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Stability of nonlinear systems with two time scales over a single communication channel*

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***Abstract*— This paper studies the stabilisation problem for a class of nonlinear systems with two time scales, where only a single communication channel is available to allocate both low and high-frequency transmissions from slow and fast subsystems, respectively. A clock mechanism is proposed to govern the transmissions, and the closed-loop system is modelled by a hybrid singularly perturbed system. Singular perturbation-based analysis is used to obtain individual maximum allowable transmission intervals for both slow and fast transmissions, and also to guarantee semi-global practical asymptotic stability with respect to the minimum allowable transmission interval of slow transmissions. We illustrate the results via a numerical example.**

I. INTRODUCTION

Networked control systems (NCSs) are feedback control systems whose loops are closed through real-time communication networks. The rapid development of network technologies provides NCSs with widespread Internet of Things (IoT) application scenarios where two or more time scales are often involved, such as manufacturing automation, smart transportation, telemedicine and space and terrestrial exploration [1]. Most of the state-of-the-art NCS design methodologies, e.g., [2], [3], [4], do not directly exploit the multiple-scale structure due to the oversimplified system models, as a consequence, they potentially require excessive transmission rates for stability. As IoT devices are often wireless and battery-supported, with limited resources such as bandwidth, a high transmission rate for the overall system may be infeasible. By exploiting the multiple time-scale property of the system with the singularly perturbed method [5], it is possible to obtain stability and robustness guarantees while mitigating redundant transmissions for the slow dynamics. Singularly Perturbed NCSs (SPNCSs) have garnered significant attention in recent years due to their practical importance in various engineering applications. Researchers have proposed a few control and analysis approaches for both linear and nonlinear SPNCSs. For instance, [6] and [7] proposed sliding mode control strategies for linear discrete-time SPNCSs, while [8] explored ultimate boundedness control for linear discrete SPNCSs with communication constraints and deception attacks. In the case of nonlinear SPNCSs, [9] developed a stabilizing event-triggered feedback law for SPNCSs with fast plant dynamics assumed to be stable. Meanwhile, [10] introduced sufficient conditions that guarantee stability for time-triggered nonlinear SPNCSs, where only plant output is transmitted via network. More precisely, [9] and [10] adopted an emulation-based approach proposed in [2] to design the NCS. That is, a controller is initially designed to guarantee stability in the absence of communication constrains. Then, an event/time-triggered condition is determined to preserve the stability of the closed-loop system when implemented over the network. Additionally, they cast the overall problem as a hybrid SPS with the formalism of [11]. While [9] only transmit slow states by assuming stable fast plant dynamics, [10] made no assumption on the stability of the plant. Moreover, it required two separate channels to transmit fast and slow states respectively, which may be restrictive since two separate channels may not be available in practice. These studies highlight the importance of considering both communication constraints and the two time scale nature of SPNCSs in developing effective control strategies for practical applications.

In this paper, we propose an emulation-based approach for the design of two-time-scale nonlinear SPNCS, where only a single channel is available to transmit both slow and fast signals, which is easier to implement in practice. Our approach leads to three main contributions. Firstly, we introduce a mechanism, which reduces the demand on resources. It involves two clocks to govern the transmissions of slow and fast signals over a single channel. Secondly, we present a stability analysis of hybrid SPS when its flow and jump sets depend on the time scale separation parameter, which is commonly denoted by ϵ in the literature. Thirdly, we consider more general nonlinear NCS scenarios than those considered in [10]. While only plant states are transmitted in [10], we consider the transmission of both plant output and controller input in our NCS. In addition to the plant and state-feedback controller form introduced in [10], we consider a dynamical output feedback controller, which draws inspiration from the linear works [12], [13] and the nonlinear work [14] that study dynamic controllers for SPSs in the absence of network. We also consider NCS with scheduling protocols, whereas [7] and [9] only assumed the sampled-data structure. Moreover, while [6]-[9] assumed periodic transmissions, we allow transmissions to be aperiodic. As a summary, our proposed approach provides a more general and flexible framework for designing SPNCSs

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with single communication channel.

Notation: The sets of real numbers and integers larger than or equal to a real number n are denoted by $\mathbb{R}_{\geq n}$ and $\mathbb{Z}_{\geq n}$, respectively. For vectors $v_i \in \mathbb{R}^n$, $i \in \{1, 2, \cdots, N\}$, we denote the vector $[v_1^T \ v_2^T \ \cdots \ v_N^T]^T$ by $(v_1, v_2, \cdots, v_N)$, and the inner product by $\langle \cdot, \cdot \rangle$. Given a vector $x \in \mathbb{R}^{n_x}$ and a non-empty closed set $\mathcal{A} \subseteq \mathbb{R}^{n_x}$, the distance from x to $\mathcal{A}$ is denoted by $|x|_{\mathcal{A}}$, and it is defined by $|x|_{\mathcal{A}} := \min_{y \in \mathcal{A}} |x - y|$. We use U° to denote the Clarke generalized derivative [15, Eqn. (20)] of a function U. We denote the n by m zero matrix by $\mathbf{0}_{n \times m}$.

II. Problem Setting

In this paper, we consider a two-time-scale nonlinear NCS that is designed by an emulation-based approach. Specifically, a dynamical output-feedback controller is initially designed to guarantee the stability of a *reduced system* (slow) and a *boundary-layer system* (fast) in the absence of a network. Subsequently, this controller is implemented over the network, and the design consists in selecting two different *maximum allowable transmission intervals* (MATIs) to preserve the stability of the networked reduced and boundary-layer systems, respectively. Figure 1 illustrates the configuration of the considered time-triggered SPNCS. Subsequent sections will elaborate on each individual element.

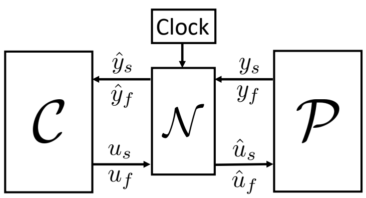

Fig. 1: NCS Block Diagram

Plant model ($\mathcal{P}$): We model the plant as the following singularly perturbed system, parameterized by $0 < \epsilon \ll 1$,

$$\mathcal{P}: \begin{cases} \dot{x}_p &= f_p(x_p, z_p, \hat{u}) \\ \epsilon \dot{z}_p &= g_p(x_p, z_p, \hat{u}) \\ y_p &= (y_s, y_f) = \big(k_{p_s}(x_p), k_{p_f}(x_p, z_p)\big), \end{cases} \tag{1}$$

where $x_p \in \mathbb{R}^{n_{x_p}}$ and $z_p \in \mathbb{R}^{n_{z_p}}$ denote the slow and fast plant states, respectively, while $y_s \in \mathbb{R}^{n_{y_s}}$ and $y_f \in \mathbb{R}^{n_{y_f}}$ represent the slow and fast outputs. Additionally, $\hat{u} = (\hat{u}_s, \hat{u}_f)$ refers to the latest received control input from the network. It is assumed that k_{p_s} and k_{p_f} are continuously differentiable, and f_p and g_p are locally Lipschitz in their arguments.

Controller ($\mathcal{C}$): We consider a class of dynamic controllers of the form

$$\mathcal{C}: \begin{cases} \dot{x}_c &= f_c(x_c, z_c, \hat{y}_p) \\ \epsilon \dot{z}_c &= g_c(x_c, z_c, \hat{y}_p) \\ u &= (u_s, u_f) = \big(k_{c_s}(x_c), k_{c_f}(x_c, z_c)\big), \end{cases} \tag{2}$$

where ϵ comes from (1), and $x_c \in \mathbb{R}^{n_{x_c}}$ and $z_c \in \mathbb{R}^{n_{z_c}}$ denote the slow and fast controller states, respectively. Additionally, $u_s \in \mathbb{R}^{n_{u_s}}$ and $u_f \in \mathbb{R}^{n_{u_f}}$ are control inputs that depend on the slow and fast controller states, respectively, while $\hat{y}_p = (\hat{y}_s, \hat{y}_f)$ refers to the most recently received output of the plant transmitted after the network. Lastly, k_{c_s} and k_{c_f} are continuously differentiable, and f_c and g_c are locally Lipschitz in their arguments. We also assume that $n_{y_s} + n_{u_s} \in \mathbb{Z}_{\geq 1}$ and $n_{y_f} + n_{u_f} \in \mathbb{Z}_{\geq 1}$, which ensures that both fast and slow signals are present in the system.

Description of communication ($\mathcal{N}$): To provide a mathematical representation of the single channel network, we first assume that the slow inputs and outputs will never be transmitted at the same instant as the fast inputs and outputs. Define $\mathcal{T}^s := \{t_0^s, t_1^s, t_2^s, \cdots\}$ as the unbounded set of transmission times at which slow inputs and outputs are transmitted, and $\mathcal{T}^f := \{t_0^f, t_1^f, t_2^f, \cdots\}$ as the unbounded set of transmission times at which fast inputs and outputs are transmitted, such that $\mathcal{T}^s \cap \mathcal{T}^f = \emptyset$. Then, let $\mathcal{T} := \mathcal{T}^s \cup \mathcal{T}^f = \{t_0, t_1, t_2, \cdots\}$ denote the set of all transmission instances, with its elements arranged in ascending time order. Moreover, we assume that the transmission intervals satisfy

$$\begin{aligned} \tau_{\text{miati}}^s \leq t_{k+1}^s - t_k^s \leq \tau_{\text{mati}}^s, \ \forall t_k^s, t_{k+1}^s \in \mathcal{T}^s, k \in \mathbb{Z}_{\geq 0}, \\ \tau_{\text{miati}} \leq t_{\ell+1} - t_\ell \leq \tau_{\text{mati}}, \ \forall t_\ell, t_{\ell+1} \in \mathcal{T}, \ell \in \mathbb{Z}_{\geq 0}, \end{aligned} \tag{3}$$

where $0 < \tau_{\text{miati}} \leq \tau_{\text{mati}}$ denote, respectively, the *minimum allowable transmission interval* (MIATI) and MATI between any two consecutive transmissions. Similarly, τ_{miati}^s and τ_{mati}^s are the MIATI and MATI between two consecutive slow updates. Note that, $\tau_{\text{miati}} \leq \frac{1}{2}\tau_{\text{mati}}^s$, $\tau_{\text{mati}} < \tau_{\text{miati}}^s$ and (3) prevent successive transmissions of slow inputs/outputs in $\mathcal{T}$, which implies fast transmissions exist between every two consecutive slow transmissions. We further assume $\tau_{\text{miati}}^s \leq \tau_{\text{mati}}^s - \tau_{\text{miati}}$ to simplify the analysis.

A change of variable useful for analysis is the so-called *network-induced error*, which we define as $e_{y_s} := \hat{y}_s - y_s$, $e_{y_f} := \hat{y}_f - y_f$, $e_{u_s} := \hat{u}_s - u_s$ and $e_{u_f} := \hat{u}_f - u_f$. For simplicity, $(\hat{y}_s, \hat{y}_f, \hat{u}_s, \hat{u}_f)$ are assumed to be constant between any two successive transmission times (i.e. zero-order hold behaviour). Other type of network-processing may be implemented if desired, see, e.g., [2]. Define $x := (x_p, x_c) \in \mathbb{R}^{n_x}$, $z := (z_p, z_c) \in \mathbb{R}^{n_z}$, $e_s := (e_{y_s}, e_{u_s}) \in \mathbb{R}^{n_{e_s}}$ and $e_f := (e_{y_f}, e_{u_f}) \in \mathbb{R}^{n_{e_f}}$, with $n_x := n_{x_p} + n_{x_c}$, $n_z := n_{z_p} + n_{z_c}$, $n_{e_s} := n_{y_s} + n_{u_s}$ and $n_{e_f} := n_{y_f} + n_{u_f}$.

A channel may consist of multiple *network nodes*, each representing a group of sensor and/or actuator states. However, only one node can transmit its data at each transmission time, and the access to the channel is regulated by the scheduling protocol of the channel. At each transmission time t_k^s for slow updates, a node, which is a group of elements in y_s and u_s are sampled and transmitted, and the values $(\hat{y}_s, \hat{y}_f, \hat{u}_s, \hat{u}_f)$ are updated according to

$$\begin{aligned} \big(\hat{y}_s((t_k^s)^+), \hat{u}_s((t_k^s)^+)\big) &= \big(y_s(t_k^s), u_s(t_k^s)\big) + h_s(k, e_s(t_k^s)) \\ \big(\hat{y}_f((t_k^s)^+), \hat{u}_f((t_k^s)^+)\big) &= \big(\hat{y}_f(t_k^s), \hat{u}_f(t_k^s)\big), \end{aligned}$$

where $h_s : \mathbb{Z}_{\geq 0} \times \mathbb{R}^{n_{e_s}} \to \mathbb{R}^{n_{e_s}}$ models the scheduling protocol for the slow updates. Similarly, for each $t_\ell^f \in \mathcal{T}^f$, we have that

$$\begin{aligned} \big(\hat{y}_s((t_\ell^f)^+), \hat{u}_s((t_\ell^f)^+)\big) &= \big(\hat{y}_s(t_\ell^f), \hat{u}_s(t_\ell^f)\big) \\ \big(\hat{y}_f((t_\ell^f)^+), \hat{u}_f((t_\ell^f)^+)\big) &= \big(y_f(t_\ell^f), u_f(t_\ell^f)\big) + h_f\big(\ell, e_f(t_\ell^f)\big), \end{aligned}$$

where the function $h_f : \mathbb{Z}_{\geq 0} \times \mathbb{R}^{n_{e_f}} \to \mathbb{R}^{n_{e_f}}$ is the scheduling protocol for the update of fast components.

III. A hybrid model for the SPNCS

A. Closed-loop System

We now present a hybrid system model for the SPNCS described in Section II, in the formalism of [11]. For this purpose, we introduce two clocks and two counters, namely $\tau_s, \tau \in \mathbb{R}_{\geq 0}$ and $\kappa_s, \kappa_f \in \mathbb{Z}_{\geq 0}$. In particular, τ_s records the time elapsed since the last slow transmission, and τ describes the inter-transmission time between any two successive transmissions, therefore, τ_s resets to zero at each slow transmission, and τ resets to zero at any transmission. Meanwhile, κ_s and κ_f count the number of slow and fast transmissions, respectively. Moreover, f_x, g_z, f_{e_s} and g_{e_f} are defined in (5) in the next page, where we use $f_{x,\iota}$ and $g_{z,\iota}$, $\iota \in \{1, 2\}$, to denote the ι–th component of f_x and g_z, respectively. Let $\xi := (x, e_s, \tau_s, \kappa_s, z, e_f, \tau, \kappa_f) \in \mathbb{X}$, with $\mathbb{X} := \mathbb{R}^{n_x} \times \mathbb{R}^{n_{e_s}} \times \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_{e_f}} \times \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, denote the full state of the hybrid system. Consequently, the SPNCS can now be expressed as the following hybrid model

$$\mathcal{H}_1 : \begin{cases} \dot{\xi} = F(\xi), & \xi \in \mathcal{C}_1^\epsilon, \\ \xi^+ \in G(\xi), & \xi \in \mathcal{D}_s^\epsilon \cup \mathcal{D}_f^\epsilon, \end{cases} \tag{4}$$

where $F(\xi) := \big(f_x(x, z, e_s, e_f), f_{e_s}(x, z, e_s, e_f), 1, 0, \frac{1}{\epsilon} g_z(x, z, e_s, e_f), \frac{1}{\epsilon} g_{e_f}(x, z, e_s, e_f, \epsilon), \frac{1}{\epsilon}, 0\big)$, and

$$G(\xi) := \begin{cases} G_s(\xi), & \xi \in \mathcal{D}_s^\epsilon \setminus \mathcal{D}_f^\epsilon, \\ G_f(\xi), & \xi \in \mathcal{D}_f^\epsilon \setminus \mathcal{D}_s^\epsilon, \\ \{G_s(\xi), G_f(\xi)\}, & \xi \in \mathcal{D}_s^\epsilon \cap \mathcal{D}_f^\epsilon. \end{cases}$$

The jump maps are defined such that $G_s(\xi) := (x, h_s(\kappa_s, e_s), 0, \kappa_s + 1, z, e_f, 0, \kappa_f)$ and $G_f(\xi) := (x, e_s, \tau_s, \kappa_s, z, h_f(\kappa_f, e_f), 0, \kappa_f + 1)$. The jump and flow sets are defined as

$$\begin{aligned} \mathcal{D}_s^\epsilon &:= \{\xi \in \mathbb{X} \mid \tau_s \in [\tau_{\text{miati}}^s, \tau_{\text{mati}}^s] \wedge \epsilon\tau \in [\tau_{\text{miati}}, \tau_{\text{mati}}]\}, \\ \mathcal{D}_f^\epsilon &:= \{\xi \in \mathbb{X} \mid \tau_s \in [\tau_{\text{miati}}, \tau_{\text{mati}}^s - \tau_{\text{miati}}] \\ &\qquad\qquad \wedge \epsilon\tau \in [\tau_{\text{miati}}, \tau_{\text{mati}}]\}, \\ \mathcal{C}_1^\epsilon &:= \mathcal{D}_s^\epsilon \cup \mathcal{D}_f^\epsilon \cup \mathcal{C}_{1,a}^\epsilon \cup \mathcal{C}_{1,b}^\epsilon \end{aligned}$$

where $\wedge$ denotes the logical conjunction, $\mathcal{C}_{1,a}^\epsilon := \{\xi \in \mathbb{X} \mid \tau_s \in [0, \tau_{\text{miati}}] \wedge \epsilon\tau \in [0, \tau_s + (\tau_{\text{mati}} - \tau_{\text{miati}})]\}$ and $\mathcal{C}_{1,b}^\epsilon := \{\xi \in \mathbb{X} \mid \tau_s \in [\tau_{\text{miati}}, \epsilon\tau + (\tau_{\text{mati}}^s - \tau_{\text{miati}})] \wedge \epsilon\tau \in [0, \tau_{\text{miati}}]\}$. Fig. 2 depicts all the components of $\mathcal{C}_1^\epsilon$, and the jump sets $\mathcal{D}_s^\epsilon$ and $\mathcal{D}_f^\epsilon$ are indicated by the orange and green regions, respectively. Additionally, $\mathcal{C}_{1,a}^\epsilon$ and $\mathcal{C}_{1,b}^\epsilon$ are the regions where a jump is not allowed due to a recent transmission of slow and fast signals, respectively.

In contrast to the two-channel case where the fast and slow jumps are essentially determined solely by their fast and slow timer [10], in the single-channel case these are determined by both the fast and slow timers (i.e., τ and τ_s). Similarly, for the single-channel case, the conditions on τ and τ_s in the flow set depend on both timers, e.g., $\tau_s \in [\max\{0, \epsilon\tau - (\tau_{\text{mati}} - \tau_{\text{miati}})\}, \tau_{\text{miati}}]$, whereas the fast and slow timers are decoupled in the two-channel case. This can be seen from the fact that slow and fast signals are transmitted over the single-channel, and there exist a minimum of τ_{miati} time units between any two consecutive transmissions.

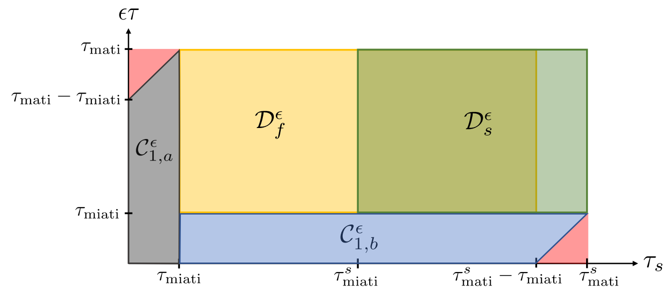

Fig. 2: Flow set and jump set

To simplify the analysis, we introduce $\mathcal{H}_2$ as the hybrid system with dynamics as per (4), i.e., same dynamics as $\mathcal{H}_1$, but with the "patched" flow set defined as $\mathcal{C}_2^\epsilon := \{\xi \in \mathbb{X} \mid \tau_s \in [0, \tau_{\text{mati}}^s] \wedge \epsilon\tau \in [0, \tau_{\text{mati}}]\}$, which aligns with the entirety of the colored area depicted in Fig. 2. We note that $\mathcal{H}_2$ *contains* $\mathcal{H}_1$ in the sense that all solutions of $\mathcal{H}_1$ are also solutions to $\mathcal{H}_2$, since $\mathcal{C}_1^\epsilon \subseteq \mathcal{C}_2^\epsilon$ and they have identical flow map, jump map and jump set [11, Section 3.4].

B. Boundary Layer System and Reduced System of $\mathcal{H}_2$

The goal of this stage is to study the stability of $\mathcal{H}_2$. Since this is a hybrid SPS, we adopt a similar approach to [5, Section 11.5] to show stability, but generalised to hybrid systems. Particularly, we will first derive a system $\mathcal{H}_2^y$ by changing the z–coordinate of $\mathcal{H}_2$, and determining its stability through a *boundary layer* and *reduced system*.

To that end, we now derive the *quasi-steady-state* of $\mathcal{H}_2$, under the following assumption.

Standing Assumption 1 *For any $\overline{x} \in \mathbb{R}^{n_x}$, $\overline{e}_s \in \mathbb{R}^{n_{e_s}}$ and $\overline{z} \in \mathbb{R}^{n_z}$, equation $0 = g_z(\overline{x}, \overline{z}, \overline{e}_s, 0)$ has a unique real solution $\overline{z} = \overline{H}(\overline{x}, \overline{e}_s)$, where $\overline{H}$ is continuously differentiable and $0 = \overline{H}(0, 0)$.*

The quasi-steady-state equilibrium $\overline{z}$ and $\overline{e}_f$ for the fast states z and e_f obtained as follows: $\overline{e}_f$ is equal to zero, as for sufficiently high frequency of fast-output transmissions, e_f converges to zero; and $\overline{z}$ corresponds to the unique solution $\overline{z} = \overline{H}(\overline{x}, \overline{e}_s)$ as per S.A. 1. Next, to derive $\mathcal{H}_2^y$, we define $y := z - \overline{H}(x, e_s)$ and the full state of $\mathcal{H}_2^y$, namely $\xi^y := (\xi_s, \xi_f) := \big((x, e_s, \tau_s, \kappa_s), (y, e_f, \tau, \kappa_f)\big)$, where $\xi^y \in \mathbb{X}, \xi_s \in \mathbb{R}^{n_{\xi_s}} := \mathbb{R}^{n_x} \times \mathbb{R}^{n_{e_s}} \times \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ and $\xi_f \in \mathbb{R}^{n_{\xi_f}} := \mathbb{R}^{n_z} \times \mathbb{R}^{n_{e_f}} \times \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Then, $\mathcal{H}_2^y$ is given by

$$\mathcal{H}_2^y : \begin{cases} \dot{\xi}^y = F^y(\xi^y), & \xi^y \in \mathcal{C}_2^{y,\epsilon}, \\ \xi^{y+} \in G^y(\xi^y), & \xi^y \in \mathcal{D}_s^{y,\epsilon} \cup \mathcal{D}_f^{y,\epsilon}. \end{cases}$$

The flow map is $F^y(\xi^y) = (\dot{\xi}_s, \dot{\xi}_f) = \big(F_s^y(x, y, e_s, e_f), \frac{1}{\epsilon} F_f^y(x, y, e_s, e_f, \epsilon)\big)$, where $F_s^y(x, y, e_s, e_f) := \big(f_x(x, y + \overline{H}(x, e_s), e_s, e_f), f_{e_s}(x, y + \overline{H}(x, e_s), e_s, e_f), 1, 0\big)$, $F_f^y(x, y, e_s, e_f, \epsilon) := \big(\epsilon \frac{\partial y}{\partial t}, g_{e_f}(x, y + \overline{H}(x, e_s), e_s, e_f, \epsilon), 1, 0\big)$ and $\epsilon \frac{\partial y}{\partial t} = g_z(x, y + \overline{H}(x, e_s), e_s, e_f) - \epsilon \frac{\partial \overline{H}}{\partial \xi_s} F_s^y(x, y, e_s, e_f)$.

$$
\begin{aligned}
f_x(x,z,e_s,e_f) &:= \big(f_p(x_p,z_p,(k_{c_s}(x_c)+e_{u_s},k_{c_f}(x_c,z_c)+e_{u_f})), f_c(x_c,z_c,(k_{p_s}(x_p)+e_{y_s},k_{p_f}(x_p,z_p)+e_{y_f}))\big)\\
g_z(x,z,e_s,e_f) &:= \big(g_p(x_p,z_p,(k_{c_s}(x_c)+e_{u_s},k_{c_f}(x_c,z_c)+e_{u_f})), g_c(x_c,z_c,(k_{p_s}(x_p)+e_{y_s},k_{p_f}(x_p,z_p)+e_{y_f}))\big)\\
f_{e_s}(x,z,e_s,e_f) &:= \Big(-\tfrac{\partial k_{p_s}(x_p)}{\partial x_p} f_{x,1}(x,z,e_s,e_f), -\tfrac{\partial k_{c_s}(x_c)}{\partial x_c} f_{x,2}(x,z,e_s,e_f)\Big)\\
g_{e_f}(x,z,e_s,e_f,\epsilon) &:= \Big(-\epsilon\tfrac{\partial k_{p_f}(x_p,z_p)}{\partial x_p} f_{x,1}(x,z,e_s,e_f) - \tfrac{\partial k_{p_f}(x_p,z_p)}{\partial z_p} g_{z,1}(x,z,e_s,e_f),\\
&\qquad -\epsilon\tfrac{\partial k_{c_f}(x_c,z_c)}{\partial x_c} f_{x,2}(x,z,e_s,e_f) - \tfrac{\partial k_{c_f}(x_c,z_c)}{\partial z_c} g_{z,2}(x,z,e_s,e_f)\Big).
\end{aligned} \tag{5}
$$

Additionally, we have

$$
G^y(\xi^y) := \begin{cases} G^y_s(\xi^y), & \xi^y \in \mathcal{D}^{y,\epsilon}_s \setminus \mathcal{D}^{y,\epsilon}_f,\\ G^y_f(\xi^y), & \xi^y \in \mathcal{D}^{y,\epsilon}_f \setminus \mathcal{D}^{y,\epsilon}_s,\\ \{G^y_s(\xi^y), G^y_f(\xi^y)\}, & \xi^y \in \mathcal{D}^{y,\epsilon}_s \cap \mathcal{D}^{y,\epsilon}_f, \end{cases}
$$

where $G^y_s(\xi_y) := \big(x, h_s(\kappa_s,e_s), 0, \kappa_s+1, y, e_f, 0, \kappa_f\big)$ and $G^y_f(\xi_y) := \big(x, e_s, \tau_s, \kappa_s, y, h_f(\kappa_f,e_f), 0, \kappa_f+1\big)$. For analysis purposes, we write $\tau_{\text{mati}} = \epsilon T^*$ for some $T^* \in \mathbb{R}_{>0}$, $\tau_{\text{miati}} = a\tau_{\text{mati}}$ for some $a \in (0,1]$. Since $\epsilon > 0$, $\epsilon\tau \in [\tau_{\text{miati}}, \tau_{\text{mati}}]$ is equivalent to $\tau \in [aT^*, T^*]$. Then the jump and flow sets are defined by

$$
\begin{aligned}
\mathcal{D}^{y,\epsilon}_s &:= \{\xi^y \in \mathbb{X} \mid \tau_s \in [\tau^s_{\text{miati}}, \tau^s_{\text{mati}}] \wedge \tau \in [aT^*, T^*]\},\\
\mathcal{D}^{y,\epsilon}_f &:= \{\xi^y \in \mathbb{X} \mid \tau_s \in [\epsilon a T^*, \tau^s_{\text{mati}} - \epsilon a T^*]\\
&\qquad\qquad \wedge\ \tau \in [aT^*, T^*]\},\\
\mathcal{C}^{y,\epsilon}_2 &:= \{\xi^y \in \mathbb{X} \mid \tau_s \in [0, \tau^s_{\text{mati}}] \wedge \tau \in [0, T^*]\}.
\end{aligned}
$$

We have changed the coordinate from z to y, and we are now ready to derive the reduced system $\mathcal{H}_r$ and boundary layer system $\mathcal{H}_{bl}$ associated with $\mathcal{H}^y_2$. We first define the fast time scale $\sigma = \frac{t}{\epsilon}$, where $\frac{\partial}{\partial\sigma} = \epsilon\frac{\partial}{\partial t}$. Then we set $\epsilon = 0$. In the perspective of $\mathcal{H}_{bl}$ (i.e., fast dynamics), the slow dynamics are now frozen. Meanwhile, the jump and flow sets of $\mathcal{H}_{bl}$ contain the condition $\tau_s \in [0, \tau^s_{\text{mati}}]$, which can always be satisfied. Therefore, the jumps and flows of $\mathcal{H}_{bl}$ are essentially only determined by τ. We thus write

$$
\mathcal{H}_{bl}: \begin{cases} (\frac{\partial\xi_s}{\partial\sigma}, \frac{\partial\xi_f}{\partial\sigma}) = (\mathbf{0}_{n_{\xi_s}\times 1}, F^y_f(x,y,e_s,e_f,0)), & \xi^y \in \mathcal{C}^{y,0}_{2,bl},\\ \xi^{y+} = G^y_f(\xi^y), & \xi^y \in \mathcal{D}^{y,0}_f, \end{cases}
$$

where $\mathcal{C}^{y,0}_{2,bl} := \{\xi^y \in \mathbb{X} \mid \tau \in [0, T^*]\}$ and $\mathcal{D}^{y,0}_f := \{\xi^y \in \mathbb{X} \mid \tau \in [aT^*, T^*]\}$. From the perspective of $\mathcal{H}_r$ (i.e., slow dynamics), the fast dynamics evolve infinitely fast. Therefore, for any $\tau_s \in [0, \tau^s_{\text{mati}}]$, the waiting time for the condition $\tau \in [aT^*, T^*]$ to be satisfied approaches to zero, and the flows and jumps of $\mathcal{H}_r$ are essentially determined only by τ_s. We assume $\mathcal{H}_{bl}$ is asymptotically stable at its quasi-steady state, which we formalise later. Then in $\mathcal{H}_r$, $y = 0$ and $e_f = 0$, that is

$$
\mathcal{H}_r: \begin{cases} \dot{\xi}_s = F^y_s(x,0,e_s,0), & \xi^y \in \mathcal{C}^{y,0}_{2,r},\\ \xi^+_s = (x, h_s(\kappa_s,e_s), 0, \kappa_s+1), & \xi^y \in \mathcal{D}^{y,0}_s, \end{cases}
$$

where $\mathcal{C}^{y,0}_{2,r} := \{\xi^y \in \mathbb{X} \mid \tau_s \in [0, \tau^s_{\text{mati}}]\}$ and $\mathcal{D}^{y,0}_s := \{\xi^y \in \mathbb{X} \mid \tau_s \in [\tau^s_{\text{miati}}, \tau^s_{\text{mati}}]\}$.

IV. Stability analysis

In this section, we start with the assumptions and preliminaries that are necessary to ensure stability property for $\mathcal{H}_1$. Particularly, Assumptions 1 and 2 introduced below provide sufficient conditions to guarantee asymptotic stability properties for $\mathcal{H}_r$ and $\mathcal{H}_{bl}$, respectively. These assumptions are commonly adopted in the NCS literature, see [3], [10].

Assumption 1 *There exist a function $W_s : \mathbb{Z}_{\geq 0} \times \mathbb{R}^{n_{e_s}} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument uniformly in its first argument, a continuous function $H_s : \mathbb{R}^{n_x} \times \mathbb{R}^{n_{e_s}} \to \mathbb{R}_{\geq 0}$, $\mathcal{K}_\infty$-functions $\underline{\alpha}_{W_s}, \overline{\alpha}_{W_s}$, constants $\lambda_s \in [0,1)$ and $L_s \geq 0$ such that, for all $\kappa_s \in \mathbb{Z}_{\geq 0}$ and $e_s \in \mathbb{R}^{n_{e_s}}$, the following properties hold:*

$$
\underline{\alpha}_{W_s}(|e_s|) \leq W_s(k_s,e_s) \leq \overline{\alpha}_{W_s}(|e_s|), \tag{6}
$$

$$
W_s(\kappa_s+1, h_s(\kappa_s,e_s)) \leq \lambda_s W_s(\kappa_s,e_s). \tag{7}
$$

For all $x \in \mathbb{R}^{n_x}, \kappa_s \in \mathbb{Z}_{\geq 0}$ and almost all $e_s \in \mathbb{R}^{n_{e_s}}$,

$$
\begin{aligned}
\Big\langle \tfrac{\partial W_s(\kappa_s,e_s)}{\partial e_s}, f_{e_s}(x, \overline{H}(x,e_s), e_s, 0)\Big\rangle&\\
\leq L_s W_s(\kappa_s,e_s) + H_s(x,e_s).&
\end{aligned} \tag{8}
$$

Moreover, there exist a locally Lipschitz, positive definite and radially unbounded function $V_s : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, positive definite function ρ_s, and real number $\gamma_s > 0$, such that for all $e_s \in \mathbb{R}^{n_{e_s}}$, all $\kappa_s \in \mathbb{Z}_{\geq 0}$, and almost all $x \in \mathbb{R}^{n_x}$, the following inequality holds

$$
\begin{aligned}
\Big\langle \tfrac{\partial V_s(x)}{\partial x}, f_x(x, \overline{H}(x,e_s), e_s, 0)\Big\rangle \leq -\rho_s(|x|)&\\
- \rho_s(W_s(\kappa_s,e_s)) - H^2_s(x,e_s) + \gamma^2_s W^2_s(\kappa_s,e_s).&
\end{aligned} \tag{9}
$$

Assumption 2 *There exist a function $W_f : \mathbb{Z}_{\geq 0} \times \mathbb{R}^{n_{e_f}} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument uniformly in its first argument, a continuous function $H_f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_{e_f}} \to \mathbb{R}_{\geq 0}$, $\mathcal{K}_\infty$-functions $\underline{\alpha}_{W_f}, \overline{\alpha}_{W_f}$, constants $\lambda_f \in [0,1)$ and $L_f \geq 0$ such that, for all $\kappa_f \in \mathbb{Z}_{\geq 0}$ and $e_f \in \mathbb{R}^{n_{e_f}}$, the following properties hold:*

$$
\underline{\alpha}_{W_f}(|e_f|) \leq W_f(k_f,e_f) \leq \overline{\alpha}_{W_f}(|e_f|), \tag{10}
$$

$$
W_f(\kappa_f+1, h_f(\kappa_f,e_f)) \leq \lambda_f W_f(\kappa_f,e_f). \tag{11}
$$

For all $x \in \mathbb{R}^{n_x}, \kappa_f \in \mathbb{Z}_{\geq 0}$ and almost all $e_f \in \mathbb{R}^{n_{e_f}}$,

$$
\begin{aligned}
\Big\langle \tfrac{\partial W_f(\kappa_f,e_f)}{\partial e_f}, g_{e_f}(x, y+\overline{H}(x,e_s), e_s, e_f, 0)\Big\rangle&\\
\leq L_f W_f(\kappa_f,e_f) + H_f(y,e_f).&
\end{aligned} \tag{12}
$$

Moreover, there exists a locally Lipschitz, positive definite *and radially unbounded function* $V_f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}_{\geq 0}$*, positive definite function* ρ_f*, and real number* $\gamma_f > 0$*, such that for all* $e_f \in \mathbb{R}^{n_{e_f}}$*, all* $\kappa_f \in \mathbb{Z}_{\geq 0}$*, and almost all* $y \in \mathbb{R}^{n_z}$*, the following holds*

$$\left\langle \frac{\partial V_f(x,y)}{\partial y}, g_z(x, y + \overline{H}(x, e_s), e_s, e_f) \right\rangle \leq -\rho_f(|y|) - \rho_f\left(W_f(\kappa_f, e_f)\right) - H_f^2(y, e_f) + \gamma_f^2 W_f^2(\kappa_f, e_f). \quad (13)$$

In Assumption 1 (similarly with Assumption 2), (6)-(7) relate to exponentially stable protocols [2], and (9) relates to the $\mathcal{L}_2$ stability of $\mathcal{H}_r$ from W_s to H_s, which is typically ensured at the first stage of emulation. We refer the reader to [2, Proposition 3] for more details on how to find Lyapunov functions to satisfy Assumptions 1 and 2.

We next provide a lemma as a preliminary to our main result. We define Lyapunov functions $U_s : \mathbb{R}^{n_{\xi_s}} \to \mathbb{R}_{\geq 0}$ and $U_f : \mathbb{R}^{n_{\xi_s}} \times \mathbb{R}^{n_{\xi_f}} \to \mathbb{R}_{\geq 0}$, just as in [3, Eqn. (25)]

$$U_s(\xi_s) = V_s(x) + \gamma_s \phi_s(\tau_s) W_s^2(\kappa_s, e_s) \quad (14a)$$

$$U_f(\xi_s, \xi_f) = V_f(x, y) + \gamma_f \phi_f(\tau) W_f^2(\kappa_f, e_f) \quad (14b)$$

where $\dot{\phi}_\star = -2L_\star \phi_\star - \gamma_\star(\phi_\star^2 + 1)$, $\phi_\star(0) = 1/\lambda_\star^*$, $\star \in \{s, f\}$. By abuse of notation, we write $U_f(\xi^y) = U_f(\xi_s, \xi_f)$.

Lemma 1 *Suppose Assumptions 1 and 2 hold. For any* $L \geq 0$*,* $\lambda \in (0, 1)$ *and* $\gamma > 0$*, we define the following mapping:*

$$T(L, \gamma, \lambda) := \begin{cases} \frac{1}{Lr} \tan^{-1}\left(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L} - 1\right) + 1 + \lambda}\right), & \gamma > L \\ \frac{1}{L} \frac{1-\lambda}{1+\lambda}, & \gamma = L \\ \frac{1}{Lr} \tanh^{-1}\left(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L} - 1\right) + 1 + \lambda}\right), & \gamma < L, \end{cases}$$

where $r := \sqrt{\left|\left(\frac{\gamma}{L}\right)^2 - 1\right|}$*. Let* $(L_s, \gamma_s, \lambda_s)$ *and* $(L_f, \gamma_f, \lambda_f)$ *come from Assumption 1 and 2, respectively, and* U_s *and* U_f *come from* (14) *with some* $\lambda_s^* \in (\lambda_s, 1)$ *and* $\lambda_f^* \in (\lambda_f, 1)$*. For all* $\tau_{mati}^s \leq T(L_s, \gamma_s, \lambda_s^*)$ *and* $T^* \leq T(L_f, \gamma_f, \lambda_f^*)$*, there exist* $\mathcal{K}_\infty$*-functions* $\underline{\alpha}_{U_s}, \overline{\alpha}_{U_s}, \underline{\alpha}_{U_f}, \overline{\alpha}_{U_f}$*, continuous positive definite functions* ψ_1, ψ_2 *and positive constants* a_1, a_2 *such that (15a) holds for all* $\xi_s \in \mathcal{C}_{2,r}^{y,0} \cup \mathcal{D}_s^{y,0}$*, (15b) holds for all* $\xi_s \in \mathcal{C}_{2,r}^{y,0}$*, (15c) holds for all* $\xi_s \in \mathcal{D}_s^{y,0}$*, (16a) holds for all* $\xi_f \in \mathcal{C}_{2,bl}^{y,0} \cup \mathcal{D}_f^{y,0}$*, (16b) holds for all* $\xi_f \in \mathcal{C}_{2,bl}^{y,0}$ *and (16c) holds for all* $\xi_f \in \mathcal{D}_f^{y,0}$*,*

$$\underline{\alpha}_{U_s}\left(|(x, e_s)|\right) \leq U_s(\xi_s) \leq \overline{\alpha}_{U_s}\left(|(x, e_s)|\right), \quad (15a)$$

$$U_s^\circ(\xi_s; F_s^y(x, 0, e_s, 0)) \leq -a_1 \psi_1^2\left(|(x, e_s)|\right), \quad (15b)$$

$$U_s((x, h_s(\kappa_s, e_s)), 0, \kappa_s + 1)) \leq U_s(\xi_s), \quad (15c)$$

$$\underline{\alpha}_{U_f}\left(|(y, e_f)|\right) \leq U_f(\xi_s, \xi_f) \leq \overline{\alpha}_{U_f}\left(|(y, e_f)|\right), \quad (16a)$$

$$U_f^\circ\left((\xi_s, \xi_f); (\mathbf{0}_{n_{\xi_s} \times 1}, F_f^y(x, y, e_s, e_f, 0))\right) \leq -a_2 \psi_2^2\left(|(y, e_f)|\right), \quad (16b)$$

$$U_f(G_f^y(\xi^y)) \leq U_f(\xi_s, \xi_f). \quad (16c)$$

Proof: The proof follows similarly to [16, Theorem 1]. ■

Lemma 1 asserts that, under Assumptions 1 and 2, we can establish upper bounds on τ_{mati}^s and T^* in a manner such that, when both bounds are met, we can construct Lyapunov functions U_s and U_f that guarantee stability for $\mathcal{H}_r$ and $\mathcal{H}_{bl}$, respectively. These Lyapunov functions will play a crucial role in the proof of our main result (namely Theorem 1 below), since we will conclude stability of $\mathcal{H}_2^y$ by considering $\mathcal{H}_r$, $\mathcal{H}_{bl}$, and their interconnection induced by nonzero ϵ. Assumption 3 specifies the *interconnection condition* between the slow and fast dynamics during flow, which is analogous to the continuous-time case as described in [5, pp. 451].

Assumption 3 *There exist* $b_1, b_2, b_3 \geq 0$ *such that*

$$\left\langle \frac{\partial U_s}{\partial \xi_s}, F_s^y(x, y, e_s, e_f) - F_s^y(x, 0, e_s, 0) \right\rangle \leq b_1 \psi_1\left(|(x, e_s)|\right) \psi_2\left(|(y, e_f)|\right), \quad (17a)$$

$$\left\langle \frac{\partial U_f}{\partial \xi_s} - \frac{\partial U_f}{\partial y}\frac{\partial \overline{H}}{\partial \xi_s} - \frac{\partial U_f}{\partial e_f}\frac{\partial \tilde{k}}{\partial \xi_s}, F_s^y(x, y, e_s, e_f) \right\rangle \leq b_2 \psi_1\left(|(x, e_s)|\right) \psi_2\left(|(y, e_f)|\right) + b_3 \psi_2^2\left(|(y, e_f)|\right) \quad (17b)$$

hold for almost all $\xi^y \in \mathcal{C}_2^{y,\epsilon}$*, where* $\tilde{k}(x, z) := (k_{pf}(x_p, z_p), k_{cf}(x_c, z_c))$.

By introducing the set $\mathcal{E} := \{\xi \in \mathbb{X} \mid x = 0 \wedge e_s = 0 \wedge z = 0 \wedge e_f = 0\}$, we are now in a position to state our main result.

Theorem 1 *Considering system* $\mathcal{H}_1$ *and suppose Assumptions 1-3 hold. Let* b_1*,* b_2*, and* b_3 *come from Assumption 3 and* a_1 *and* a_2 *come from Lemma 1. Then, there exists* $\epsilon^* = \frac{a_1 a_2}{a_1 b_3 + b_1 b_2}$ *such that for all* $0 < \epsilon < \epsilon^*$*,* $\tau_{mati}^s \leq T(L_s, \gamma_s, \lambda_s^*)$ *and* $\tau_{mati} \leq \epsilon T(L_f, \gamma_f, \lambda_f^*)$*, the following statement holds:*

There exists a $\mathcal{KL}$*-function* β*, such that for all* $\Delta, \nu > 0$*, there exists a* $\tau_{miati}^{s,*} > 0$*, such that if* $\tau_{miati}^{s,*} \leq \tau_{mati}^s - \tau_{miati}$*, any solution* ξ *with* $|\xi(0,0)|_\mathcal{E} < \Delta$ *satisfies* $|\xi(t,j)|_\mathcal{E} \leq \beta(|\xi(0,0)|_\mathcal{E}, t + j) + \nu$ *for any* $(t, j) \in \mathrm{dom}\,\xi$*.*

Remark 1 *From Theorem 1, we note that, to stabilise a SPNCS (in a semi-global practical sense) over one single communication channel, not only fast and slow transmissions need to be sent sufficiently fast, but also slow transmissions should not occur too often. Otherwise, there may not be enough bandwidth to stabilise the fast dynamics. This is in contrast to the two-channel case presented in [10], where transmitting fast and slow dynamics sufficiently fast over two separate channels was enough for stability.*

V. Illustrative Example

This section provides an example to show how the quasi-steady state, and the reduced and boundary layer systems can be obtained, and how to determine the stability of the system using Theorem 1. Consider an SPNCS with the plant and the controller from [17, pp. 150, Example 7.4], given by

$$\mathcal{P} : \begin{cases} \dot{x}_p = x_p + z_1 + 2\hat{u} \\ \epsilon \dot{z}_1 = -z_1 - \hat{u} \\ \epsilon \dot{z}_2 = x_p - z_1 - z_2 - \hat{u} \\ y_p = x_p + z_2 \end{cases} \qquad \mathcal{C} : \begin{cases} \dot{x}_c = -a x_c + a \hat{y}_p \\ u = -k x_c \end{cases}$$

where $a = 2.088$ and $k = 0.7563$. We define the network induced error as $e_s := \hat{u} - u$ and $e_f := \hat{y}_p - y_p$. Then the flow map of the hybrid model $\mathcal{H}_1$ is given by

$$\begin{cases} \dot{x}_p = x_p + z_1 + 2(-kx_c + e_s) \\ \dot{x}_c = -ax_c + a(x_p + z_2 + e_f) \\ \epsilon\dot{z}_1 = -z_1 - (-kx_c + e_s) \\ \epsilon\dot{z}_2 = x_p - z_1 - z_2 - (-kx_c + e_s) \\ \dot{e}_s = -akx_c + ak(x_p + z_2 + e_f) \\ \epsilon\dot{e}_f = -\epsilon\big(x_p + z_1 + 2(-kx_c + e_s)\big) \\ \qquad\qquad - \big(x_p - z_1 - z_2 - (-kx_c + e_s)\big) \\ \dot{\tau}_s = 1,\ \epsilon\dot{\tau} = 1,\ \dot{\kappa}_s = 0,\ \dot{\kappa}_f = 0 \end{cases}$$

and the flow set is $\mathcal{C}_1^\epsilon$. The enlarged system $\mathcal{H}_2$ will have the same flow map as $\mathcal{H}_1$ and a flow set $\mathcal{C}_2^\epsilon$. Moreover, since both u and y_p are one dimensional variables, we have $h_s(\kappa_s, e_s) = 0$ and $h_f(\kappa_f, e_f) = 0$.

By setting $\epsilon = 0$, we have the quasi-steady states $\overline{e}_f = 0$ and $\overline{H} = (\overline{z}_1, \overline{z}_2) = (kx_c - e_s, x_p)$. We define $(y_1, y_2) := (z_1 - \overline{z}_1, z_2 - \overline{z}_2)$, then we can derive the boundary layer system $\mathcal{H}_{bl}$ of $\mathcal{H}_2$ by setting $\epsilon = 0$ and substituting $(z_1, z_2) = (y_1 + \overline{z}_, y_2 + \overline{z}_2)$ into $\mathcal{H}_2$, which is given by

$$\mathcal{H}_{bl}: \begin{cases} \left.\begin{array}{l} \frac{\partial x}{\partial \sigma} = 0,\ \frac{\partial e_s}{\partial \sigma} = 0,\ \frac{\partial \tau_s}{\partial \sigma} = 0,\ \frac{\partial \kappa_s}{\partial \sigma} = 0 \\ \frac{\partial y_1}{\partial \sigma} = -y_1, \quad \frac{\partial y_2}{\partial \sigma} = -y_1 - y_2 \\ \frac{\partial e_f}{\partial \sigma} = y_1 + y_2,\ \frac{\partial \tau}{\partial \sigma} = 1,\ \frac{\partial \kappa_f}{\partial \sigma} = 0 \end{array}\right\} \begin{array}{l}\text{when} \\ \xi^y \in \mathcal{C}_{2,bl}^{y,0}\end{array} \\ \left.\begin{array}{l} x^+ = x,\ e_s^+ = e_s,\ \tau_s^+ = \tau_s,\ \kappa_s^+ = \kappa_s \\ y^+ = y,\ e_f^+ = 0,\ \tau^+ = 0,\ \kappa_f^+ = \kappa_f + 1 \end{array}\right\} \begin{array}{l}\text{when} \\ \xi^y \in \mathcal{D}_f^{y,0}.\end{array} \end{cases}$$

Moreover, the reduced system $\mathcal{H}_r$ can be obtained by setting $\epsilon = 0$ and substituting $\overline{e}_f$ and $\overline{H}$ into $\mathcal{H}_2$, and it is given by

$$\mathcal{H}_r: \begin{cases} \left.\begin{array}{l} \dot{x}_p = x_p + (-kx_c + e_s) \\ \dot{x}_c = 2ax_p - ax_c,\ \dot{e}_s = 2akx_p - akx_c \\ \dot{\tau}_s = 1,\ \dot{\kappa}_s = 0 \end{array}\right\} \begin{array}{l}\text{when} \\ \xi^y \in \mathcal{C}_{2,r}^{y,0}\end{array} \\ \left.\begin{array}{l} x^+ = x,\ e_s^+ = 0 \\ \tau_s^+ = \tau_s,\ \kappa_s^+ = \kappa_s + 1 \end{array}\right\} \begin{array}{l}\text{when} \\ \xi^y \in \mathcal{D}_s^{y,0}.\end{array} \end{cases}$$

We now verify each adopted assumption.

Assumption 1: Let $W_s(\kappa_s, e_s) := |e_s|$, then (6)-(8) hold for $\underline{\alpha}_{W_s}(s) = \overline{\alpha}_{W_s}(s) = s$, $\lambda_s = 0$, $L_s = 0$ and $H_s(x, e_s) = |A_{21}x|$, where $A_{21} = \left[\, 2ak\ -ak \,\right]$ and $x = \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}\right]$. Let $P_s = \left[\begin{smallmatrix} 16.476 & -5.615 \\ -5.615 & 3.037 \end{smallmatrix}\right]$, $\rho_s(s) = s$ and $\gamma_s = 11.45$, then (9) is satisfied with $V_s(x) = x^T P_s x$.

Assumption 2: Let $W_f(\kappa_f, e_f) := |e_f|$, then (10)-(12) hold for $\underline{\alpha}_{W_f}(s) = \overline{\alpha}_{W_f}(s) = s$, $\lambda_f = 0$, $L_f = 0$ and $H_f(y, e_f) = |y_1 + y_2|$, where $y = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$. Let $P_f = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $\rho_f(s) = 0.5s$ and $\gamma_f = \sqrt{0.5}$, then (13) is satisfied with $V_f(y) = y^T P_f y$.

Assumption 3: Let $\phi_\star = -2L_\star\phi_\star - \gamma_\star(\phi_\star^2 + 1)$, $\phi_\star(0) = \lambda_\star^*$ and $U_\star = V_\star + \gamma_\star\phi_\star(\tau_\star)W_\star^2(e_\star)$ for $\star \in \{s, f\}$, and we let $\lambda_s^* = 0.4$ and $\lambda_f^* = 0.6$. By Assumptions 1, 2 and the definition of $\phi_\star$, inequalities (15) and (16) hold with $a_1 = 1$, $a_2 = 0.5$, $\psi_1(s) = s$, $\psi_2(s) = s$. Then inequality (17) holds with $b_1 = 293.61, b_2 = 11.25, b_3 = 4.36$.

Now that we have verified all the assumptions, we can compute $\epsilon^* = 0.000151$, $T(L_s, \gamma_s, \lambda_s^*) = 0.0707$ and $T(L_f, \gamma_f, \lambda_f^*) = 0.6929$, and the required MATIs to stabilise the system are given by $\tau_{\text{mati}}^s \leq 0.0707$ and $\tau_{\text{mati}} \leq 0.6929\epsilon$.

VI. Conclusion

We studied stability of two-time-scale Singularly Perturbed Networked Control Systems (SPNCSs) under one single communication channel. Compared to previous works that assumed availability of two channels to transmit fast and slow dynamics separately, we consider a more general setting and provide a more resource-aware strategy to stabilise the SPNCS when only one channel is available. Future work will focus on event-triggered strategies to manage fast and slow dynamics over a single-channel.

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