e) + \frac{\bar{\mu}_i \gamma_i}{\tilde{v}_i} \varrho_{W_i}^2(|w|) \\ &\leq -\delta_i(v_i) - J_i(x, e, w) + \frac{\bar{\mu}_i \gamma_i}{\tilde{v}_i} \varrho_{W_i}^2(|w|) + \varrho_{V,i}(|w|) \\ &\leq \frac{\bar{\mu}_i^* \gamma_i}{\tilde{v}_i} \varrho_{W_i}^2(|w|) + \varrho_{V,i}(|w|). \end{aligned} \quad (30)$$

Case III. $i \in N_{III}$: In view of Lemma 2 and (30), in this case, $Z_i(q, w) + S_i^\circ(q; F(q, w)) \leq \frac{\bar{\mu}_i \gamma_i}{\tilde{v}_i} \varrho_{W_i}^2(|w|) + \varrho_{V,i}(|w|)$.

In view of Cases I–III, we have

$$\begin{aligned} U^\circ(q; F(q, w)) &\leq -\alpha_V(|x|) - \alpha_W(|e|) + \sum_{i \in \bar{N}} (Z_i(q, w) \\ &\quad + S_i^\circ(q; F(q, w))) \leq -\alpha_V(|x|) - \alpha_W(|e|) \\ &\quad + \sum_{i \in \bar{N}} \left(\frac{\bar{\mu}_i^* \gamma_i}{\tilde{v}_i} \varrho_{W_i}^2(|w|) + \varrho_{V,i}(|w|) \right). \end{aligned} \quad (31)$$

In view of item a), there exists $\alpha_U \in \mathcal{K}_\infty$ such that item b) holds with $\theta_F(s) := \sum_{i \in \bar{N}} \theta_{F_i}(s)$, and $\theta_{F_i}(s) := \frac{\bar{\mu}_i^* \gamma_i}{\tilde{v}_i} \varrho_{W_i}^2(s) + \varrho_{V,i}(s)$ for all $s \geq 0$.

Proof of item c): Let $q \in D$, $i \in \bar{N}$. We distinguish two cases whether a transmission occurs. When a transmission occurs, $W_i(\chi_i(e_i, \kappa_i), \kappa_i + 1) \leq \rho_i W_i(e_i, \kappa_i)$ according to item (ii) of Assumption 1. Note that $\phi_i(0) < \bar{\mu}_i^* = 1/\bar{\rho}_i$, $\phi_i(T_i) > \underline{\mu}_i^* = \bar{\rho}_i$, and $\bar{\rho}_i \geq \rho_i$ in view of Lemma 3. Let $g \in \mathcal{G}_i(q)$, and

$$\begin{aligned} S_i(g) &= \max \{ \gamma_i \phi_i(0) W_i^2(\chi_i(e_i, \kappa_i), \kappa_i + 1), \lambda_i \delta_i(v_i) \} \\ &\leq \max \left\{ \gamma_i \frac{1}{\bar{\rho}_i} \rho_i^2 W_i^2(e_i, \kappa_i), \lambda_i \delta_i(v_i) \right\} \\ &\leq \max \{ \gamma_i \bar{\rho}_i W_i^2(e_i, \kappa_i), \lambda_i \delta_i(v_i) \} \\ &\leq \max \{ \gamma_i \phi_i(T_i) W_i^2(e_i, \kappa_i), \lambda_i \delta_i(v_i) \} \leq S_i(q). \end{aligned} \quad (32)$$

When no transmission occurs, it follows from (7), (13), and item (ii) of Assumption 1 that $\gamma_i W_i^2(\chi_i(e_i, \kappa_i), \kappa_i) = \gamma_i W_i^2(e_i, \kappa_i) \leq \lambda_i \bar{\rho}_i \delta_i(v_i)$. Since $\phi_i(0) < \bar{\mu}_i^* = 1/\bar{\rho}_i$ according to Lemma 3, we have that

$$\begin{aligned} S_i(g) &= \max \left\{ \gamma_i \phi_i(0) W_i^2(e_i, \kappa_i), \lambda_i \delta_i(v_i) \right\} \\ &\leq \max \left\{ \frac{1}{\bar{\rho}_i} \lambda_i \bar{\rho}_i \delta_i(v_i), \lambda_i \delta_i(v_i) \right\} = \lambda_i \delta_i(v_i) \leq S_i(q). \end{aligned}$$

As a result, for all $q \in D$ and $g \in G(q)$, we have

$$U(g) \leq V(x) + \sum_{i=1}^N S_i(g) \leq V(x) + \sum_{i=1}^N S_i(q) = U(q)$$

and item c) holds.

Let (φ, w) be a solution to (9). Note that U is locally Lipschitz in x, e, τ from item a). In view of [32, p. 99], for any $\nu \in (0, 1)$, all $j \in \mathbb{Z}_{\geq 0}$ such that there exists $t \in \mathbb{R}_{\geq 0}$ with $(t, j) \in \text{dom } \varphi$, and almost all $s \in I^j := \{\bar{t} : (\bar{t}, j) \in \text{dom } \phi\}$, we have

$$\begin{aligned} \frac{d}{ds} U(\varphi(s, j)) &\leq U^\circ(\varphi(s, j); F(\varphi(s, j), w(s, j))) \\ &\leq -(1-\nu)\alpha_U U(\varphi(s, j)) \\ &\quad - \nu\alpha_U (U(\varphi(s, j)) + \theta_F(|w(s, j)|)). \end{aligned}$$

By invoking standard ISS arguments, we derive that there exists $\bar{\beta} \in \mathcal{KL}$ such that for all $(t, j) \in \text{dom } \varphi$, we have

$$U(\varphi(t, j)) \leq \bar{\beta}(U(\varphi(t, t_j), t - t_j) + \alpha_U^{-1} \left(\frac{1}{\nu} \theta_F(\|w\|_{(t, j)}) \right))$$

where $t_j := \inf I_j$. It follows from item c) that, for all j such that $(t, j) \in \text{dom } \varphi$, we have

$$U(\varphi(t_{j+1}, j+1)) \leq U(\varphi(t_{j+1}, j)).$$

Let $\varepsilon := \min_{i \in \bar{N}} \varepsilon_i$, where $\varepsilon_i > 0$ is the minimum intersampling time corresponding to network \mathcal{N}_i . Let $(t, j) \in \text{dom } \varphi$. The integer j represents the total number of transmissions over the N networks; we can, therefore, write it as $j = j_1 + \dots + j_N$, where j_i is the number of transmissions that has occurred so far on network \mathcal{N}_i . In view of the definition of the jump set in (9), $t \geq \varepsilon(j_i - 1)$. Consequently, $\frac{t}{N} \geq \frac{\varepsilon}{N}(j_i - 1)$ and $t = N \frac{t}{N} \geq \frac{\varepsilon}{N}(j_1 + \dots + j_N) - \varepsilon = \frac{\varepsilon}{N}j - \varepsilon$. Since $t \geq 0$, we have that $t \geq t/2 + \varepsilon/2 \max\{\frac{j}{N} - 1, 0\}$. Consequently, for any $(t, j) \in \text{dom } \varphi$, we have

$$\begin{aligned} U(\varphi(t, j)) &\leq \bar{\beta}(U(\varphi(0, 0), t) + \alpha_U^{-1} \left(\frac{1}{\nu} \theta_F(\|w\|_{(t, j)}) \right)) \\ &\leq \bar{\beta} \left(U(\varphi(0, 0)), 0.5t + 0.5\varepsilon \max \left\{ \frac{j}{N} - 1, 0 \right\} \right) \\ &\quad + \alpha_U^{-1} \left(\frac{1}{\nu} \theta_F(\|w\|_{(t, j)}) \right). \end{aligned}$$

Since $\alpha(s_1 + s_2) \leq \alpha(2s_1) + \alpha(2s_2)$ for any $\alpha \in \mathcal{K}_\infty$ and $s_1, s_2 \geq 0$ (see [33, eq. (7)]), we deduce from item a) to have that $|\varphi(t, j)|_A \leq \beta(|\varphi(0, 0)|_A, t + j) + \psi(\|w\|_{(t, j)})$, for all $(t, j) \in \text{dom } \varphi$, where $\beta(s_1, s_2) := \underline{\alpha}_U^{-1}(2\bar{\beta}(\bar{\alpha}_U(s_1), 0.5 \min\{\varepsilon, 1\} \max\{\frac{s_2}{N} - 2, 0\}))$ and $\psi(s) := \underline{\alpha}_U^{-1}(2\alpha_U^{-1}(\frac{1}{\nu} \theta_F(\|w\|_{(t, j)})))$.

B. Sketched Proof of Theorem 2

For any $q \in C \cup D$, let $U(q) := V(x) + \max\{\gamma\phi(\tau)W^2(e, \kappa), \lambda V(x)\}$, with W and V coming from Assumption 3. The function $\phi : [0, T] \rightarrow [\underline{\mu}, \bar{\mu}]$ is defined as in Lemma 5 in Appendix B.

The proof Theorem 2 follows the same steps as the proof of Theorem 1. We only explicitly prove the flow property corresponding to item b) in Appendix VIII-A in the following. In particular, we need to show that item b) holds under Assumption 3. Let $q \in C$ and $w \in \mathbb{R}^{n_w}$, and we distinguish three cases according to Lemma 2.

Case I. $\gamma\phi(\tau)W^2(e, \kappa) < \lambda V(x)$: We have that $U(q) = (\lambda + 1)V(x)$ in this case. According to Lemma 5, $\phi(\tau) \geq \bar{\rho} > \frac{\gamma\lambda}{a_V}$. Hence, there exists $\nu \in (0, 1)$ such that $\bar{\rho} \geq \frac{1}{1-\nu} \frac{\gamma\lambda}{a_V}$. Thus, $\gamma^2 W^2(e, \kappa) \leq \frac{\gamma\lambda V(x)}{\bar{\rho}} \leq (1-\nu)a_V V(x)$ and $U^\circ(q; F(q, w)) \leq (\lambda + 1)(-a_V V(x) - \alpha_W(|e|) + \gamma^2 W^2(e, \kappa) + \rho_V(|w|)) \leq (\lambda + 1)(-\nu V(x) - \alpha_W(|e|) + \rho_V(|w|))$.

Case II. $\gamma\phi(\tau)W(e, \kappa) > \lambda V(x)$: In this case, we have that $U(q) = V(x) + \gamma\phi(\tau)W^2(e, \kappa)$. From item (ii) of Assumption 3, the facts that $\phi(\tau) \leq \bar{\mu}$ in view of Lemma 5, and $2ab \leq ca^2 + 1/cb^2$ for all $a, b \geq 0$ and $c > 0$, we have

$$\begin{aligned} U^\circ(q; F(q, w)) &= -a_V V(x) - \alpha_W(|e|) \\ &\quad + \gamma^2 W^2(e, \kappa) + \rho_V(|w|) \\ &\quad + \gamma \left(-(2L_W + \tilde{\nu})\phi - \gamma \left(\frac{L_V^2}{a_V - \tilde{\nu}} \phi^2 + 1 \right) \right) W^2(e, \kappa) \\ &\quad + 2\gamma\phi W(e, \kappa)(L_W W(e, \kappa) + L_V \sqrt{V(x)} + \rho_W(|w|)) \\ &\leq -a_V V(x) - \alpha_W(|e|) + \gamma^2 W^2(e, \kappa) + \rho_V(|w|) \\ &\quad - \gamma(2L_W + \tilde{\nu})\phi W^2(e, \kappa) - \gamma^2 \left(\frac{L_V^2}{a_V - \tilde{\nu}} \phi^2 + 1 \right) W^2(e, \kappa) \\ &\quad + \gamma^2 \frac{L_V^2}{a_V - \tilde{\nu}} \phi^2 W^2(e, \kappa) + (a_V - \tilde{\nu})V(x) \\ &\quad + 2\gamma L_W \phi W^2(e, \kappa) + \gamma\phi \left(\tilde{\nu} W^2(e, \kappa) + \frac{1}{\tilde{\nu}} \rho_W^2(|w|) \right) \\ &\leq -\tilde{\nu}V(x) - \alpha_W(|e|) + \frac{\bar{\mu}\gamma}{\tilde{\nu}} \rho_W^2(|w|) + \rho_V(|w|) \end{aligned}$$

where $\tilde{\nu} > 0$ comes from Lemma 5.

Case III. $\gamma\phi(\tau)W^2(e, \kappa) = \lambda V(x)$: In view of Lemma 2, and cases I and II, $U^\circ(q; F(q, w)) \leq -\min\{\nu, \tilde{\nu}\}V(x) - \alpha_W(|e|) + \rho_V(|w|) + \frac{\bar{\mu}\gamma}{\tilde{\nu}} \rho_W^2(|w|)$.

In view of cases I–III, $U^\circ(q; F(q, w)) \leq -\min\{\nu, \tilde{\nu}\}V(x) - \alpha_W(|e|) + \rho_V(|w|) + \frac{\bar{\mu}\gamma}{\tilde{\nu}} \rho_W^2(|w|)$ for all $q \in C$ and $w \in \mathbb{R}^{n_w}$. Then, there exists $\alpha_U \in \mathcal{K}_\infty$ such that item b) holds with $\theta_F(s) := \frac{\bar{\mu}\gamma}{\tilde{\nu}} \rho_W^2(|w|) + \rho_V(|w|)$ for all $s \geq 0$. The rest of proof follows the same steps as the proof of Theorem 1 and is, therefore, omitted.

C. Sketched Proof of Theorem 3

We define the storage function as (27) and prove the desired result by ensuring the satisfaction of the next properties.

- a) U is locally Lipschitz in x , e , and τ , and, for all $q \in C \cup D$, $\underline{\alpha}_U(|q|_{\mathcal{A}}) \leq U(q) \leq \bar{\alpha}_U(|q|_{\mathcal{A}})$, where $\bar{\alpha}_U, \underline{\alpha}_U \in \mathcal{K}_\infty$;
- b) for all $q \in C$ and $w \in \mathbb{R}^{n_w}$, $U^\circ(q; F(q, w)) \leq \mu(\theta^p |w|^p - |z|^p)$;
- c) for all $q \in D$, $w \in \mathbb{R}^{n_w}$ and $g \in G(q)$, $U(g) \leq U(q)$.

The proof of items a) and c) follows the same steps as the proof of Theorem 1. We, therefore, prove the flow property corresponding to item b) in Appendix VIII-A. Recall that there are three cases to consider when $q \in C$: *Case I*: $\gamma_i \phi_i(\tau_i) W_i^2(e_i, \kappa_i) < \lambda_i \delta_i(v_i)$; *Case II*: $\gamma_i \phi_i(\tau_i) W_i^2(e_i, \kappa_i) > \lambda_i \delta_i(v_i)$; and *Case III*: $\gamma_i \phi_i(\tau_i) W_i^2(e_i, \kappa_i) = \lambda_i \delta_i(v_i)$, and they, respectively, hold for $i \in N_I$, $i \in N_{II}$, and $i \in N_{III}$. We derive that the following hold.

Case I. $i \in N_I$: In this case, $S_i^\circ(q; F(q, w)) \leq \lambda_i(L_{\delta_i} \delta_i(v_i) + \tilde{H}_i^2(x, e, w) + J_i(x, e, w))$. Let $Z_i(q, w) = \gamma_i^2 W_i^2(e_i) - \tilde{H}_i^2(x, e, w) - J_i(x, e, w) - \delta_i(v_i)$. We have that $Z_i(q, w) + S_i^\circ(q; F(q, w)) \leq \gamma_i^2 W_i^2(e_i, \kappa_i) - (1 - \lambda_i L_{\delta_i}) \delta_i(v_i) - (1 - \lambda_i)(\tilde{H}_i^2(x, e, w) + J_i(x, e, w)) \leq -(1 - \lambda_i) \tilde{H}_i^2(x, e, w) - (1 - \lambda_i) J_i(x, e, w)$, $i \in \bar{N}$, since $\lambda_i L_{\delta_i} < 1$ from (14) and $\gamma_i^2 W_i^2(e_i, \kappa_i) < (1 - \lambda_i L_{\delta_i}) \delta_i(v_i)$.

Case II. $i \in N_{II}$: In Case II, $S_i^\circ(q; F(q, w)) = \gamma_i(-2L_{W_i} + \tilde{\nu}_i) \phi_i - \gamma_i(\phi_i^2 + 1) W_i^2(e_i, \kappa_i) + 2\gamma_i \phi_i W_i(e_i, \kappa_i)(L_{W_i} W_i(e_i, \kappa_i) + \tilde{H}_i^2(x, e, w)) \leq -\gamma_i^2 W_i^2(e_i, \kappa_i) + \tilde{H}_i^2(x, e, w)$ and $Z_i(q, w) + S_i^\circ(q; F(q, w)) \leq -\delta_i(v_i) - J_i(x, e, w)$, $i \in \bar{N}$.

Case III. $i \in N_{III}$: In view of Lemma 2, and Cases I and II, $Z_i(q, w) + S_i^\circ(q; F(q, w)) \leq \max\{-(1 - \lambda_i) \tilde{H}_i^2(x, e, w) - (1 - \lambda_i) J_i(x, e, w), -\delta_i(v_i) - J_i(x, e, w)\}$.

Consequently, $U^\circ(q; F(q, w)) \leq \mu(\theta^p |w|^p - |z|^p) + \sum_{i \in \bar{N}} Z_i(q, w) + S_i^\circ(q; F(q, w)) \leq \mu(\theta^p |w|^p - |z|^p) - \sum_{i \in N_{II}} (\delta_i(v_i) + J_i(x, e, w)) - \sum_{i \in N_I} (1 - \lambda_i)(\tilde{H}_i^2(x, e, w) + J_i(x, e, w)) - \sum_{i \in N_{III}} \max\{(1 - \lambda_i)(\tilde{H}_i^2(x, e, w) + J_i(x, e, w)) + \delta_i(v_i) + J_i(x, e, w)\} \leq \mu(\theta^p |w|^p - |z|^p)$ since $\lambda_i < \lambda_i^* \leq 1$ in view of (14). The satisfaction of items a) and c) follows by applying similar arguments as in the proof of Theorem 1.

Let (φ, w) be a solution to (9). In view of [32, p. 99], for all $j \in \mathbb{Z}_{\geq 0}$ such that there exists $t \in \mathbb{R}_{\geq 0}$ with $(t, j) \in \text{dom } \varphi$ and almost all $s \in I^j := \{\bar{t} : (\bar{t}, j) \in \text{dom } \varphi\}$, $\frac{d}{ds} U(\varphi(s, j)) \leq U^\circ(\varphi(s, j); F(\varphi(s, j), w(s, j))) \leq -|z(s, j)|^p + \theta^p |w(s, j)|^p$. We then follow similar lines as the proof of Theorem 1 and [26, Th. IV.7], and we obtain that $\|z\|_{\mathcal{L}_p} \leq \tilde{\gamma}(|\varphi(0, 0)|_{\mathcal{A}}) + \theta \|w\|_{\mathcal{L}_p}$, where $\tilde{\gamma} := \bar{\alpha}_U$.

D. Proof of Proposition 1

Let matrix P , a_V , a_W , $\tilde{\theta}$, $\epsilon_i > 0$ and $\nu_i \geq a_W + \frac{\varpi_i^2}{\bar{\alpha}_i} |\bar{\mathcal{B}}_{2,i}|^2$, $i \in \bar{N}$, be given such that (24) holds. Let $i \in \bar{N}$, since $|\frac{\partial W_i(e_i, \kappa_i)}{\partial e_i}| \leq \varpi_i$ for almost all $e_i \in \mathbb{R}^{n_{e_i}}$ and $\kappa_i \in \mathbb{Z}_{\geq 0}$ [as the protocol is UGES, see discussions above (21)], we have that for all $x \in \mathbb{R}^{n_x}$ and almost all $e \in \mathbb{R}^{n_e}$, $\langle \frac{\partial W_i(e_i, \kappa_i)}{\partial e_i}, -\bar{\mathcal{C}}_i(\mathcal{A}_1 x + \mathcal{B}_1 e + \mathcal{D}_1 \bar{\psi}(x) + \mathcal{E}_1 w) \rangle \leq \varpi_i |\bar{\mathcal{C}}_i \mathcal{A}_1 x + \bar{\mathcal{C}}_i \mathcal{B}_1 e + \bar{\mathcal{C}}_i \mathcal{D}_1 \bar{\psi}(x) + \bar{\mathcal{C}}_i \mathcal{E}_1 w| \leq \varpi_i |\bar{\mathcal{C}}_i \mathcal{A}_1 x + \bar{\mathcal{B}}_{2,i} e + \bar{\mathcal{C}}_i \mathcal{D}_1 \bar{\psi}(x) + \bar{\mathcal{C}}_i \mathcal{E}_1 w| + |\bar{\mathcal{C}}_i \mathcal{B}_1 \bar{\Psi}_i e|$ and $|\bar{\mathcal{C}}_i \mathcal{D}_1 \bar{\psi}(x)| \leq L |\bar{\mathcal{C}}_i \mathcal{D}_1| \|x\|$ in view of the properties of ψ . Note that $\bar{\mathcal{B}}_{2,i} e$ is independent of e_i , and $\mathcal{B}_1 \bar{\Psi}_i e$ depends only on e_i according to the definition of $\bar{\mathcal{B}}_{2,i}$ and $\bar{\Psi}_i$. This implies that item

(iii) of Assumption 1 holds with $L_{W_i} = \varpi_i / a_{W_i} |\bar{\mathcal{C}}_i \mathcal{B}_1 \bar{\Psi}_i|$, $\tilde{H}_i(x, e, w) = \varpi_i (|\bar{\mathcal{C}}_i \mathcal{A}_1 x + \bar{\mathcal{B}}_{2,i} e + \bar{\mathcal{C}}_i \mathcal{E}_1 w| + L |\bar{\mathcal{C}}_i \mathcal{D}_1| \|x\|)$.

Recall that $\delta_i(v_i) = \epsilon_i^2 |v_i|^2$, $v_i = \bar{\mathcal{C}}_i x$, and note that $|x^T (\bar{\mathcal{C}}_i^T \bar{\mathcal{C}}_i \mathcal{A}_1 + \mathcal{A}_1^T \bar{\mathcal{C}}_i^T \bar{\mathcal{C}}_i) x| = 2 |\langle \bar{\mathcal{C}}_i x, \bar{\mathcal{C}}_i \mathcal{A}_1 x \rangle| \leq 2 |\bar{\mathcal{C}}_i x| |\bar{\mathcal{C}}_i \mathcal{A}_1 x| \leq |x^T \bar{\mathcal{C}}_i^T \bar{\mathcal{C}}_i x| + |x^T \mathcal{A}_1^T \bar{\mathcal{C}}_i^T \bar{\mathcal{C}}_i \mathcal{A}_1 x|$. Hence, $\langle \nabla \delta_i(v_i), \bar{\mathcal{C}}_i(\mathcal{A}_1 x + \mathcal{B}_1 e + \mathcal{D}_1 \bar{\psi}(x) + \mathcal{E}_1 w) \rangle \leq \epsilon_i^2 |x^T (\bar{\mathcal{C}}_i^T \bar{\mathcal{C}}_i \mathcal{A}_1 + \mathcal{A}_1^T \bar{\mathcal{C}}_i^T \bar{\mathcal{C}}_i) x| + 2 \epsilon_i^2 (|x^T \bar{\mathcal{C}}_i^T \bar{\mathcal{C}}_i (\mathcal{B}_1 e + \mathcal{E}_1 w)| + L |\bar{\mathcal{C}}_i^T \bar{\mathcal{C}}_i \mathcal{D}_1| \|x\|^2) \leq L_{\delta_i} \delta_i + J_i(x, e, w)$, where $L_{\delta_i} = 1$ and $J_i(x, e, w) = \epsilon_i^2 (|\mathcal{A}_1^T \bar{\mathcal{C}}_i^T \bar{\mathcal{C}}_i \mathcal{A}_1| + 2L |\bar{\mathcal{C}}_i^T \bar{\mathcal{C}}_i \mathcal{D}_1|) \|x\|^2 + 2 \epsilon_i^2 |x^T \bar{\mathcal{C}}_i^T \bar{\mathcal{C}}_i (\mathcal{B}_1 e + \mathcal{E}_1 w)|$; hence, item (iii) of Assumption 2 holds.

Let $V(x) = x^T P x$. Item (i) of Assumption 2 holds with $\underline{\alpha}_V(s) := \lambda_{\min}(P) s^2$, $\bar{\alpha}_V(s) := \lambda_{\max}(P) s^2$ for all $s \geq 0$. Note that, for all $x \in \mathbb{R}^{n_x}$ and almost all $e \in \mathbb{R}^{n_e}$, $\langle \nabla V(x), \mathcal{A}_1 x + \mathcal{B}_1 e + \mathcal{D}_1 \bar{\psi}(x) + \mathcal{E}_1 w \rangle = x^T (\mathcal{A}_1^T P + P \mathcal{A}_1) x + 2x^T P \mathcal{B}_1 e + 2L |\mathcal{D}_1| x^T P x + 2x^T P \mathcal{E}_1 w$. We now post- and premultiply LMI in (24), respectively, by the state vector (x, e, w) and its transpose, rearrange its terms, and derive from (24) and the definitions of \mathcal{A}_2 , \mathcal{B}_2 , and \mathcal{E}_2 defined after (22) that item (ii) of Assumption 2 holds with functions \tilde{H}_i , J_i , δ_i , $\varrho_{V,i}$, and $\gamma_i > 0$ given in Proposition 1.

E. Proof of Lemma 1

The proof agrees with the one to Proposition 1 with replacing $\psi(x)$ by $\psi(y, e)$, and the property $|\bar{\psi}(y, e)| \leq L|e|$ in view of properties of ψ . Since $2L|P x| |\mathcal{D}_1| |e| \leq a_V |x|^2 + 1/a_V (L|P| |\mathcal{D}_1| |e|)^2$ for all $x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_e}$ and any $a_V > 0$, and hence, $\langle \nabla V(x), \mathcal{A}_1 x + \mathcal{B}_1 e + \mathcal{D}_1 \bar{\psi}(y, e) + \mathcal{E}_1 w \rangle = x^T (\mathcal{A}_1^T P + P \mathcal{A}_1) x + 2x^T P \mathcal{B}_1 e + 2L|P x| |\mathcal{D}_1| |e| + 2x^T P \mathcal{E}_1 w$. We then follow similar lines as in the proof of Proposition 1 to show that Assumptions 1 and 2 hold. On the other hand, (24) always has a solution in this case since \mathcal{A}_1 is Hurwitz, which ensures $\Sigma_{11} < 0$ and (24) follows by taking sufficiently large $\tilde{\theta}$, ν and small enough $\epsilon > 0$.

APPENDIX B TECHNICAL LEMMAS

The next statements corresponds to [34, Lemma II.1].

Lemma 2: Consider two functions $U_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ and $U_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ that have well-defined Clarke derivatives for all $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$. Introduce three sets $A := \{x : U_1(x) > U_2(x)\}$, $B := \{x : U_1(x) < U_2(x)\}$, $\Gamma := \{x : U_1(x) = U_2(x)\}$. Then, for any $v \in \mathbb{R}^n$, the function $U(x) := \max\{U_1(x), U_2(x)\}$ satisfies $U^\circ(x; v) = U_1^\circ(x; v)$ for all $x \in A$, $U^\circ(x; v) = U_2^\circ(x; v)$ for all $x \in B$, and $U^\circ(x; v) \leq \max\{U_1^\circ(x; v), U_2^\circ(x; v)\}$ for all $x \in \Gamma$. ■

Lemma 3: Let $i \in \bar{N}$, $\lambda_i \in [0, \lambda_i^*)$, and $T_i < T_{\text{MASP},i}(\lambda_i)$ with λ_i^* and $T_{\text{MASP},i}(\lambda_i)$ defined in (14) and (15), respectively. Let $\bar{\mu}_i^* := 1/\bar{\rho}_i$, $\underline{\mu}_i^* = \bar{\rho}_i$, with $\bar{\rho}_i$ defined after (13). There exist $\bar{\mu}_i > \underline{\mu}_i > 0$, satisfying $0 < \underline{\mu}_i^* < \underline{\mu}_i < \bar{\mu}_i < \bar{\mu}_i^*$, and $\tilde{\nu}_i > 0$ such that the solution ϕ_i to $\dot{\phi}_i = -(2L_{W_i} + \tilde{\nu}_i) \phi_i - \gamma_i(\phi_i^2 + 1)$, $\phi_i(0) = \bar{\mu}_i$, verifies $\phi_i(t) \in [\underline{\mu}_i, \bar{\mu}_i]$ for all $t \in [0, T_i]$, where $L_{W_i} \geq 0$ comes from Assumption 1 and $\gamma_i > 0$ from Assumption 2. ■

Proof of Lemma 3: Let $i \in \bar{N}$, $\lambda_i \in [0, \lambda_i^*)$, and $T_i < T_{\text{MASP},i}(\lambda_i)$. We first show that the following fact holds.

Fact 1: $0 \leq \frac{\gamma_i \lambda_i}{1 - \lambda_i L_{\delta_i}} < 1$, $i \in \bar{N}$. ■

Fact 1 holds, since $0 \leq \frac{\gamma_i \lambda_i}{1 - \lambda_i L_{\delta_i}} \leq \frac{\gamma_i \lambda_i}{1 + \gamma_i \lambda_i} < 1$ when $L_{\delta_i} \leq -\gamma_i$, and $\lambda_i < \frac{1}{L_{\delta_i} + \gamma_i}$ when $L_{\delta_i} > -\gamma_i$, in view of the definition of λ_i^* in (14).

Fact 1 leads to $\bar{\rho}_i = \max\{\rho_i, \frac{\gamma_i \lambda_i}{1 - \lambda_i L_{\delta_i}}\} \in [0, 1)$, since $\rho_i \in [0, 1)$. Hence, $0 \leq \underline{\mu}_i^* < \bar{\mu}_i^*$. Denote by $\mathcal{T}(\bar{\mu}_i, \underline{\mu}_i, \tilde{\nu}_i)$ the time it takes for the solution ϕ_i to decrease from $\bar{\mu}_i$ to $\underline{\mu}_i$ for a given $\tilde{\nu}_i \geq 0$. In view of the dynamics of ϕ_i , the function \mathcal{T}_i is continuous in all its arguments, increasing in $\bar{\mu}_i$, decreasing in $\underline{\mu}_i$, and decreasing in $\tilde{\nu}_i$. By following similar lines as in [28, proof of Lemma 2], we have that $T_{\text{MASP},i}$ defined in (15) satisfies $T_{\text{MASP},i}(\lambda_i) = \mathcal{T}_i(\lambda_i, \bar{\mu}_i^*, \underline{\mu}_i^*, 0)$. Since $T_i < T_{\text{MASP},i}(\lambda_i)$, by continuity of \mathcal{T}_i and in view of its increasing/decreasing properties, there exists a triplet $(\bar{\mu}_i, \underline{\mu}_i, \tilde{\nu}_i)$ with $\bar{\mu}_i < \bar{\mu}_i^*$, $\underline{\mu}_i > \underline{\mu}_i^*$ and $\tilde{\nu}_i > 0$, such that $T_i = \mathcal{T}_i(\lambda_i, \bar{\mu}_i, \underline{\mu}_i, \tilde{\nu}_i)$ and $\phi_i(t) \in [\underline{\mu}_i, \bar{\mu}_i]$ holds for all $t \in [0, T_i]$. ■

Lemma 4: For any $\alpha_1, \dots, \alpha_N \in \mathcal{K}$, $\underline{\alpha}(\sum_{i=1}^N s_i) \leq \sum_{i=1}^N \alpha_i(s_i) \leq \bar{\alpha}(\sum_{i=1}^N s_i)$ holds for all $s_i \geq 0$ and $i \in \bar{N}$, where $\underline{\alpha}, \bar{\alpha} : s \mapsto \min\{\alpha_1(\frac{s}{N}), \dots, \alpha_N(\frac{s}{N})\} \in \mathcal{K}$ and $\bar{\alpha} := \sum_{i=1}^N \alpha_i$. ■

Proof of Lemma 4: Let $\alpha_1, \dots, \alpha_N \in \mathcal{K}$ and $s_1, \dots, s_N \geq 0$. Since $\alpha_1, \dots, \alpha_N$ are increasing functions, $\sum_{i=1}^N \alpha_i(s_i) \leq \sum_{i=1}^N \alpha_i(s_1 + \dots, s_N) = \bar{\alpha}(\sum_{i=1}^N s_i)$. Without loss of generality, we assume that $s_j = \max\{s_1, \dots, s_N\}$ for some $j \in \bar{N}$. It then follows that $\frac{1}{N} \sum_{i=1}^N s_i \leq s_j$ and $\sum_{i=1}^N \alpha_i(s_i) \geq \alpha_j(s_j) \geq \alpha_j(\frac{1}{N} \sum_{i=1}^N s_i) \geq \min\{\alpha_1(\frac{1}{N} \sum_{i=1}^N s_i), \dots, \alpha_N(\frac{1}{N} \sum_{i=1}^N s_i)\}$, which completes the proof. ■

The proof of Lemma 5 follows the proof to Lemma 3 and is, therefore, omitted.

Lemma 5: Let $\lambda \in [0, \lambda^*)$ and $T < T_{\text{MASP}}(\lambda)$ with λ^* and $T_{\text{MASP}}(\lambda)$ defined as (17) and (18), respectively. There exist $\bar{\mu} > \underline{\mu} > 0$, satisfying $0 < \bar{\rho} < \underline{\mu} < \bar{\mu} < 1/\bar{\rho}$, and $\tilde{\nu} \in (0, a_V)$ such that the solution ϕ to $\dot{\phi} = -(2L_W + \tilde{\nu})\phi - \gamma(\frac{L_V^2}{a_V - \tilde{\nu}}\phi^2 + 1)$, $\phi(0) = \bar{\mu}$, verifies $\phi(t) \in [\underline{\mu}, \bar{\mu}]$ for all $t \in [0, T]$, where $\bar{\rho} := \max\{\rho, \frac{\gamma\lambda}{a_V}\}$, $\rho \in [0, 1)L_W, L_V \geq 0$ and $a_V, \gamma > 0$ come from Assumption 3. ■

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