Extension and Evaluation of Model-Based Periodic Event-Triggered Control

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Abstract— **Periodic event-triggered control (PETC) is a control strategy that combines ideas from conventional periodic sampled-data control and event-triggered control. By communicating periodically sampled sensor and controller data only when needed to guarantee stability and performance properties, PETC is capable of reducing the number of transmissions significantly, while still retaining a satisfactory closed-loop behavior. In this paper, we provide an extension of an existing model-based PETC strategy for linear systems by including an (approximate) disturbance model. This extension can further enhance communication savings in the presence of disturbances. In addition, we evaluate the extended model-based PETC strategy by comparing this strategy to the standard model-based PETC and to a model-based periodic time-triggered control (PTTC) strategy. In this PTTC strategy, data is transmitted at fixed sampling times. For the evaluation, we present techniques for stability and ℓ_2-gain performance analysis for both the PETC strategy and the PTTC strategy. Finally, the advantage of the (extended) PETC strategy over the PTTC strategy will be demonstrated by providing numerical examples.**

I. INTRODUCTION

In digital control systems, sampling of the outputs of the plant, and computation and implementation of the control inputs to the plant are in most cases executed periodically. Despite the fact that periodic sampling may be preferable from a design perspective, it may be less preferable from a resource utilization point of view. When there are no disturbances acting on the system and the system is operating desirably, executing a control task is clearly a waste of communication resources. To mitigate the unnecessary waste of communication resources, it is of interest to consider an alternative control paradigm, namely event-triggered control (ETC) [1]–[8]. ETC is a control strategy in which the control task is executed after the occurrence of an event rather than after the elapse of a certain fixed period of time. In this way, ETC is capable of significantly reducing the number of control task executions, while retaining a satisfactory closed-loop performance in many cases. In this paper, we study a class of event-triggered control algorithms that aims at integrating ideas from conventional periodic time-triggered control and ETC paradigms. This results in so-called periodic event-triggered control (PETC) [4], [7]–[15], in which the

event-triggered condition is verified periodically, and at every sampling time it is decided whether or not to transmit new measurement and control values. The network resources are used only when needed to guarantee stability or certain performance properties.

In this paper, we study periodic event-triggered control for discrete-time systems as in [4], [8], [9], [13]–[15], although the resulting PETC strategy can also be considered in a continuous-time framework, as e.g. in [11], [12]. The current paper has two main contributions. First of all, we present an extension to the model-based periodic event-triggered mechanisms proposed in [13]. In [13] it was shown that for both sensor-to-controller and the controller-to-actuator channels significant reductions of communications can be achieved by this PETC strategy, while preserving desirable stability and (ℓ_2) performance properties. The objective of this paper is to provide an extension to the existing model-based PETC strategy by including an (approximate) disturbance model [6], [10]. The advantage of including a disturbance model is that, in case we have a good estimate of the disturbance, we can further reduce communication recourses. The second contribution entails the evaluation of the existing and extended model-based PETC strategies by comparing them to a so-called model-based periodic time-triggered control (PTTC) strategy. The PTTC strategy differs from the PETC strategy only in terms of the triggering mechanism. Indeed, in the PTTC strategy new measurement information is transmitted at fixed sampling times, instead of at the occurrence of well designed events. Using numerical examples we provide an assessment of the mentioned three strategies and show under which conditions they perform well.

The remainder of this paper is structured as follows. In Section II, the problem formulation and the objectives of this paper are given. In Section III, the existing model-based PETC strategy of [13] will be briefly summarized. This existing strategy will be extended by including (approximate) disturbance models, and, we will provide a closed-loop model for this extended PETC strategy for performance and stability analysis. In Section IV, we will study the model-based PTTC strategy. The performance and stability analysis of both the PETC and PTTC strategies will be discussed in Section V. In Section VI, the model-based (extended) PETC strategy will be evaluated by comparing this strategy with the PTTC strategy. Finally, in Section VII, we will conclude the paper and provide recommendations for future work.

Notations: For a vector $x \in \mathbb{R}^n$, we denote by $||x|| := \sqrt{x^\top x}$ its 2-norm. For vectors $x^i \in \mathbb{R}^{n_i}$, $i \in \{1, \dots, N\}$, the

vector $x = \text{col}(x^1, x^2, \ldots, x^N) \in \mathbb{R}^n$, where $n = \sum_{i=1}^{N} n_i$, is given by $\left[(x^1)^\top, \ (x^2)^\top, \ \ldots, \ (x^N)^\top \right]^\top$. We call a matrix $P \in \mathbb{R}^{n \times n}$ positive definite and write $P \succ 0$, if P is symmetric and $x^\top P x > 0$ for all $x \neq 0$. Similarly, we use $P \succeq 0$ to denote that P is positive semidefinite. A matrix $A \in \mathbb{R}^{n \times n}$ is called Schur, if all its eigenvalues are within the open unit circle of the complex plane. The set ℓ_2^n consists of all infinite sequences $w = \{w_k\}_{k \in \mathbb{N}}$ with $w_k \in \mathbb{R}^n$, $k \in \mathbb{N}$, and $\sum_{k \in \mathbb{N}} ||w_k||^2 < \infty$. We denote by $\mathbb{N}_{>0} = \{1, 2, 3, \ldots\}$ the set of all natural numbers excluding zero.

II. PROBLEM FORMULATION

In this paper, we will study the networked control configuration as depicted in Fig. 1. The plant is given by a discrete-time linear time-invariant (LTI) model of the form

$$\mathcal{P}: \quad \begin{cases} x_{k+1} & = & Ax_k + Bu_k + Ew_k \\ y_k & = & Cx_k, \end{cases} \quad (1)$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, $w_k \in \mathbb{R}^{n_w}$, and $y_k \in \mathbb{R}^{n_y}$ denote the state, control input, disturbance and measured output, respectively, at discrete-time instant $k \in \mathbb{N}$.
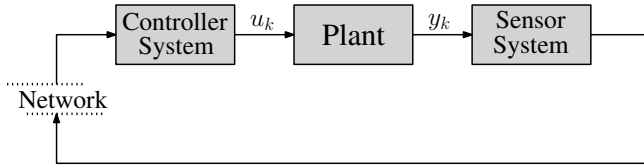

Fig. 1: Control configuration with the shared network in sensor-to-controller channel.

The sensor system measures the output of the plant and transmits the measurement information to the controller system over a shared and possibly wireless network. For this shared network, communication and/or energy resources are limited. Therefore, it is desirable to reduce the number of transmissions from sensor to controller as much as possible, while desirable closed-loop stability and performance are still guaranteed. To address this objective, first an existing PETC strategy [13] is briefly discussed, next to a novel extension that further enhances communication savings in the presence of disturbances.

III. MODEL-BASED PETC STRATEGY

In this section, we first briefly recall the existing model-based PETC strategy as proposed in [13]. This strategy is then extended by including (approximate) disturbance models. Finally, we provide a closed-loop piecewise linear (PWL) model for the extended model-based PETC strategy, which is amendable for stability and performance analysis.

A. Existing Model-Based PETC Strategy

In this section, we will start by discussing the existing model-based periodic event-triggered control (PETC) strategy for the configuration in Fig. 1, see Fig. 2.

In this configuration, the sensor system consists of a Lu-enburger observer $\mathcal{O}$, a predictor $\mathcal{Pr}$ and an event-triggering
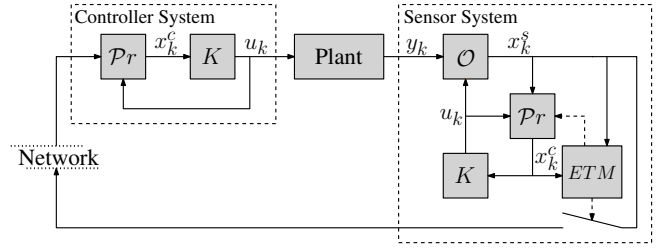

Fig. 2: Model-based PETC strategy.

mechanism ETM that determines when information should be transmitted to the controller system. The Luenberger observer will be of the form

$$\mathcal{O}: \quad x_{k+1}^s = Ax_k^s + Bu_k + L(y_k - Cx_k^s) \quad (2)$$

in which x_k^s denotes the estimated state at the sensor system at time $k \in \mathbb{N}$, and the matrix L is a suitably designed observer gain. The predictor $\mathcal{Pr}$ is given by

$$\mathcal{Pr}: \quad x_{k+1}^c = \begin{cases} Ax_k^c + Bu_k, & \text{when } x_k^s \text{ is not sent} \\ Ax_k^s + Bu_k, & \text{when } x_k^s \text{ is sent}, \end{cases} \quad (3)$$

where x_k^c is the predicted state. Finally, the ETM is given at time $k \in \mathbb{N}$ by

$$ETM: \quad x_k^s \text{ is sent} \quad \Leftrightarrow \quad ||x_k^s - x_k^c|| > \sigma_s ||x_k^s||, \quad (4)$$

where $\sigma_s \geq 0$ is a chosen design parameter. The controller system consists of the same predictor $\mathcal{Pr}$, and a proportional state feedback controller with gain K. In particular, the control signal is given by

$$u_k = \begin{cases} Kx_k^c, & \text{when } x_k^s \text{ is not sent} \\ Kx_k^s, & \text{when } x_k^s \text{ is sent}. \end{cases} \quad (5)$$

The sensor system runs a copy of the predictor $\mathcal{Pr}$ (both initialized at the same initial predicted state x_0^c), i.e., the sensor system is aware of the estimated state x_k^c of the controller system. As a consequence of this, the sensor system can determine u_k to compute the state estimate x_k^s according to (2).

The rationale of the proposed scheme can be explained as follows. In the control configuration of Fig. 2, both the Luenberger observer $\mathcal{O}$ and the predictor $\mathcal{Pr}$ produce estimates x_k^s and x_k^c, respectively, of the state x_k. Typically, the estimates of the Luenberger observer $\mathcal{O}$, located in the sensor system, are better than the estimates of the predictor, as $\mathcal{O}$ has access to all measurements y_k, $k \in \mathbb{N}$, while the predictor provides open-loop predictions only with some sporadic new information when x_k^s is sent. As already mentioned, the sensor system runs a copy of $\mathcal{Pr}$. Hence, the sensor system is aware of the predicted state x_k^c the controller system is using for the computation of the inputs.

As long as $||x_k^s - x_k^c|| \leq \sigma_s ||x_k^s||$, the controller system does not receive new state estimates x_k^s from the sensor system, thereby saving costly communications between sensor and controller. If at a certain time instant $k \in \mathbb{N}$, $||x_k^s - x_k^c|| > \sigma_s ||x_k^s||$, i.e., the estimation x_k^s deviates significantly from

the prediction x_k^c (measured relatively to size x_k^s), then the estimate x_k^s is sent to the controller system. Subsequently, the predictor $\mathcal{P}r$ and control signal u_k are updated according to the second case in (3) and (5), respectively.

B. Including disturbance models

In this section, we provide an extension for the model-based PETC strategy as discussed in Section III-A. To be more precise, an (approximate) disturbance model will be included in the observer and predictor models, (2) and (3), respectively. The disturbance model is given by

$$\hat{w}_{k+1} = S\hat{w}_k, \tag{6}$$

where $\hat{w}_k$ is an approximation of w_k as in (1) at $k \in \mathbb{N}$ and S an appropriately chosen matrix representing the disturbance dynamics. The choice of S is based on prior knowledge about the actual disturbance w_k. Note that $\hat{w}_k$ is not necessarily equal to w_k, $k \in \mathbb{N}$, making w_k still an external disturbance. Including this approximate disturbance model results in an extended Luenberger observer $\mathcal{O}_w$ of the form

$$\mathcal{O}_w: \quad \eta_{k+1}^s = \hat{A}\eta_k^s + \hat{B}u_k + \hat{L}(y_k - \hat{C}\eta_k^s) \tag{7}$$

with

$$\hat{A} = \begin{bmatrix} A & E \\ 0 & S \end{bmatrix}, \hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \hat{L} = \begin{bmatrix} L \\ L_w \end{bmatrix}, \hat{C} = \begin{bmatrix} C & 0 \end{bmatrix},$$

in which $\eta_k^s = \mathrm{col}(x_k^s, \hat{w}_k^s)$ with $\hat{w}_k^s$ an estimate of the disturbance w_k, $k \in \mathbb{N}$, in the sensor system. Note that L_w is now an additional observer gain for the estimation of the disturbance. The extended predictor $\mathcal{P}r_w$ becomes now

$$\mathcal{P}r_w: \quad \eta_{k+1}^c = \begin{cases} \hat{A}\eta_k^c + \hat{B}u_k, & \text{when } \eta_k^s \text{ is not sent} \\ \hat{A}\eta_k^s + \hat{B}u_k, & \text{when } \eta_k^s \text{ is sent,} \end{cases} \tag{8}$$

where $\eta_k^c = \mathrm{col}(x_k^c, \hat{w}_k^c)$ are the estimates of the state and disturbance at the controller system. Finally, the event-triggering mechanism remains the same as in (4), except that η_k^s is sent rather than x_k^s, i.e.,

$$ETM_w: \quad \eta_k^s \text{ is sent} \quad \Leftrightarrow \quad ||x_k^s - x_k^c|| > \sigma_s||x_k^s||. \tag{9}$$

The advantage of including a disturbance model is that with a good estimate of the disturbance, we can further reduce the communication between the sensor and controller system. To be more precise, a good approximation of w_k results in better state estimations x_k^s and x_k^c, which lead to less triggering and thereby to a reduction of the number of transmissions from the sensor system to the controller system.

C. Closed-Loop Piecewise Linear Modelling

In this section, we provide a closed-loop model in the form of a piecewise linear (PWL) system for the model-based PETC strategy including disturbance model. To obtain the closed-loop PWL model, we will use the definitions $e_k^s := x_k^s - x_k$ and $e_k^c := x_k^c - x_k$, and adopt the state variable $\vartheta_k := \mathrm{col}(x_k, e_k^s, e_k^c, \hat{w}_k^s, \hat{w}_k^c) \in \mathbb{R}^{n_\vartheta}$ with $n_\vartheta := 3n_x + 2n_w$. Based

on this definition, we obtain the following closed-loop model

$$\vartheta_{k+1} = \begin{cases} \hat{\mathcal{A}}_1\vartheta_k + \hat{\mathcal{E}}w_k, & \vartheta_k^\top \hat{Q}_s \vartheta_k > 0, \\ \hat{\mathcal{A}}_2\vartheta_k + \hat{\mathcal{E}}w_k, & \vartheta_k^\top \hat{Q}_s \vartheta_k \le 0, \end{cases} \tag{10}$$

with

$$\hat{\mathcal{A}}_1 = \left[\begin{array}{ccc|cc} A+BK & BK & 0 & 0 & 0 \\ 0 & A-LC & 0 & E & 0 \\ 0 & A & 0 & E & 0 \\ 0 & -L_wC & 0 & S & 0 \\ 0 & 0 & 0 & S & 0 \end{array} \right] =: \left[\begin{array}{c|c} \mathcal{A}_1 & E_1 \\ \hline \mathcal{C} & \mathcal{S}_1 \end{array} \right], \tag{11a}$$

$$\hat{\mathcal{A}}_2 = \left[\begin{array}{ccc|cc} A+BK & 0 & BK & 0 & 0 \\ 0 & A-LC & 0 & E & 0 \\ 0 & 0 & A & 0 & E \\ 0 & -L_wC & 0 & S & 0 \\ 0 & 0 & 0 & 0 & S \end{array} \right] =: \left[\begin{array}{c|c} \mathcal{A}_2 & E_2 \\ \hline \mathcal{C} & \mathcal{S}_2 \end{array} \right], \tag{11b}$$

$$\hat{\mathcal{E}} = \left[\begin{array}{c} E \\ -E \\ -E \\ \hline 0 \\ 0 \end{array} \right] =: \left[\begin{array}{c} \mathcal{E} \\ \hline 0 \end{array} \right], \tag{11c}$$

$$\hat{Q}_s = \left[\begin{array}{ccc|cc} -\sigma_s^2 I & -\sigma_s^2 I & 0 & 0 & 0 \\ -\sigma_s^2 I & (1-\sigma_s^2)I & -I & 0 & 0 \\ 0 & -I & I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] =: \left[\begin{array}{c|c} Q_s & 0 \\ \hline 0 & 0 \end{array} \right]. \tag{11d}$$

Note that $\vartheta_k^\top \hat{Q}_s \vartheta_k > 0$ is equivalent to $||e_k^s - e_k^c|| > \sigma_s||x_k + e_k^s||$, which is the same as the event-triggering condition in (9). Hence, the discrete-time PWL model (10) describes the closed-loop behavior of the plant and model-based PETC strategy including the disturbance model given by (1), (5), (7)-(9).

Remark 1: Note that the closed-loop model for the model-based PETC strategy without disturbance model as discussed in Section III-A can be retrieved from the above by omitting $\hat{w}_k^s$ and $\hat{w}_k^c$ in ϑ_k and reducing the matrices of the PWL model in (10) to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{E}, Q_s$ as specified in (11), see also [13].

Before providing stability and performance analysis of the newly proposed PETC scheme, in Section V, we will provide a periodic time-triggered control (PTTC) strategy in the next section for comparison reasons later.

IV. MODEL-BASED PTTC STRATEGY

In this section, we will study an alternative strategy aiming at reducing the number of required transmissions between sensor and controller, namely a model-based periodic time-triggered control (PTTC) strategy as depicted in Fig. 3.

For this strategy, we will use a (periodic) time-triggered mechanism TTM that determines when information is transmitted to the controller system. The TTM is given at time $k \in \mathbb{N}$ by

$$TTM: \quad x_k^s \text{ is sent} \quad \Leftrightarrow \quad \frac{k}{h} \in \mathbb{N}, \tag{12}$$

where $h \in \mathbb{N}_{>0}$ is a chosen transmission/update period. The PTTC strategy only differs from the (standard) PETC strategy in terms of the trigger mechanism. Hence, the Luenburger observer $\mathcal{O}$, the predictor $\mathcal{P}r$ and the controller
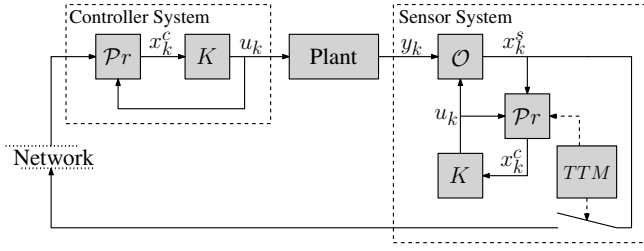
Fig. 3: Model-based PTTC strategy.

gain as depicted in Fig. 3 are equal to (2), (3) and (5), respectively. In this strategy, the sensor system periodically transmits measurement information to the controller (i.e., when $\frac{k}{h} \in \mathbb{N}$ holds, the estimate x_k^s is sent to the controller system) and corresponding updates of the prediction x_{k+1}^c and control signal u_k are made.

The closed-loop behavior of the PTTC configuration can be described by a periodic switched linear model. Indeed, we will adopt the state variable $\xi_k := \text{col}(x_k, e_k^s, e_k^c) \in \mathbb{R}^{n_\xi}$ with $n_\xi := 3n_x$, which results in the following closed-loop model

$$
\xi_{k+1} = \begin{cases} \mathcal{A}_1 \xi_k + \mathcal{E} w_k, & \frac{k}{h} \in \mathbb{N}, \\ \mathcal{A}_2 \xi_k + \mathcal{E} w_k, & \frac{k}{h} \notin \mathbb{N}, \end{cases} \tag{13}
$$

with $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{E}$ as defined in (11). The discrete-time periodic switched linear model (13) describes the closed-loop behavior of the plant and the model-based PTTC strategy given by (1)-(3), (5) and (12).

Remark 2: In case the model-based PTTC strategy is extended with an (approximate) disturbance model (6), one should include $\hat{w}_k^s$ and $\hat{w}_k^c$ in ξ_k, resulting in the state variable ϑ_k, and in (13) use instead of $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{E}$ respectively, the matrices $\hat{\mathcal{A}}_1$, $\hat{\mathcal{A}}_2$ and $\hat{\mathcal{E}}$ as defined in (11).

V. STABILITY AND ℓ_2-GAIN ANALYSIS

In this section, we will analyze the stability and the ℓ_2-gain performance of the closed-loop PETC strategy (with or without disturbance model) based on the closed-loop PWL system (10). Furthermore, the stability and the performance of the closed-loop PTTC strategy will be analyzed. To do so, let us first introduce the performance output z_k, $k \in \mathbb{N}$, as

$$
z_k = C x_k. \tag{14}
$$

The definitions of stability and performance used in this paper are given by

Definition 1: The system (10) with $w = 0$ is said to be globally exponentially stable (GES), if there exist constants $c \in \mathbb{R}_{\geq 0}$ and $\rho \in [0, 1)$ such that for all initial states $\vartheta_0 \in \mathbb{R}^{n_\vartheta}$ and $w_k = 0$, $k \in \mathbb{N}$, the corresponding solution to (10) satisfies for all $k \in \mathbb{N}$

$$
||\vartheta_k|| \leq c \rho^k ||\vartheta_0||. \tag{15}
$$

Definition 2: The system given by (10) and (14) is said to have an ℓ_2-gain form w to z smaller than or equal to $\gamma \in \mathbb{R}_{\geq 0}$, if there is a function $\beta : \mathbb{R}^{n_\vartheta} \to \mathbb{R}$ such that for all initial states $\vartheta_0 \in \mathbb{R}^{n_\vartheta}$ and all inputs $w \in \ell_2^{n_\vartheta}$, the corresponding output z satisfies for all $k \in \mathbb{N}$

$$
||z||_{\ell_2} \leq \beta(\vartheta_0) + \gamma ||w||_{\ell_2}. \tag{16}
$$

The ℓ_2-gain from w to z is defined as the infimum of all values γ that satisfy the above property.

A. Analysis of Model-Based PETC Strategy

For the stability and performance analysis of the PETC strategy (with or without disturbance model) essentially a PWL model as in (10) has to be analyzed appended with the performance output (14). To do so, we will utilize a piecewise quadratic (PWQ) Lyapunov function of the form

$$
V(\vartheta_k) = \begin{cases} \vartheta_k^\top P_1 \vartheta_k, & \text{when } \vartheta_k^\top \hat{Q}_s \vartheta_k > 0, \\ \vartheta_k^\top P_2 \vartheta_k, & \text{when } \vartheta_k^\top \hat{Q}_s \vartheta_k \leq 0. \end{cases} \tag{17}
$$

Enforcing that V is positive definite in combination with the satisfaction of the dissipation inequality

$$
V(\vartheta_{k+1}) - V(\vartheta_k) \leq -\varepsilon ||\vartheta_k||^2 - ||z_k||^2 + \gamma^2 ||w_k||^2, \tag{18}
$$

we can guarantee a finite ℓ_2-gain that is smaller or equal to γ, and, in case $w = 0$, GES of (10) with (14). In fact, the conditions in (18) plus the positive definiteness of V can be translated into LMI conditions as provided in [13]. Hence, one can minimize the upper bound γ on the ℓ_2-gain subject to the LMIs in [13] given a value of σ_s. For a more elaborate discussion on stability and ℓ_2-gain performance, we refer the interested reader to [13].

B. Analysis of Model-Based PTTC Strategy

We will now present the stability and performance analysis of the PTTC strategy based on both a lifted system description [16, Ch. 8] and the periodic switched linear model (13).

In the PTTC strategy, the time-triggering mechanism, TTM represents an underlying clock with base period h. Besides that, the input and output signals, w_k and z_k, respectively, change every time unit $k \in \mathbb{N}$. Hence, the PTTC strategy exhibits the behavior of a multi-rate system. Lifting is a useful technique to analyze the behavior of this kind of systems. The lifting process has the purpose of extending the input and output spaces in order to obtain a LTI system. Applying the lifting technique to the PTTC closed-loop system, given a fixed base period h, results in the lifted system representation

$$
\bar{\xi}_{l+1} = \underline{A} \bar{\xi}_l + \underline{B} \underbrace{\begin{bmatrix} w_k \\ w_{k+1} \\ \vdots \\ w_{k+h-1} \end{bmatrix}}_{\underline{w}_l} \tag{19a}
$$

$$\underbrace{\begin{bmatrix} z_k \\ z_{k+1} \\ \vdots \\ z_{k+h-1} \end{bmatrix}}_{\underline{z}_l} = \underline{C}\bar{\xi}_l + \underline{D}\underbrace{\begin{bmatrix} w_k \\ w_{k+1} \\ \vdots \\ w_{k+h-1} \end{bmatrix}}_{\underline{w}_l}, \qquad (19b)$$

where $\bar{\xi}_l := \xi_{lh} \in \mathbb{R}^{n_\xi}$, $\underline{w}_l \in \mathbb{R}^{hn_w}$ and $\underline{z}_l \in \mathbb{R}^{hn_z}$, $l \in \mathbb{N}$, denote the state of the lifted system, lifted input and lifted output, respectively. Note that the states of the LTI system (19) will now be mapped with the base period h. The resulting LTI system $(\underline{A}, \underline{B}, \underline{C}, \underline{D})$ is given by

$$\left[\begin{array}{c|c} \underline{A} & \underline{B} \\ \hline \underline{C} & \underline{D} \end{array} \right] = \left[\begin{array}{c|cccc} \mathcal{A}_2^{h-1}\mathcal{A}_1 & \mathcal{A}_2^{h-1}\mathcal{E} & \mathcal{A}_2^{h-2}\mathcal{E} & \cdots & \mathcal{E} \\ \hline C & 0 & 0 & \cdots & 0 \\ \bar{C}\mathcal{A}_1 & \bar{C}\mathcal{E} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{C}\mathcal{A}_2^{h-2}\mathcal{A}_1 & \bar{C}\mathcal{A}_2^{h-2}\mathcal{E} & \bar{C}\mathcal{A}_2^{h-3}\mathcal{E} & \cdots & 0 \end{array} \right], \quad (20)$$

where $\bar{C} = \begin{bmatrix} C & 0 & 0 \end{bmatrix}$. It is not difficult to see, and, in fact, proven in for instance, [16] that the lifted system description is GES if the lifted state matrix $\underline{A}$ is a Schur matrix and that the ℓ_2-gain of the lifted LTI system (19) is equal to the ℓ_2-gain of the periodic switched linear system (13). Moreover, it is known from linear system theory that the $\mathcal{H}_\infty$-norm of the transfer function $\underline{G}$ of a LTI system equals the the exact bound γ on the ℓ_2-gain. The ℓ_2-gain of the PTTC strategy is now given by the $\mathcal{H}_\infty$-norm of $\underline{G}$, i.e.,

$$||\underline{G}||_\infty := \max_{\theta \in [0,\pi)} \sigma_{\max}(\underline{C}(e^{j\theta}I - \underline{A})^{-1}\underline{B} + \underline{D}) = \gamma. \quad (21)$$

An alternative way to analyse the stability and performance of the PTTC strategy, is by directly using the periodic linear model in (13). Using this model and a parameter-dependent quadratic Lyapunov function of the form

$$V(\xi_k, \sigma_k) = \xi_k^\top P_{\sigma_k} \xi_k, \quad \sigma_k \in \{1, 2, 3, \dots, h\}, \qquad (22)$$

we can calculate exactly the same exact bound γ on the ℓ_2-gain of the PTTC system (13) with (14) as in (21) using LMI conditions guaranteeing the dissipation inequality (18).

VI. COMPARISON

In this section, we will evaluate the extended model-based PETC strategy of Section III-B by comparing this strategy to the existing model-based PETC and the model-based PTTC strategy of Section IV. In particular, this will show the beneficial effect of including an (approximate) disturbance model to the model-based PETC strategy as discussed in Section III-B. For both the PETC (with or without disturbance model) as well as the PTTC strategy, we are interested in plotting the (average) transmission period versus the upper bound γ on the ℓ_2-gain that can be guaranteed based on the discussed conditions. Clearly, the PTTC strategy has a fixed transmission period h. For the proposed PETC strategies, the average transmission times h_{avg} are determined by numerical simulations using different disturbance signals.

In the evaluation, we consider a well-known benchmark example in the networked control system literature, namely a discretized model of a chemical batch reactor [17], with

sampling period $\tau = 0.15$. For the disturbance input matrix E, and the observer and state feedback gains K, L and $\hat{L}$ we take the values

$$E = \begin{bmatrix} 0.2 & 0 & 0.2 & 0 \\ 0 & 0.1 & 0 & 0.1 \end{bmatrix}^\top, \qquad (23a)$$

$$K = \begin{bmatrix} 0.1780 & -0.3056 & 0.2677 & -0.2446 \\ 1.2729 & 0.3096 & 1.7253 & -1.4448 \end{bmatrix}, \qquad (23b)$$

$$L = \begin{bmatrix} 1.5328 & -0.0649 & 0.2007 & 0.0665 \\ 0.0515 & 0.6414 & 0.6099 & 0.3476 \end{bmatrix}^\top, \qquad (23c)$$

$$\hat{L} = \begin{bmatrix} 1.5337 & -0.0931 & 0.5843 & 0.1531 & 1.1015 & -0.2884 \\ 0.0752 & 0.8434 & 1.3551 & 1.6796 & -0.7151 & 1.6864 \end{bmatrix}^\top, \qquad (23d)$$

where $\hat{L}$ is designed based on the eigenvalues of $A - LC$ and two additional eigenvalues for the disturbance estimation. The advantage of the model-based PETC strategies compared to the model-based PTTC strategy can be illustrated by applying the disturbance type $w_k = \begin{bmatrix} w_k^1 \\ w_k^2 \end{bmatrix}$ with $w_k^1 = w_k^2$ as depicted in Fig. 4. For this disturbance type, we define the following

$$\text{disturbance percentage} = \frac{q}{p} \times 100\%, \qquad (24)$$

where p denotes the period time of the disturbance and q the duration in which the signal attains a non-zero value. Furthermore, the amplitude of the block is given by $a = 20$.
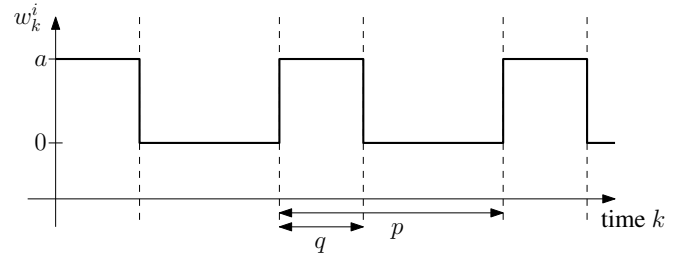

Fig. 4: The applied disturbance $w_k^i, i = 1, 2$, as a function of time k.

Based on this block signal disturbance type, as shown in Fig. 4, the average transmission time h_{avg} of the (extended) PETC strategy will be determined by simulation and using the initial conditions $x_0 = x_0^s = x_0^c = 0$.

We will now first evaluate the existing model-based PETC strategy without disturbance model by comparing this strategy with the model-based PTTC strategy. For the PETC strategy, we will analyse the averaged transmission time h_{avg} (related to the disturbance signal used) versus the upper bound γ on the ℓ_2-gain given a value of σ_s, see Fig. 5a. The analysis as discussed in Section V-A is used to find the upper bound γ corresponding to different values of σ_s. Clearly, different values of σ_s correspond to different γ and also result in different h_{avg}. The different curves as shown Fig. 5a have been obtained using different disturbance percentages as defined in (24), namely 5%, 25% and 50%. For the PTTC strategy, we will analyse the fixed transmission time h versus
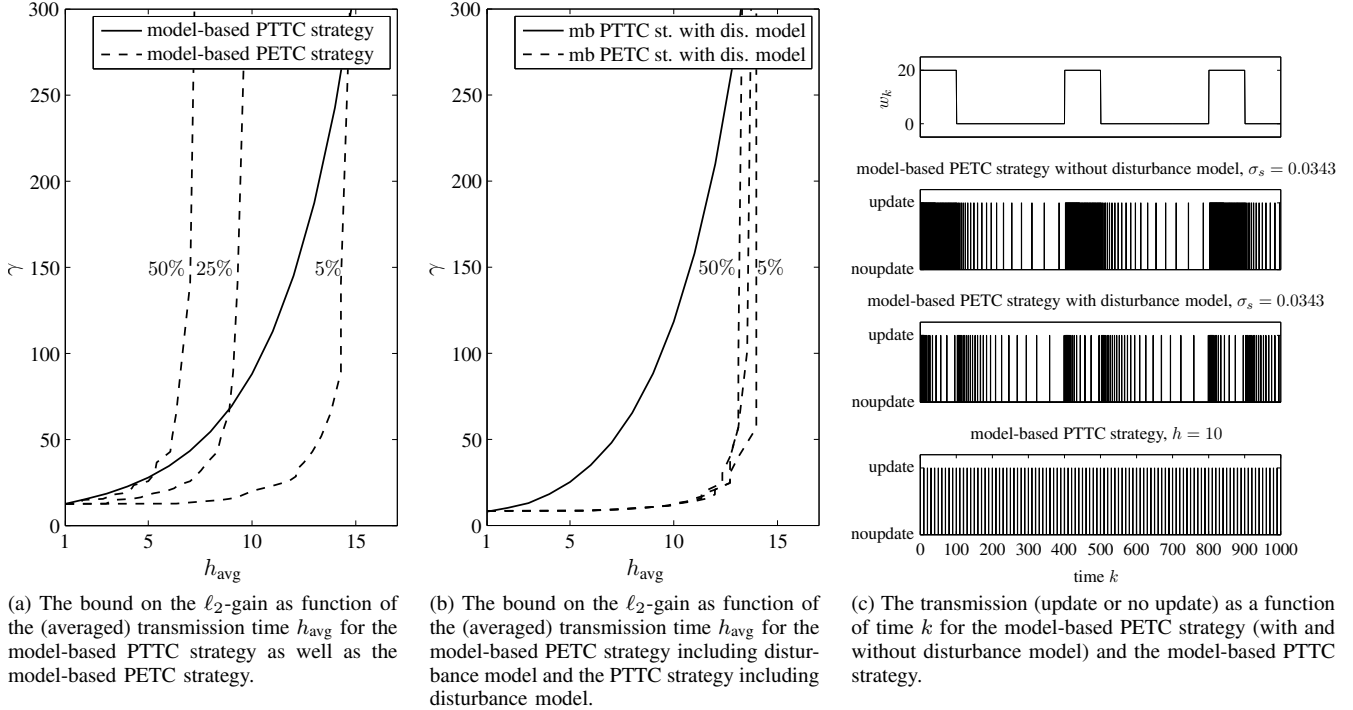
(a) The bound on the ℓ_2-gain as function of the (averaged) transmission time h_{avg} for the model-based PTTC strategy as well as the model-based PETC strategy.

(b) The bound on the ℓ_2-gain as function of the (averaged) transmission time h_{avg} for the model-based PETC strategy including disturbance model and the PTTC strategy including disturbance model.

(c) The transmission (update or no update) as a function of time k for the model-based PETC strategy (with and without disturbance model) and the model-based PTTC strategy.

Fig. 5: Evaluation and comparison of the PETC strategy and the PTTC strategy.

the exact bound γ on the ℓ_2-gain, see Fig. 5a. The analysis in Section V-B has been used to find the ℓ_2-gain γ. Fig. 5a shows that for both strategies (the bound on) the ℓ_2-gain increases as the transmission time h_{avg} or h increases. Note that for $\sigma_s = 0$, the upper bound γ on the ℓ_2-gain for the model-based PETC strategy is equal to the exact bound γ on the ℓ_2-gain for the PTTC strategy if we set $h = 1$. This figure shows that in case the disturbance percentage in (24) is low (5%), i.e., when the disturbance acting on the system is very sporadic, the averaged transmission time h_{avg} increases significantly, while resulting in only a minor degradation of the closed-loop performance in terms of the ℓ_2-gain. But whenever the disturbance percentage is high (50%), the model-based PETC strategy looses its advantage over the model-based PTTC strategy. This can be explained by the fact that the predictor $\mathcal{P}r$ generates good estimates x_k^c of the state x_k in the absence of disturbances ($w_k = 0$), because the predictor is based on the plant model (1) without disturbance input.

Let us now consider the extended model-based PETC strategy including a disturbance model as discussed in Section III-B. Again we will analyze the averaged transmission time h_{avg} versus the upper bound γ on the ℓ_2-gain given a value of σ_s, see Fig. 5b. Again different curves are depicted, obtained using the disturbance percentages 5%, 25%, and 50%. Because we apply the block shaped disturbance to determine h_{avg}, we choose $S = I$ in (6) to approximate the disturbance dynamics. Note that this model

is not exact due to the discontinuities in the block shaped disturbance. The inclusion of the disturbance model in the observer and predictor, (7) and (8), respectively, reduces the communication between sensor and controller when the block shaped disturbance is active, as can be concluded by comparing Fig. 5a and Fig. 5b. As a result the averaged inter-transmission time increases tremendously. Even when the disturbance ratio is very high (50%), the amount of data transmitted over the network is reduced significantly, resulting in a substantial increase of the average transmission time h_{avg}, while there is hardly any reduction in the closed-loop ℓ_2-gain performance. In Fig. 5b, also the model-based PTTC strategy with disturbance model (see Remark. 2) is shown. Clearly, this extension of the PTTC strategy does have little effect on the (h, γ)-curve compared to the curve for PTTC as in Fig. 5a without the disturbance model.

To provide further insights, we also plot the transmission (update or no update) as a function of time k for the model-based PETC strategy (with and without disturbance model) and the model-based PTTC strategy, when the disturbance type w_k as in Fig. 4 is applied, see Fig. 5c. For the applied disturbance, a disturbance percentage of 25% is chosen. Clearly, the model-based PETC strategy without disturbance model triggers frequently when the non-zero part of the disturbance occurs, i.e., when $w_k = 20$. If the disturbance becomes equal to zero, $w_k = 0$, the model-based PETC strategy reduces its updates over time. When the (approximate) disturbance model is included in the PETC

strategy, the number of transmissions significantly reduces when the non-zero part of the disturbance occurs ($w_k = 20$). Only at discontinuities some time is needed to obtain a good estimation again. Also the transmissions as a function of time k are shown for the model-based PTTC strategy, where the transmission period is chosen as $h = 10$. Hence, from these plots it is clear that the model-based PETC strategy is particularly effective with sporadic disturbances. The extension provided in this paper can deal with non-sporadic disturbance as well, while guaranteeing good performance and limited usage of communication recourses as long as adopted disturbance models are rather accurate. Only when the disturbance model is not accurate (e.g. around the discontinuities in the block shaped signal, see Fig. 5c) more communications are temporarily needed.

VII. CONCLUSIONS

In this paper, we provided two main contributions in the area of periodic event-triggered control. First of all, we provided an extension to an existing model-based PETC strategy, which further reduced communication between the sensor and controller in the presence of disturbances. This reduction was achieved by including an (approximate) disturbance model in the observer/predictor structure. Secondly, we provided an evaluation of the (extended) model-based PETC strategy by comparing this strategy to a model-based PTTC strategy and we demonstrated the potential advantages of the PETC strategy over the PTTC strategy.

The existing model-based PETC strategy is based on exploiting a Luenberger observer in the sensor system and two identical predictors in both the controller and the sensor system. This sensor system only transmits an estimation of the state, produced by the observer, to the controller system, if the estimation of the predictor deviates significantly from observer estimation in a relative manner. An extension for the model-based PETC strategy was provided by including an approximate disturbance model in the Luenberger observer and predictor models. To analyse this (extended) model-based PETC strategy, we provided a closed-loop model in the form of a piecewise linear (PWL) system.

In order to evaluate the (extended) model-based PETC strategy, a model-based PTTC strategy was also discussed. The PTTC strategy only differs from the existing PETC strategy in terms of the trigger mechanism. In this strategy, the estimated state is sent at fixed sampling times by the sensor system to the controller system.

For both the (extended) PETC strategy and the PTTC strategy we provided stability and ℓ_2-gain performance analysis. The stability and performance of the closed-loop model-based PETC strategy was analyzed based on the PWL system, which eventually resulted in LMI-based conditions for global exponential stability and guaranteed ℓ_2-gains. Moreover, the stability and performance of the closed-loop PTTC strategy was analyzed based on a lifted system description, resulting in a linear time-invariant (LTI) system, for which stability and ℓ_2-gains are guaranteed in terms of eigenvalue and $\mathcal{H}_\infty$-norm calculations, respectively.

Via numerical examples we illustrated the communication savings and performance that can be accomplished by the (extended) model-based PETC strategy compared to the (model-based) PTTC strategy. The examples show that the implementation of the model-based PETC strategy can result in a tremendous reduction of the number of transmission and thereby saving communication and/or energy resources, while desirable closed-loop stability and performance is still guaranteed. The existing model-based PETC strategy is particulary effective (compared to the PTTC strategy), when the disturbances are only sporadically active. When the model-based PETC strategy is extended with a good estimation of the disturbance, good performance together with significant communication savings are obtained, even in the case the disturbances are not sporadic.

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