2018 IEEE Conference on Decision and Control (CDC)
Miami Beach, FL, USA, Dec. 17-19, 2018

Periodic event-triggered output feedback control of nonlinear systems

W. Wang, R. Postoyan, D. Nešić and W.P.M.H. Heemels

Abstract— **We investigate the stabilization of perturbed nonlinear systems using output-based periodic event-triggered controllers. Thus, the communication between the plant and the controller is triggered by a mechanism, which evaluates an output- and input-dependent rule at given sampling instants. We address the problem by emulation. Hence, we assume the knowledge of a continuous-time output feedback controller, which robustly stabilizes the system in the absence of network. We then implement the controller over the network and model the overall system as a hybrid system. We design the event-triggered mechanism to ensure an input-to-state stability property. An explicit bound on the maximum allowable sampling period at which the triggering rule is evaluated is provided. The analysis relies on the construction of a novel hybrid Lyapunov function. The results are applied to a class of Lipschitz nonlinear systems, for which we formulate the required conditions as linear matrix inequalities. The effectiveness of the scheme is illustrated via simulations of a nonlinear example.**

I. Introduction

Networked control systems (NCSs) refer to systems for which the control loop is closed through a communication network. Typically, sensors and/or actuators transmit their data at instants defined by a clock, we talk of *time-triggered* transmissions. While this paradigm is easy to implement and convenient for the analysis, it may unnecessarily overload the network. In contrast, *event-triggered control* (ETC) adapts the transmission instants to the current state of the plant. The underlying idea is to use the channel only when this is needed, thus limiting transmission delays and the occurrence packet losses, which may destroy the desired closed-loop system properties. Various ETC strategies are available in the literature and most of them focus on *continuous event-triggered control* (CETC), in the sense that the triggering condition has to be evaluated at all times, see for instance [1]–[5]. The continuous evaluation of the triggering condition is not realistic when we implement the CETC on a digital platform. In this case, it is more natural, and in fact, necessary, to evaluate the triggering rule at discrete sampling instants, leading to so-called *periodic event-triggered control* (PETC), see [6], [7].

When CETC controllers are directly sampled without further adjustments or re-design, the stability properties guaranteed by CETC are preserved only in a semiglobal and practical manner provided the sampling period of the triggering rule is sufficiently small. Therefore, research on PETC has concentrated on overcoming two limitations of the results in [8] by: (i) preserving *global asymptotic* properties; (ii) providing *explicit* bounds on the sampling period at which the triggering condition is evaluated. Most works along these lines are dedicated to linear systems, see for instance [6], [9]–[11]. PETC results on nonlinear systems are more rare, with some exceptions given in [7, Chapter 6.5], [12]–[14]. In [7, Chapter 6.5] and [12], it is explained how to convert general continuous *state-feedback* event-triggered controllers to periodic event-triggered ones, while (approximately) preserving the properties of the former. The work in [13] develops observer-based output-feedback controllers for a class of nonlinear Lipschitz systems and a practical stability property is ensured at the end. In [14], output-feedback PETC is designed for a class of polynomial nonlinear systems to ensure a global asymptotic stability. Obviously, PETC for nonlinear systems is at its early stage and a lot remains to be done. In particular, there is a need for systematic design frameworks, which are flexible enough to cope with output feedbacks as well as exogenous disturbances. The primary aim of this paper is to address this challenge.

In this paper, we aim at designing periodic event-triggered controllers for nonlinear systems, which are applicable in the context of output feedback control and which are robust to exogenous disturbances. The emulation approach is pursued for this purpose. We thus assume that an output feedback control law is given and ensures an input-to-state stability property for the closed-loop system in the absence of network. At this stage, any continuous-time design techniques can be applied. We next implement the controller over the network. Based on assumptions we make on the original closed-loop system, we design the event-triggered rule and we provide an explicit bound on the maximum allowable sampling period (MASP) at which the former is evaluated. We model the overall system as a hybrid system using the formalism of [15], [16] and study its input-to-state stability. Afterwards, we show how to apply the results to a class of nonlinear Lipschitz systems and in which case the required conditions are formulated as a linear matrix inequality (LMI).

This work extends our previous study in [17], where PETC with state-feedbacks for unperturbed systems was addressed, to output feedback control and we deal with exogenous

This work is supported by the Australian Research Council under the Discovery Project DP170104099, the ANR under the grant COMPACS (ANR-13- BS03-0004-02), and the Innovational Research Incentives Scheme under the VICI grant "Wireless control systems: A new frontier in automation" (No. 11382) awarded by NWO (The Netherlands Organisation for Scientific Research) and STW (Dutch Technology Foundation).

W. Wang and D. Nešić are with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, 3010, Victoria, Australia wei.wang@unimelbedu.au, dnesic@unimelb.edu.au

R. Postoyan is with the Université de Lorraine, CNRS, CRAN, F-54000 Nancy, France romain.postoyan@univ-lorraine.fr

W.P.M.H. Heemels is with the Department of Mechanical Engineering, Eindhoven University of Technology, Eindhoven, The Netherlands, m.heemels@tue.nl

disturbances here. Compared to [7, Chapter 6.5] and [12], we address output feedback control and consider disturbances, in addition, our results explicitly reveal a link between the triggering condition and the sampling instants. Compared to [13], we consider exogenous disturbances, we do not restrict our attention to nonlinear systems with a specific structure, and we ensure global asymptotic stability in the absence of perturbations. Our results are consistent with time-triggered output-based results for NCS [18], as we recover [18] as a special case.

In the journal version of this work [19], we investigate a decentralized scenario in which several networks, with their own triggering mechanisms, are used to connect the plant and the controller. The two main contributions with respect to [19] are that: (i) we provide tailored results when there is a single network and (ii) we present case studies on Lipschitz nonlinear systems for which the results can be applied when a LMI holds.

The paper is organized as follows. The notation and preliminaries are given in Section II. We state the problem and present the hybrid model in Section III. The main results are provided in Section IV and the case study is given in Section V. Simulation results for a nonlinear system are given in Section VI and conclusions are summarized in Section VII. The proofs are omitted due to space limitations.

II. PRELIMINARIES

Let $\mathbb{Z}_{>0} := \{1, 2, \ldots\}$, $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \ldots\}$, $\mathbb{R} := (-\infty, \infty)$ and $\mathbb{R}_{\geq 0} := [0, \infty)$. Let $|x|$ denote the Euclidean norm of the vector $x \in \mathbb{R}^n$. The notation I_n stands for the identity matrix of dimension n. Let $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the minimum and maximum eigenvalues of real symmetric positive definite matrix P, respectively. For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, (x, y) stands for $[x^T, y^T]^T$. Given a set $\mathcal{A} \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define the distance of x to $\mathcal{A}$ as $|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|$. For $x, v \in \mathbb{R}^n$ and locally Lipschitz $U : \mathbb{R}^n \to \mathbb{R}$, $U^\circ(x; v)$ is the Clarke derivative of the function U at x in the direction v, *i.e.* $U^\circ(x; v) := \limsup_{y \to x, \lambda \downarrow 0} \dfrac{U(y + \lambda v) - U(y)}{\lambda}$. This notion will be useful as we will be working with locally Lipschitz Lyapunov functions, which are not differentiable everywhere.

Consider the following hybrid system [15], [16]

$$\begin{aligned} \dot{q} &= \mathcal{F}(q, w) \quad & q \in C \\ q^+ &\in \mathcal{G}(q) \quad & q \in D, \end{aligned} \tag{1}$$

where $q \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^m$ is the input, $C, D \subset \mathbb{R}^n$ are respectively the flow and the jump sets. We assume that the sets C and D are closed, $\mathcal{F} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function, $\mathcal{G} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semi-continuous and locally bounded, and $\mathcal{G}(q)$ is nonempty for each $q \in D$. Solutions to (1) are defined in [15], [16].

We study input-to-state stability (ISS) for system (1), as defined next.

Definition 1: Set $\mathcal{S} \subset \mathcal{X}$ is input-to-state stable (ISS) for system (1) if there exist $\beta \in \mathcal{KL}$ and $\widetilde{\psi} \in \mathcal{K}_\infty$ such that any solution pair (φ, w) satisfies[1] $|\varphi(t, j)|_{\mathcal{S}} \leq \beta(|\varphi(0, 0)|_{\mathcal{S}}, t + j) + \widetilde{\psi}(||w||_\infty)$ for all $(t, j) \in \text{dom } \varphi$. We say that $\mathcal{S}$ is exponentially ISS with a linear gain when $\beta(s_1, s_2) = ks_1 \exp(-cs_2)$ and $\widetilde{\psi}(s) = \gamma s$ for some $k, c, \gamma > 0$ and any $s_1, s_2, s \geq 0$. □

III. PETC SETUP

We consider the plant model

$$\begin{aligned} \dot{x}_p &= f_p(x_p, u, w) \\ y &= g_p(x_p), \end{aligned} \tag{2}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state, $w \in \mathbb{R}^{n_w}$ is the exogenous disturbance, $y \in \mathbb{R}^{n_y}$ is the plant output, and $u \in \mathbb{R}^{n_u}$ is the control input which is generated by the controller

$$\begin{aligned} \dot{x}_c &= f_c(x_c, y) \\ u &= g_c(x_c), \end{aligned} \tag{3}$$

with $x_c \in \mathbb{R}^{n_c}$ being the state of the controller. We assume that functions f_p and f_c are continuous, and g_p and g_c are continuously differentiable and zero at zero.

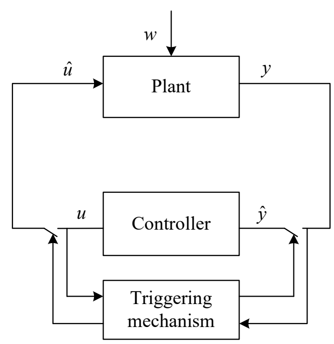

Fig. 1: Block diagram of the setup

We consider the scenario where plant (2) and controller (3) communicate with each other via a network, as illustrated in Figure 1. We assume that the triggering mechanism has access to both u and y at each sampling instant. This can be difficult to achieve in practice. Nevertheless, this formulation allows covering in a unified way the cases where only y or u is transmitted over the network, which is more realistic, as explained in Section 3.1 in [20]. We address the more general case, where y and u are sent separately via multiple networks with their own triggering mechanism in [19].

Define a sequence of sampling instants s_j, $j \in \mathbb{Z}_{\geq 0}$, which satisfy

$$\varepsilon \leq s_{j+1} - s_j \leq T \tag{4}$$

with $T > 0$ being the upper bound on the sampling period and $\varepsilon \in (0, T]$ being the minimum time between two successive evaluations of the triggering condition. Parameter ε reflects the minimum achievable transmission interval given by the hardware constraints. At each sampling instant s_i, the triggering mechanism (i) collects the current value of the

[1] See Section 2.1 in [15] for the definition of $||w||_\infty$.

control signal u and the output measurement y, (ii) compares these with their respective values at the last transmission instant, and (iii) decides whether a transmission over the network is needed. The design of the triggering condition and of the sampling instants is addressed in Section IV-B. Consequently, the sequence of transmission instants, t_i, $i \in \mathcal{I} \subseteq \mathbb{Z}_{\geq 0}$, is a subsequence of s_j, $j \in \mathbb{Z}_{\geq 0}$, and two successive transmissions are spaced by at least ε units of time in view of (4), thereby avoiding Zeno phenomena.

In this context, plant (2) no longer has access to u, but to its networked version $\hat{u}$. Similarly, controller (3) has access to $\hat{y}$, the networked version of y. We let $v := (y, u) \in \mathbb{R}^{n_y+n_u}$ and $\hat{v} := (\hat{y}, \hat{u})$. Zero-order-hold devices are used for implementation. Hence, $\dot{\hat{v}} = 0$, for almost all $t \in (s_j, s_{j+1})$ and $j \in \mathbb{Z}_{\geq 0}$.

We now introduce the network-induced error $e := \hat{v} - v \in \mathbb{R}^{n_e}$, where $n_e := n_y + n_u$. At each sampling instant s_j, $j \in \mathbb{Z}_{\geq 0}$, a triggering criterion defined through the function Υ, which depends on v and e, is evaluated. The expression of Υ is given in Section IV-B. A transmission is triggered depending on the sign of Υ, which leads to the update law for $\hat{v}$ given by

$$\hat{v}(s_j{}^+) \in \begin{cases} v(s_j) & \text{when } \Upsilon(e(s_j), v(s_j)) > 0 \\ \hat{v}(s_j) & \text{when } \Upsilon(e(s_j), v(s_j)) < 0 \\ \{\hat{v}(s_j), v(s_j)\} & \text{when } \Upsilon(e(s_j), v(s_j)) = 0, \end{cases} \tag{5}$$

where $x := (x_p, x_c) \in \mathbb{R}^{n_x}$ and $n_x := n_p + n_c$. We can see from (5) that no transmission occurs when Υ is strictly negative and a transmission is triggered when it is strictly positive. When Υ is equal to 0, both situations can arise as a transmission may occur or not. This construction ensures that the jump map given above is outer semi-continuous, which is essential for the hybrid model presented below to be (nominally) well-posed, see Chapter 6 in [16] for more details. We deduce from (5) that the variable e satisfies

$$e(s_j{}^+) \in h\left(x(s_j), e(s_j)\right), \tag{6}$$

where $h(x, e) := \big(1 - \Gamma(e, v)\big)e$. The function $\Gamma : \mathbb{R}^{2n_e} \rightrightarrows \{0, 1\}$ indicates whether a transmission occurs. Based on the discussion above, $\Gamma(e, v) = \{1\}$ when $\Upsilon(e, v) > 0$, which corresponds to a transmission and $h(x, e) = 0$ in this case. When $\Upsilon(e, v) < 0$, $\Gamma(e, v) = \{0\}$ and this corresponds to no transmission and $h(x, e) = e$. When $\Upsilon(e, v) = 0$, $\Gamma(e, v) = \{0, 1\}$ covers the above two possibilities. In agreement with [21], we call (6) the *protocol map*.

We now write the impulsive model given above as a hybrid system using the formalism of [15], [16]. We introduce for this purpose a clock variable $\tau \in \mathbb{R}_{\geq 0}$ to keep track of the time elapsed since the last evaluation of the triggering criterion. We model the overall closed-loop system as hybrid system (1) with letting $q := (x, e, \tau)$,

$$\begin{aligned} C &:= \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times [0, T] \\ D &:= \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times [\varepsilon, T]. \end{aligned} \tag{7}$$

The mapping F in (1) is defined as, for $q \in C$, $F(q, w) := (f(x, e, w), g(x, e, w), 1)$, where $f(x, e, w) := \begin{pmatrix} f_p(x_p, g_c(x_c) + e_u, w) \\ f_c(x_c, g_p(x_p) + e_y) \end{pmatrix}$, $g(x, e, w) := -\dfrac{\partial \bar{g}}{\partial x} f(x, e, w)$, and $\bar{g}(x) := \big(g_p(x_p), g_c(x_c)\big)$ with g_p and g_c coming from (2) and (3), respectively.

The set-valued mapping G is defined, for $q \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}_{\geq 0}$, as $G(q) := (x, h(x, e), \tau)$ with $h(x, e)$ from (6). The map G describes how e jumps when a transmission occurs. In particular, e is reset to 0 when an event is triggered, otherwise it keeps the same value. The timer τ is always reset to 0.

Our objective is to design triggering function Υ and to provide an explicit bound on the sampling period T to ensure an input-to-state stability property for the system defined by (1) and (7).

IV. Main Results

In this section, we first state the assumption we make on the system defined by (1) and (7), based on which we then construct the triggering function Υ and the bound on T. Finally, we present the stability guarantees.

A. Assumptions

We assume that the system defined by (1) and (7) satisfies the following.

Assumption 1: There exist a locally Lipschitz function $W : \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$, a continuous function $H : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$, $\underline{\alpha}_W, \overline{\alpha}_W, \varrho_W \in \mathcal{K}_\infty$ and $L_W \geq 0$ such that:

(i) For all $e \in \mathbb{R}^{n_e}$, $\underline{\alpha}_W(|e|) \leq W(e) \leq \overline{\alpha}_W(|e|)$.
(ii) For almost all $e \in \mathbb{R}^{n_e}$ and all $(x, w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_w}$, $\langle \nabla W(e), g(x, e, w) \rangle \leq L_W W(e) + H(x, e) + \varrho_W(|w|)$. □

Function W in Assumption 1 can be interpreted as a Lyapunov function for the e-system. In particular, item (ii) of Assumption 1 is an exponential growth condition of W on flows, which is commonly used in the related literature, see for instance [4], [21], [22].

We assume that controller (3) has been designed to robustly stabilize system (2) in the following sense.

Assumption 2: There exist a locally Lipschitz function $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, $\underline{\alpha}_V, \overline{\alpha}_V, \alpha_V, \alpha_W, \varrho_V \in \mathcal{K}_\infty$, locally Lipschitz functions $\delta : \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ satisfying $\delta(0) = 0$, and $J : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_w} \to \mathbb{R}_{\geq 0}$, $\gamma > 0$ and $L_\delta \in \mathbb{R}$ such that the following holds.

(i) For all $x \in \mathbb{R}^{n_x}$, $\underline{\alpha}_V(|x|) \leq V(x) \leq \overline{\alpha}_V(|x|)$.
(ii) For almost all $x \in \mathbb{R}^{n_x}$ and all $(e, w) \in \mathbb{R}^{n_e} \times \mathbb{R}^{n_w}$, $\langle \nabla V(x), f(x, e, w) \rangle \leq -\alpha_V(|x|) - \alpha_W(|e|) + \gamma^2 W^2(e) - H^2(x, e) - J(x, e, w) - \delta(v) + \varrho_V(|w|)$, where W and H come from Assumption 1.
(iii) For almost all $x \in \mathbb{R}^{n_x}$ and all $(e, w) \in \mathbb{R}^{n_e} \times \mathbb{R}^{n_w}$, $\langle \nabla \delta(v), -g(x, e, w) \rangle \leq L_\delta \delta(v) + H^2(x, e) + J(x, e, w)$, where g is defined below (7). □

Function V is a Lyapunov function used to prove the stability of the closed-loop system (2)-(3) in the absence of the network. In particular, items (i) and (ii) of Assumption 2 imply that controller (3) robustly stabilizes system (2) with respect to the network-induced error e and the perturbation w. Indeed, when there is no network ($e = 0$), we derive from item (ii) of Assumption 2 and item (i) of Assumption 1 that

$\langle \nabla V(x), f(x,0,w)\rangle \leq -\alpha_V(|x|) + \varrho_V(|w|)$, which means that the closed-loop system defined by (2) and (3) is ISS since V is positive definite and radially unbounded according to item (i) of Assumption 2. Function δ in Assumption 2 will be used in the event-triggering condition design and item (iii) is an exponential growth condition on δ. In Section V, we formulate these conditions as LMIs for two classes of nonlinear systems.

B. Triggering mechanism

We define Υ in (5), for $v, e \in \mathbb{R}^{n_e}$, as

$$\Upsilon(e,v) = \gamma W^2(e) - \rho(\lambda)\delta(v), \tag{8}$$

where W and δ come from Assumptions 1 and 2, respectively, $\rho(\lambda) := \dfrac{\gamma\lambda^2}{1-L_\delta\lambda}$ with $L_\delta \in \mathbb{R}$, $\gamma > 0$ coming from Assumption 2, and $\lambda \geq 0$ is a design parameter. The denominator in the definition of $\rho(\lambda)$ is guaranteed to be strictly positive in the following by suitably selecting λ. The triggering condition (8) is similar to those proposed in [1], [2] for continuous event-triggering control in different contexts.

We select λ such that $\lambda \in [0, \lambda^*)$ where λ^* is defined as

$$\lambda^* := \begin{cases} 1 & \text{when } L_\delta \leq -\gamma \\ \min\left\{1, \dfrac{1}{L_\delta+\gamma}\right\} & \text{when } L_\delta > -\gamma. \end{cases} \tag{9}$$

Given $\lambda \in [0,\lambda^*)$, we select T in (7) such that $T < T_{\mathrm{MASP}}(\lambda)$, where $T_{\mathrm{MASP}}(\lambda)$ is defined as

$$T_{\mathrm{MASP}}(\lambda) := \begin{cases} \dfrac{1}{L_W r}\arctan(\vartheta), & \text{when } \gamma > L_W \\ \dfrac{1}{L_W}\dfrac{1-\overline{\rho}(\lambda)}{1+\overline{\rho}(\lambda)}, & \text{when } \gamma = L_W \\ \dfrac{1}{L_W r}\operatorname{arctanh}(\vartheta), & \text{when } \gamma < L_W \end{cases} \tag{10}$$

where $\overline{\rho}(\lambda) := \dfrac{\gamma\lambda}{1-L_\delta\lambda}$, $r := \sqrt{\left|\left(\dfrac{\gamma}{L_W}\right)^2 - 1\right|}$, $\vartheta := \dfrac{r(1-\overline{\rho}(\lambda))}{2\frac{\overline{\rho}(\lambda)}{1+\overline{\rho}(\lambda)}\left(\frac{\gamma}{L_W}-1\right)+1+\overline{\rho}(\lambda)}$, $L_W \geq 0$ and $\gamma > 0$ come from Assumptions 1 and 2, respectively. Note that $T_{\mathrm{MASP}}(\lambda) > 0$ for all $\lambda \in [0,\lambda^*)$ as $\overline{\rho}(\lambda) \in [0,1)$, which holds since $0 \leq \overline{\rho}(\lambda) = \dfrac{\gamma\lambda}{1-L_\delta\lambda} \leq \dfrac{\gamma\lambda}{1+\gamma\lambda} < 1$ when $L_\delta \leq -\gamma$, and $0 \leq \gamma\lambda < 1 - L_\delta\lambda$ as $\lambda < \dfrac{1}{L_\delta+\gamma}$ when $L_\delta > -\gamma$ according to (9).

The bound in (10) is decreasing in λ. In other words, the larger the λ, the smaller T_{MASP} and vice versa.

Remark 1: The expression of T_{MASP} in (10) agrees with [18] if we replace $\overline{\rho}(\lambda)$ by a parameter $\rho \in (0,1)$. In [18], multiple nodes are scheduled by a protocol to transmit their data and ρ characterizes the stability properties of the protocol and it typically depends on the number of nodes in the network. Here, $\overline{\rho}(\lambda)$ is introduced to incorporate the influence of the event-triggering parameter λ. Indeed, when $\lambda = 0$, the triggering function Υ is always non-negative. Consequently, transmissions occur at every sampling instant according to (5). We then recover the time-triggered results as a special case, and the bound on the *maximal allowable transmission interval* (MATI) is the same as in [18] when $\rho = 0$ (which corresponds to the case that there is only one node) and there are no disturbances, i.e. $w = 0$. □

C. Input-to-state stability

We are ready to state the main result about the input-to-state stability.

Theorem 1: Consider the system defined by (1) and (7). Suppose that Assumptions 1 and 2 hold. Let $\lambda \in [0,\lambda^*)$ and $T < T_{\mathrm{MASP}}(\lambda)$, where λ^* and $T_{\mathrm{MASP}}(\lambda)$ are defined in (9) and (10), respectively. Then, set $\mathcal{A} := \{\xi \in C \cup D : x = 0, e = 0, \tau \in [0,T]\}$ is ISS. □

Theorem 1 states that set $\mathcal{A}$ is ISS, hence, (i) x and e globally converge to a neighborhood of the origin whose "size" depends on the $\mathcal{L}_\infty$ norm of w; (ii) the set $\mathcal{A}$ is uniformly globally asymptotically stable [16, Definition 3.6] when $w = 0$.

The next statement ensure an exponential ISS property by strengthening the conditions of Theorem 1.

Corollary 1: Consider the system defined by (1) and (7). Suppose that Assumptions 1 and 2 hold with $\underline{\alpha}_W(s) = \underline{a}_W s$, $\overline{\alpha}_W(s) = \overline{a}_W s$, $\underline{\alpha}_V(s) = \underline{a}_V s^2$, $\overline{\alpha}_V(s) = \overline{a}_V s^2$ and $\alpha_V(s) = a_V s^2$ and $\alpha_W(s) = a_W s^2$ for $s \geq 0$. Let $\lambda \in [0,\lambda^*)$ and $T < T_{\mathrm{MASP}}(\lambda)$, where λ^* and $T_{\mathrm{MASP}}(\lambda)$ are defined in (9) and (10), respectively. Then, set $\mathcal{A}$ defined in Theorem 1 is exponentially ISS with a linear gain. □

V. Case Study

A. Globally Lipschitz nonlinear systems

Consider the nonlinear plant

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u + D_p\psi(x_p) + E_p w \\ y &= C_p x_p, \end{aligned} \tag{11}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is the external disturbance, $y \in \mathbb{R}^{n_y}$ is the measured output, A_p, B_p, C_p, D_p and E_p are matrices of appropriate dimensions, (A_p, B_p) and (A_p, C_p) are stabilizable and detectable, respectively. Function $\psi : \mathbb{R}^{n_p} \to \mathbb{R}^r$, satisfies $\psi(0) = 0$ and

$$|\psi(x_1) - \psi(x_2)| \leq L|x_1 - x_2| \text{ for all } x_1, x_2 \in \mathbb{R}^{n_p} \tag{12}$$

and for some constant $L > 0$.

We focus on observer-based controllers of the form

$$\begin{aligned} \dot{x}_c &= A_p x_c + B_p u + D_p\psi(x_c) - M(C_p x_c - y) \\ u &= K x_c, \end{aligned} \tag{13}$$

where $x_c \in \mathbb{R}^{n_p}$ is the state estimate, M and K are matrices of appropriate dimensions such that $\mathcal{A} := \begin{bmatrix} A_p & B_p K \\ MC & A_p + B_p K - MC \end{bmatrix}$ is Hurwitz, which is always possible since (A_p, B_p) and (A_p, C_p) are stabilizable and detectable, respectively.

We now implement the controller (13) over the network, as explained in Section III. We obtain $v = D_v x$ with $D_v := \begin{bmatrix} C_p & 0 \\ 0 & K \end{bmatrix}$ and we derive hybrid system (1) with

$$\begin{aligned} f(x,e,w) &:= \mathcal{A}x + \mathcal{B}e + \mathcal{D}\overline{\psi}(x) + \mathcal{E}w \\ g(x,e,w) &:= -D_v f(x,e,w), \end{aligned} \tag{14}$$

where $\mathcal{B} := \begin{bmatrix} 0 & B_p \\ M & 0 \end{bmatrix}$, $\mathcal{D} := \begin{bmatrix} D_p & 0 \\ 0 & D_p \end{bmatrix}$, $\mathcal{E} := \begin{bmatrix} E_p \\ 0 \end{bmatrix}$, and $\overline{\psi}(x) := (\psi(x_p), \psi(x_c))$.

The next proposition provides a LMI, which if satisfied, ensures that the conditions of Corollary 1 hold.

Proposition 1: Suppose that there exist a positive definite symmetric matrix P, $a_V, a_W, \theta, \overline{a}_W, \epsilon > 0$ and $\eta \geq a_W$ such that the following LMI holds

$$\begin{bmatrix} \Sigma_{11} & \star & \star \\ \Sigma_{21} & \Sigma_{22} & \star \\ \Sigma_{31} & 0 & -\theta I_{n_w} + \overline{a}_W^2 \mathcal{E}^T D_v^T D_v \mathcal{E} \end{bmatrix} < 0, \tag{15}$$

where $\Sigma_{11} := \mathcal{A}^T P + P\mathcal{A} + 2L|\mathcal{D}|P + (a_V + \overline{\alpha}_W^2 L^2 |D_v \mathcal{D}|^2) I_{n_x} + \overline{\alpha}_W^2 (\mathcal{A}^T D_v^T D_v \mathcal{A} + 2L|D_v\mathcal{D}|\mathcal{A}) + \epsilon^2\big((2L|\mathcal{D}| + 1) D_v^T D_v + \mathcal{A}^T D_v^T D_v \mathcal{A}\big)$, $\Sigma_{21} := \mathcal{B}^T P + \epsilon^2 \mathcal{B}^T D_v^T D_v$, $\Sigma_{22} := -(\eta - a_W) I_{n_e}$ and $\Sigma_{31} := \mathcal{E}^T P + \epsilon^2 \mathcal{E}^T D_v^T D_v$. Then,

- Assumption 1 holds with $\underline{\alpha}_W(s) = \overline{\alpha}_W(s) = \overline{a}_W s$, $L_W = |D_v \mathcal{B}|$, $H(x,e) = \overline{a}_W(|D_v \mathcal{A} x| + L|D_v \mathcal{D}||x|)$ for all $x \in \mathbb{R}^{n_x}$ and $e \in \mathbb{R}^{n_e}$, and $\varrho_W(s) = \overline{a}_W |D_v \mathcal{E}| s$ for all $s \geq 0$.
- Assumption 2 holds with $V(x) = x^T P x$ and $\delta(v) = \epsilon^2 |v|^2$ for all $x \in \mathbb{R}^{n_x}$ and $v \in \mathbb{R}^{n_e}$, $\underline{\alpha}_V(s) := \lambda_{\min}(P) s^2$, $\overline{\alpha}_V(s) := \lambda_{\max}(P) s^2$ for all $s \geq 0$, $\gamma^2 = \eta - a_W$, $\alpha_V(s) = a_V s^2$, $\alpha_W(s) = a_W s^2$ and $\varrho_V(s) = \theta s^2$ for all $s \geq 0$, $L_\delta = 1$, $J(x,e,w) = 2\epsilon^2\big(|x^T D_v^T D_v \mathcal{B} e + x^T D_v^T D_v \mathcal{E} w| + (L|D_v^T D_v||\mathcal{D}| + \frac{1}{2}|\mathcal{A}^T D_v^T D_v \mathcal{A}|)|x|^2\big)$. □

A direct consequence of Proposition 1 is that, when (15) holds, set $\mathcal{A} := \{\xi \in C \cup D : x = 0, e = 0, \tau \in [0,T]\}$ is exponentially ISS with a linear gain for the system defined in (1) and (14) according to Corollary 1.

B. Systems with globally Lipschitz output injection terms

We consider system (11) in the case where $D_p = B_p$ and ψ is a function of the output signal. We will show that the conditions in Corollary 1 are verified as the obtained corresponding LMI is always feasible in this case. In this case, we consider the next controller instead of (13)

$$\begin{aligned} \dot{x}_c &= A_p x_c + B_p u + B_p \psi(y) - M(C_p x_c - y) \\ u &= K x_c - \psi(y). \end{aligned} \tag{16}$$

We follow similar lines to derive the hybrid system defined with (1) and (14) by letting $\mathcal{D} := \begin{bmatrix} B_p & 0 \\ 0 & 0 \end{bmatrix}$ and $\overline{\psi}(y,e) := \begin{bmatrix} \psi(y) - \psi(y + e_y) \\ 0 \end{bmatrix}$. We have the next result.

Proposition 2: There exist a positive definite symmetric matrix P, $a_V, a_W, \theta, \overline{a}_W, \epsilon > 0$ and $\eta \geq a_W - (2\overline{a}_W^2 + \epsilon^2) L^2 |D_v \mathcal{D}|^2 - L^2 |\mathcal{D}|^2 |P|^2 / a_V$ such that the following LMI holds

$$\begin{bmatrix} \Sigma_{11} & \star & \star \\ \Sigma_{21} & \Sigma_{22} & \star \\ \Sigma_{31} & 0 & -\theta I_{n_w} + \overline{a}_W^2 \mathcal{E}^T D_v^T D_v \mathcal{E} \end{bmatrix} < 0, \tag{17}$$

where $\Sigma_{11} := \mathcal{A}^T P + P\mathcal{A} + 2a_V I_{n_x} + 2\overline{a}_W^2 \mathcal{A}^T D_v^T D_v \mathcal{A} + 2\epsilon^2 D_v^T D_v + \epsilon^2 \mathcal{A}^T D_v^T D_v \mathcal{A}$, $\Sigma_{21} := \mathcal{B}^T P + \epsilon^2 \mathcal{B}^T D_v$ and $\Sigma_{22} := -\big(\eta - a_W - L^2|D_v\mathcal{D}|^2(2\overline{a}_W^2 + \epsilon^2) - L^2|\mathcal{D}|^2|P|^2/a_V\big) I_{n_e}$ and $\Sigma_{31} := \mathcal{E}^T P + \epsilon^2 \mathcal{E}^T D_v^T D_v$. Then,

- Assumption 1 holds with $\underline{\alpha}_W(s) = \overline{\alpha}_W(s) = \overline{a}_W s$, $L_W = |D_v \mathcal{B}|$, $H(x,e) = \overline{a}_W(|D_v \mathcal{A} x| + L|D_v \mathcal{D}||e|)$ for all $x \in \mathbb{R}^{n_x}$ and $e \in \mathbb{R}^{n_e}$, $\varrho_W(s) = \overline{a}_W |D_v \mathcal{E}| s$ for all $s \geq 0$.
- Assumption 2 holds with $V(x) = x^T P x$ and $\delta(v) = \epsilon^2 |v|^2$ for all $x \in \mathbb{R}^{n_x}$ and $v \in \mathbb{R}^{n_e}$, $\underline{\alpha}_V(s) = \lambda_{\min}(P) s^2$, $\overline{\alpha}_V(s) = \lambda_{\max}(P) s^2$ for all $s \geq 0$, $\gamma^2 = \big(\eta - a_W - (2\overline{a}_W^2 + \epsilon^2) L^2 |D_v \mathcal{D}|^2 - L^2 |\mathcal{D}|^2 |P|^2 / a_V\big) / \overline{a}_W^2$, $\alpha_V(s) = a_V s^2$, $\alpha_W(s) = a_W s^2$ and $\varrho_V(s) = \theta s^2$ for all $s \geq 0$, $L_\delta = 1$ and $J(x,e,w) = 2\epsilon^2 |x^T D_v^T D_v \mathcal{B} e + x^T D_v^T D_v \mathcal{E} w| + \epsilon^2 (L^2 |D_v \mathcal{D}|^2 |e|^2 + (|D_v^T D_v| + |\mathcal{A}^T D_v^T D_v \mathcal{A}|)|x|^2)$. □

Conditioon (17) is indeed always feasible since $\mathcal{A}$ is Hurwitz, which ensures $\Sigma_{11} < 0$ and (17) follows by taking sufficiently large θ, η and small enough $a_W, \epsilon > 0$. Then, we can conclude exponential ISS of the overall system using Corollary 1.

VI. Illustrative example

We consider a single-link robot arm as in [20] for which only the angle is measured. The model agrees with the one proposed in Section V-B, where $x_p \in \mathbb{R}^2$ is the state, with its two components denoting the angle and the rotational velocity, respectively, and w is the external disturbance, $A_p = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_p = [1 \;\; 0]$, $\psi(y) = \sin(y)$, $E_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The output feedback controller u comes from (16) with $M = [11 \;\; 30]$ and $K = [-2 \;\; -3]$, as given in [20].

We solve LMI (17) by selecting $\overline{a}_W = 0.1$, $\epsilon = 0.1$, $\nu = 5.08$, $a_V = a_W = 1$, $\theta = 20$. It follows from Proposition 2 that Assumptions 1 and 2 hold with $W(e) = 0.1|e|$, $V(x) = x^T P x$, where $P = \begin{bmatrix} 4.11 & -1.1 & 3.96 & -0.37 \\ -1.1 & 7.66 & -1.54 & 2.35 \\ 3.96 & -1.54 & 4 & -0.67 \\ -0.37 & 2.35 & -0.67 & 4.12 \end{bmatrix}$, $L_W = 3.2$, $L_\delta = 1$ and $\gamma = 127.1$. We then apply (9) to obtain $\lambda^* = 0.0078$ and we derive $T_{\text{MASP}}(\lambda)$ for each $\lambda \in (0, \lambda^*)$ using (10). Figure 2 illustrates the dependency of T_{MASP} as a function of λ. We consider different values of λ and T such that $T < T_{\text{MASP}}(\lambda)$ to illustrate the impact of λ and T on the number of transmissions. We have run 50 simulations over 10 seconds with initial conditions randomly

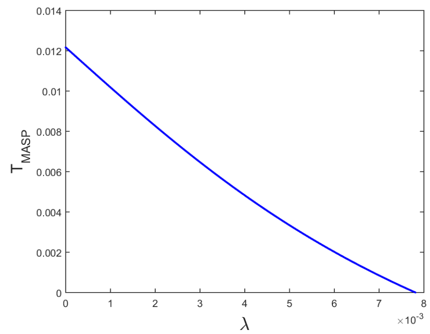

Fig. 2: Triggering parameter λ v.s. T_{MASP}

TABLE I: Average inter-transmission time

	Average inter-transmission time		
	$\lambda = 0.001$	$\lambda = 0.003$	$\lambda = 0.005$
$T = 0.0033$	0.0067	0.0093	0.0122
$T = 0.0065$	0.0118	0.0127	$\times$
$T = 0.0102$	0.0167	$\times$	$\times$

selected in $[-15, 15]$, $w = \sin(\pi t)$ and picked $\varepsilon = T$ so that the triggering rule is periodically evaluated, and the obtained average inter-transmission times are reported in Table I.

Empty boxes in Table I mean that the condition $T < T_{\mathrm{MASP}}(\lambda)$ is violated. We see from Table I that the average inter-transmission times increase when λ grows for the same sampling period T, and the same occurs when T grows for the same triggering parameter λ. This implies that, for this example and this set of simulations, setting sampling periods close to $T_{\mathrm{MASP}}(\lambda)$ uses less network bandwidth while ensuring stability.

It is important to point out that the setup we consider here is different from [20], where a triggering condition is continuously verified after waiting a fixed amount of time since the last transmission instant. In that case, a transmission is allowed to occur whenever the condition is violated. In contrast, we consider a more difficult circumstance as the triggering conditions are only checked at sampling instants and this might induce more transmission requests than [20].

VII. Conclusions

We considered periodic event-triggered control for networked nonlinear control systems subject to exogenous disturbances. An emulation-based design procedure was proposed. We started with a continuous-time controller which robustly stabilizes a continuous-time plant in the absence of communication constraints. In the next step, the controller was implemented over a communication network and transmissions were triggered when a criterion is satisfied at given discrete sampling instants. We derived a hybrid system model to describe the overall system and proposed a novel Lyapunov function to study its stability properties. We provided conditions on the controller, the event-triggering criterion and the explicit bound on the maximum allowable sampling periods, to ensure an input-to-state stability property. The effectiveness of the scheme was illustrated on simulations for a nonlinear example.

References

[1] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control*, vol. 52, pp. 1680–1685, 2007.

[2] V. Dolk, D. Borgers, and W. Heemels, "Output-based and decentralized dynamic event-triggered control with guaranteed L_p-gain performance and Zeno-freeness," *IEEE Transactions on Automatic Control*, vol. 62, pp. 34–49, 2017.

[3] J. Lunze and D. Lehmann, "A state-feedback approach to event-based control," *Automatica*, vol. 46, pp. 211–215, 2010.

[4] X. Wang and M. Lemmon, "Event-triggering in distributed networked control systems," *IEEE Transactions on Automatic Control*, vol. 56, pp. 586–601, 2011.

[5] R. Postoyan, P. Tabuada, D. Nešić, and A. Anta, "A framework for the event-triggered stabilization of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 60, pp. 982–996, 2015.

[6] W. Heemels, M. Donkers, and A. Teel, "Periodic event-triggered control for linear systems," *IEEE Transactions on Automatic Control*, vol. 58, pp. 847–861, 2013.

[7] W. Heemels, R. Postoyan, M. Donkers, A. Teel, A. Anta, P. Tabuada, and D. Nešić, "Periodic event-triggered control," in *Event-based control and signal processing*, M. Miskowicz, Ed. CRC Press/Taylor & Francis, 2015, pp. 105–119.

[8] R. Sanfelice and A. Teel, "Lyapunov analysis of sampled-and-hold hybrid feedbacks," in *Proceedings of the 45th IEEE Conference on Decision and Control*, San Diego, USA, 2006, pp. 4879–4884.

[9] X. Chen and F. Hao, "Periodic event-triggered state-feedback and outputfeedback control for linear systems," *International Journal of Control, Automation and Systems*, vol. 13, pp. 779–787, 2015.

[10] A. Selivanov and E. Fridman, "Event-triggered H_∞ control: a switching approach," *IEEE Transactions on Automatic Control*, vol. 61, pp. 3221–3226, 2016.

[11] M. Braksmayer and L. Mirkin, "Redesign of stabilizing discrete-time controllers to accommodate intermittent sampling," in *Proceedings of the 2017 IFAC World Congress*, Toulouse, France, 2017, pp. 2633–2638.

[12] D. P. Borgers, R. Postoyan, A. Anta, P. Tabuada, D. Nešić, and W. Heemels, "Periodic event-triggered control of nonlinear systems using overapproximation techniques," *Automatica*, https://doi.org/10.1016/j.automatica.2018.04.019.

[13] L. Etienne, S. D. Gennaro, and J. Barbot, "Periodic event-triggered observation and control for nonlinear Lipschitz systems using impulsive observers," *international Journal of Robust and Nonlinear Control*, vol. 27, pp. 4363–4380, 2017.

[14] E. A. Escolástico, M. Abdelrahim, M. Guinaldo, S. Dormido, and W. Heemels, "Design of periodic event-triggered control for polynomial systems: A delay system approach," in *Proceedings of the 2017 IFAC World Congress*, Toulouse, France, 2017, pp. 7887–7892.

[15] C. Cai and A. R. Teel, "Characterizations of input-to-state stability for hybrid systems," *Systems & Control Letters*, vol. 58, pp. 47–53, 2009.

[16] R. Goebel, R. Sanfelice, and A. Teel, *Hybrid dynamical systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[17] W. Wang, R. Postoyan, D. Nešić, and W. Heemels, "Stabilization of nonlinear systems using state-feedback periodic event-triggered controllers," in *Proceedings of the 55th IEEE Conference on Decision and Control*, Las Vegas, NV, USA, 2016, pp. 6808–6813.

[18] D. Carnevale, A. Teel, and D. Nešić, "A Lyapunov proof of an improved maximum allowable transfer interval for networked control systems," *IEEE Transactions on Automatic Control*, vol. 52, pp. 892–897, 2007.

[19] W. Wang, R. Postoyan, D. Nešić, and W. Heemels, "Robust periodic event-triggered control of nonlinear systems," submitted for journal publication.

[20] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nešić, "Robust event-triggered output feedback controllers for nonlinear systems," *Automatica*, vol. 75, pp. 96–108, 2017.

[21] D. Nešić and A. Teel, "Input-output stability properties of networked control systems," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1650–1667, 2004.

[22] R. Postoyan, N. van de Wouw, D. Nešić, and W. Heemels, "Tracking control for nonlinear networked control systems," *IEEE Transactions on Automatic Control*, vol. 59, pp. 1539–1554, 2014.