$\mathcal{L}_2$-gain Analysis of Periodic Event-Triggered Control and Self-Triggered Control using Lifting

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***Abstract*—We analyze the stability and $\mathcal{L}_2$-gain properties of a class of hybrid systems that exhibit time-varying linear flow dynamics, periodic time-triggered jumps and arbitrary nonlinear jump maps. This class of hybrid systems encompasses periodic event-triggered control, self-triggered control and networked control systems including time-varying communication delays. New notions on the stability and contractivity ($\mathcal{L}_2$-gain strictly smaller than 1) from the beginning of the flow and from the end of the flow are introduced and formal relationships are derived between these notions, revealing that some are stronger than others. Inspired by ideas from lifting, it is shown that the internal stability and contractivity in $\mathcal{L}_2$-sense of a continuous-time hybrid system in the framework is equivalent to the stability and contractivity in ℓ_2-sense (meaning the ℓ_2-gain is smaller than 1) of an appropriate time-varying discrete-time nonlinear system. These results recover existing works in the literature as special cases and indicate that analysing different discrete-time nonlinear systems (of the same level of complexity) than in existing works yield stronger conclusions on the $\mathcal{L}_2$-gain.**

I. INTRODUCTION

In this work we consider hybrid systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\\ \theta\end{bmatrix} = \begin{bmatrix}A_\theta\xi + B_\theta w\\ 1\\ 0\end{bmatrix}, \text{ when } \tau \in [0,1] \tag{1a}$$

$$\begin{bmatrix}\xi^+\\ \tau^+\\ \theta^+\end{bmatrix} \in \Phi_\theta(\xi)\times\{0\}\times\Psi_\theta(\xi), \text{ when } \tau = 1 \tag{1b}$$

$$z = C_\theta\xi + D_\theta w, \tag{1c}$$

where we adopted the hybrid systems framework [1]. The states of this hybrid system consist of $\xi \in \mathbb{R}^{n_\xi}$, a timer variable $\tau \in \mathbb{R}_{\geq 0}$, and a discrete mode variable $\theta \in \mathbb{T}$, with $\mathbb{T}$ a finite or countable subset of $\mathbb{N}$. The variable $w \in \mathbb{R}^{n_w}$ denotes the disturbance input and $z \in \mathbb{R}^{n_z}$ the performance output. Moreover, A_θ, B_θ, C_θ, D_θ, $\theta \in \mathbb{T}$, are constant real matrices of appropriate dimensions. When τ reaches 1, the state of the system undergoes a jump according to the set-valued map given in (1b). The jump of the state ξ is governed by $\Phi_\theta : \mathbb{R}^{n_\xi} \rightrightarrows \mathbb{R}^{n_\xi}$, $\theta \in \mathbb{T}$, an arbitrary nonlinear set-valued and possibly discontinuous map with $\Phi_\theta(0) = \{0\}$, $\theta \in \mathbb{T}$. The jump of the discrete mode θ jumps is specified by $\Psi_\theta : \mathbb{R}^{n_\xi} \rightrightarrows \mathbb{T}$, $\theta \in \mathbb{T}$. In between the jumps the system flows according to the differential equation (1a).

Interestingly, the class of hybrid systems captured by (1) is a generalization and unification of those found in [2], [3]. In [2] a class of hybrid systems is discussed with periodic jumps and time-invariant possibly nonlinear jump maps. This class of hybrid systems is included in (1), by proper time scaling of the flow dynamics, and by omitting the mode θ, i.e, $\mathbb{T} := \{0\}$. The class of hybrid systems discussed in [3] uses linear jump maps, i.e., Φ_θ is a linear mapping from $\mathbb{R}^{n_\xi}$ to $\mathbb{R}^{n_\xi}$, for all $\theta \in \mathbb{T}$. Moreover, the aperiodic inter-jump time considered in [3] can be modelled through the mode θ and proper time scaling of the linear flow dynamics. The framework (1) also extends our preliminary work in [4], in which the adapted hybrid model did not include the mode θ and its corresponding jump map Ψ_θ, thereby not capable of encompassing certain applications such as self-triggered control, see Section IV below. The hybrid systems (1) are of importance for the description of the closed-loop systems arising from periodic event-triggered control [5], networked control systems with constant transmission intervals and shared communication networks requiring medium access protocols [6], reset control systems [7]–[12] with possibly aperiodically verified reset conditions, self-triggered control systems [13]–[17] that include time-varying communication delays satisfying the small delay assumption, etc.

We are interested in analysing stability and $\mathcal{L}_2$-gain performance properties for this generalized framework compared to [2]–[4], thereby capable of handling the full range of applications just mentioned, including full proofs of the results, which were not provided in our preliminary version [4]. Exploiting ideas from lifting [3], [18]–[21], novel necessary and sufficient conditions for stability and contractivity ($\mathcal{L}_2$-gain strictly smaller than one) of (1) are derived in terms of the stability and contractivity in the ℓ_2-sense of an appropriate time-varying discrete-time nonlinear system. To arrive at the time-varying discrete-time nonlinear system using lifting, some restrictions on the initial conditions of (1) are required related to taking, $\tau(0) = 0$ or $\tau(0) = 1$, referred to *beginning of the flow* and *end of the flow*, respectively. In particular, we formally introduce three different notions of internal stability and contractivity for (1), related to taking, $\tau(0) \in [0, 1]$, $\tau(0) = 0$ or $\tau(0) = 1$. In [2], [3] the initial conditions of (1) are restricted to starting at the end of the flow, i.e., $\tau(0) = 1$, which as shown in this note, turns out to be the weakest of the three notions. Interestingly, the new results in this paper not only lead to stronger conclusions but also apply to a significantly larger class of systems than in [2]–[4] encompassing new applications in areas such as self-triggered control for which no necessary and sufficient conditions on stability and contractivity were available in the literature before.

The remainder of this paper is organized as follows. In Section II we introduce the preliminaries. In Section III we introduce new notions of stability and contractivity and derive formal relationships between them. In Section IV we show how PETC systems with a varying delay and self-triggered control system can be captured in (1). In the main result in Section V (Theorem V.3) we connect the corresponding internal stability and contractivity properties of (1) to the internal stability and contractivity of an appropriate discrete-time system. Conclusions are given in Section VI. Proofs are given in the Appendix.

II. Preliminaries

We recall a few necessary preliminaries, mostly taken from [2]. As usual, we denote by $\mathbb{R}^n$ the standard n-dimensional Euclidean space with inner product $\langle x, y\rangle = x^\top y$ and norm $|x| = \sqrt{x^\top x}$ for $x, y \in \mathbb{R}^n$. $\mathcal{L}_2^n[0,\infty)$ denotes the set of square-integrable functions defined on $\mathbb{R}_{>0} := [0,\infty)$ and taking values in $\mathbb{R}^n$ with $\mathcal{L}_2$-norm $\|x\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty |x(t)|^2 \,\mathrm{d}t}$ and inner product $\langle x, y\rangle_{\mathcal{L}_2} = \int_0^\infty x^\top y \,\mathrm{d}t$ for $x, y \in \mathcal{L}_2[0,\infty)$. If n is clear from the context we also write $\mathcal{L}_2$. A function $\beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called a $\mathcal{K}$-function if it is continuous, strictly increasing and $\beta(0) = 0$.

For X, Y Hilbert spaces with inner products $\langle \cdot,\cdot\rangle_X$ and $\langle \cdot,\cdot\rangle_Y$, respectively, a linear operator $U : X \to Y$ is called isometric if $\langle Ux_1, Ux_2\rangle_Y = \langle x_1, x_2\rangle_X$ for all $x_1, x_2 \in X$. The (Hilbert) adjoint operator is denoted by $U^* : X \to Y$ and satisfies $\langle Ux, y\rangle_Y = \langle x, U^* y\rangle_X$ for all $x \in X$ and $y \in Y$. Note that U being isometric is equivalent to $U^*U = I$ (or $UU^* = I$). The operator U is called an isomorphism if it is an invertible mapping, i.e., if it is one-to-one. The induced norm of U (provided it is finite) is denoted by $\|U\|_{X,Y} = \sup_{x\in X\setminus\{0\}} \frac{\|Ux\|_Y}{\|x\|_X}$. If the induced norm is finite we say U is a bounded linear operator. If $X = Y$ we write $\|U\|_X$ and if X, Y are clear form the context we use the notation $\|U\|$.

To an infinite sequence of Hilbert spaces $\{X_k\}_{k\in\mathbb{N}}$, we can associate a Hilbert space $\ell_2(\{X_k\}_{k\in\mathbb{N}})$ consisting of infinite sequences $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, ...)$, with $\tilde{x}_k \in X_k$, $k \in \mathbb{N}$, satisfying $\sum_{i=0}^\infty \|\tilde{x}_i\|^2_{X_i} < \infty$, and the inner product $\langle \tilde{x}, \tilde{y}\rangle_{\ell_2(\{X_k\}_{k\in\mathbb{N}})} = \sum_{i=0}^\infty \langle \tilde{x}_i, \tilde{y}_i\rangle_{X_i}$. In case $X_k = V$ for all $k \in \mathbb{N}$, we also write $\ell_2(V)$ for short. We denote $\ell_2(\mathbb{R}^n)$ by ℓ_2 when $n \in \mathbb{N}_{\geq 1}$ is clear from the context. We also use the notation $\ell(\{X_k\}_{k\in\mathbb{N}})$ to denote the set of all infinite sequences $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, ...)$ with $x_k \in X_k$, $k \in \mathbb{N}$.

Consider the discrete-time system of the form

$$\begin{bmatrix} \xi_{k+1} \\ r_k \end{bmatrix} \in \psi(\xi_k, v_k) \tag{2}$$

with $v_k \in V$, $r_k \in R$, $\xi_k \in \mathbb{R}^{n_\xi}$, $k \in \mathbb{N}$, with V and R Hilbert spaces and $\psi : \mathbb{R}^{n_\xi} \times V \rightrightarrows \mathbb{R}^{n_\xi} \times R$.

Definition II.1. The discrete-time system (2) is said to have an ℓ_2-gain from v to r smaller than $\gamma \in \mathbb{R}_{>0}$ if there exist a $\bar{\gamma} \in [0, \gamma)$ and a $\mathcal{K}$-function β such that, for any $v \in \ell_2(V)$ and any intial state $\xi_0 \in \mathbb{R}^{n_\xi}$, the corresponding solutions to (2) satisfy $\|r\|_{l_2(R)} \leq \beta(\|\xi_0\|) + \bar{\gamma}\|v\|_{l_2(V)}$. The terminology γ-contractivity is used if this property holds. Moreover, 1-contractivity is also called contractivity (in ℓ_2-sense).

Definition II.2. The discrete-time system (2) is said to be internally stable if there is a $\mathcal{K}$-function β such that, for any $v \in \ell_2(V)$ and any initial state $\xi_0 \in \mathbb{R}^{n_\xi}$, the corresponding solutions ξ to (2) satisfy $\|\xi\|_{\ell_2} \leq \beta(\max(|\xi_0|, \|v\|_{\ell_2(V)}))$.

III. Stability and Contractivity Notions

We will focus on both internal stability and whether the $\mathcal{L}_2$-gain of (1) is smaller than 1, called contractivity. Note that by proper scaling of C and D in (1), it can be determined from contractivity properties if the $\mathcal{L}_2$-gain is smaller than any other value of $\gamma \in \mathbb{R}_{>0}$ as well. In fact, three notions of internal stability and contractivity will be introduced and we will derive formal relations between the notions.

To introduce the notions, let us first remark that in this work we opted to define the domains of solutions and (input) signals to (1) as (subsets) on the nonnegative real line $\mathbb{R}_{\geq 0}$ instead of using hybrid time domains (involving both continuous time and the number of jumps) as in [1]. The reason is threefold. This notation is more in line with the existing and related works [2]–[4], [22], it is more convenient at some places and, finally, for solutions to (1) there is no ambiguity in this notation as at most one jump can take place at any time instant $t \in \mathbb{R}_{\geq 0}$. By working with left-continuous signals and adopting the notation $\xi(t^+) := \lim_{s\downarrow t} \xi(s)$ for a signal $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $t \in \mathbb{R}_{\geq 0}$, the solutions are defined unambiguously (and can, if desired, be one-to-one translated into hybrid solutions defined in hybrid time domains as used in [1]). we will use $(\xi_0, \tau_0, \theta_0)$ for the initial state value and write $\mathcal{S}_p(\xi_0, \tau_0, \theta_0, w)$ for the set of maximal solutions (ξ, τ, θ, z) including performance output z, satisfying (1) starting with $\xi(0) = \xi_0$, $\tau(0) = \tau_0$ and $\theta(0) = \theta_0$, and driven by $w \in \mathcal{L}_2$, where we assume all signals are left-continuous. See [1] for the definition of maximal solutions. If we are only interested in the states of the solutions to (1) – and not the performance output – we write $(\xi, \tau, \theta) \in \mathcal{S}(\xi_0, \tau_0, \theta_0, w)$.

Definition III.1. The hybrid system (1) is said to be contractive if there exist a $\bar{\gamma} \in [0, 1)$ and a $\mathcal{K}$-function β such that, for any $w \in \mathcal{L}_2$, $\xi_0 \in \mathbb{R}^{n_\xi}$, $\tau_0 \in [0, 1]$, $\theta_0 \in \mathbb{T}$, any $(\xi, \tau, \theta, z) \in \mathcal{S}_p(\xi_0, \tau_0, \theta_0, w)$ satisfies

$$\|z\|_{\mathcal{L}_2} \leq \beta(|\xi_0|) + \bar{\gamma}\|w\|_{\mathcal{L}_2}. \tag{3}$$

If this property holds for $\tau_0 = 0$, or $\tau_0 = 1$ the system is said to be contractive from the beginning of the flow (b-contractive) or contractive from the end of the flow (e-contractive), respectively.

Definition III.2. The hybrid system (1) is said to be internally stable if there exists a $\mathcal{K}$-function β such that, for any $w \in \mathcal{L}_2$, $\xi_0 \in \mathbb{R}^{n_\xi}$, $\tau_0 \in [0, 1]$, and $\theta_0 \in \mathbb{T}$ any $(\xi, \tau, \theta) \in \mathcal{S}(\xi_0, \tau_0, \theta_0, w)$ satisfies

$$\|\xi\|_{\mathcal{L}_2} \leq \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_2})). \tag{4}$$

If this property holds for $\tau_0 = 0$, or $\tau_0 = 1$, the system is said to be internally stable from the beginning of the flow (b-internally stable) or internally stable from the end of the flow (e-internally stable), respectively.

In the context of lifting, [18], [19], [21], it is important to work with a fixed initial time $\tau(0)$, which suggests the consideration of b-contractivity or e-contractivity, although from a system theoretic point of view one would be interested in contractivity as this gives the strongest guarantees on the system properties (as (3) holds for any $\tau(0) \in [0,1]$). Therefore, we study the relationships between these notions.

Proposition III.3. *The following statements are equivalent:*

(a) The hybrid system (1) *is internally stable.*
(b) The hybrid system (1) *is b-internally stable*
(c) The hybrid system (1) *is e-internally stable*

Proposition III.4. *Consider the following statements:*

(i) The hybrid system (1) *is contractive.*
(ii) The hybrid system (1) *is b-contractive.*
(iii) The hybrid system (1) *is e-contractive.*

Then it holds that

$$(i) \Leftrightarrow (ii) \Rightarrow (iii).$$

Moreover, it holds that (iii) *implies* (i)*, if the following condition holds:*

(A) There exists a $\mathcal{K}$*-function* α_θ *such that for all* $\xi_0 \in \mathbb{R}^{n_\xi}$*, and* $\theta_0 \in \mathbb{T}$ *there exists a response* $(\xi', \tau', \theta') \in \mathcal{S}(\xi_0', 1, \theta_0', 0)$ *with some* $\xi_0' \in \mathbb{R}^{n_\xi}$*,* $\theta' \in \mathbb{T}$ *for which*
$$\theta'(t^+) = \theta_0, \text{ and } \xi'(t^+) = \xi_0 \tag{5}$$
are satisfied for some $t \in \mathbb{N}$*, and* $|\xi_0'| \leq \alpha_\theta(|\xi_0|)$*.*

Condition (A) can be interpreted as follows: it is possible to reach any combination of state $\xi_0 \in \mathbb{R}^{n_\xi}$ and mode $\theta_0 \in \mathbb{T}$ starting at the beginning of the flow ($\tau = 0$, and thus a $t \in \mathbb{N}$), from a state $\xi_0' \in \mathbb{R}^{n_\xi}$ (with its norm bounded by $|\xi_0|$) and mode $\theta_0' \in \mathbb{T}$ starting from the end of the flow ($\tau = 1$) with zero disturbance input, i.e., $w \equiv 0$.

A sufficient condition to guarantee that condition (A) is satisfied is given in the following lemma.

Lemma III.5. *If there exists a finite* $N \in \mathbb{N}$*, such that* (5) *is satisfied with* $t \leq N$ *for each* $\xi_0 \in \mathbb{R}^{n_\xi}$ *and* $\theta_0 \in \mathbb{T}$*, then a sufficient condition to guarantee* $|\xi_0'| \leq \alpha_\theta(|\xi_0|)$ *is*

- *For each jump map* $\Phi_\theta, \theta \in \mathbb{T}$*, there exists a* $\mathcal{K}$*-function* α_θ *such that for all* $\xi_0 \in \mathbb{R}^{n_\xi}$ *there is a* $\xi_0' \in \mathbb{R}^{n_\xi}$ *with* $\Phi_\theta(\xi_0') = \xi_0$ *and* $|\xi_0'| \leq \alpha_\theta(|\xi_0|)$

Remark III.6. In the literature, only the notion e-contractivity is studied for systems as in (1), see e.g., [2]. The results of Proposition III.4 show that the notions of contractivity and b-contractivity lead to stronger conclusions.

Remark III.7. Note that for each system (1) an equivalent system can be determined for which e-contractivity coincides with b-contractivity of the original system. The equivalent system extends the original system with an initial jump equal to identity, i.e., $\Phi_{\theta_{\text{init}}}(\xi) = \xi$, corresponding to an initial mode θ_{init}, which is solely active when $t \in [0, 1-\tau_0)$. This equivalent system allows to analyse b-contractivity of the orginal system as e-contravity of the equivalent system.

IV. Applications

In this section we show that systems such as periodic event-triggered control with varying delays and self-triggered control are captured by (1). To do so, first we introduce a modelling framework similar to (1) with varying inter-jump times. Due to the latter feature, the modelling framework captures aperiodic systems such as systems with varying delays and self-triggered control systems. We will show that, for each of the systems that is captured in this modelling framework, a system of the form (1) exists with identical internal stability and contractivity properties.

A. Modelling Framework

In this section we show that systems similar to (1), which include varying timer deadlines, can be reduced to a system of the form (1) with equal internal stability and contractivity properties. Consider the hybrid systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\\ l\\ \sigma\end{bmatrix} = \begin{bmatrix}A_\sigma\xi + B_\sigma w\\ 1\\ 0\\ 0\end{bmatrix}, \text{ when } \tau \in [0, h_l] \tag{6a}$$

$$\begin{bmatrix}\xi^+\\ \tau^+\\ l^+\\ \sigma^+\end{bmatrix} \in \phi(o) \times \{0\} \times L(o) \times \psi(o), \text{ when } \tau = h_l \tag{6b}$$

$$z = C_\sigma\xi + D_\sigma w, \tag{6c}$$

The states of this hybrid system consist of $\xi \in \mathbb{R}^{n_\xi}$, a timer variable $\tau \in \mathbb{R}_{\geq 0}$, the timer deadline mode $l \in \mathbb{L}$, with $\mathbb{L}$: $\{1, 2, ..., n_h\}$, $n_h \in \mathbb{N}$, and the flow mode $\sigma \in \mathbb{S}$ with $\mathbb{S} \subseteq \mathbb{N}$. The set of time deadlines is denoted by $\mathbb{H} = \{h_1, h_2, ..., h_{n_h}\}$ where $h_l \in \mathbb{R}_{>0}$ for all $l \in \mathbb{L}$. The variable $w \in \mathbb{R}^{n_w}$ denotes the disturbance input and $z \in \mathbb{R}^{n_z}$ the performance output. Moreover, A_σ, B_σ, C_σ, D_σ, $\sigma \in \mathbb{S}$, are constant real matrices of appropriate dimensions. When τ reaches h_l the state of the system undergoes a jump according to the set-valued mapping in (6b), where $o := (\xi, l, \sigma) \to \mathbb{R}^{n_\xi} \times \mathbb{L} \times \mathbb{S}$. The state ξ jumps according to $\phi : \mathbb{R}^{n_\xi} \times \mathbb{L} \times \mathbb{S} \rightrightarrows \mathbb{R}^{n_\xi}$, an arbitrary nonlinear set-valued and possibly discontinuous map with $\phi(0, l, \sigma) = \{0\}$, for $l \in \mathbb{L}$, $\sigma \in \mathbb{S}$. The timer deadline mode l jumps according to $L : \mathbb{R}^{n_\xi} \times \mathbb{L} \times \mathbb{S} \rightrightarrows \mathbb{L}$, and the flow mode σ jumps according to $\psi : \mathbb{R}^{n_\xi} \times \mathbb{L} \times \mathbb{S} \rightrightarrows \mathbb{S}$. In between the jumps the system flows according to the differential equation (6a).

In order to determine a system with equal internal stability and contractivity properties of the form (1) the matrices A_σ, B_σ, C_σ, D_σ should be scaled, for each $\sigma \in \mathbb{S}$, such that each timer deadline $h_l \in \mathbb{H}$ is transformed to 1. This gives rise to the idea to define a mode θ for each unique combination of the flow mode σ and timer deadline mode l, i.e., $\mathbb{T} := \mathbb{N}$ when $\mathbb{L}$ or $\mathbb{S}$ (or both) are countable otherwise $\mathbb{T} := n_h \times \mathbf{card}(\mathbb{S})$. Using this definition for the mode θ a bijection mapping $T : \mathbb{L} \times \mathbb{S} \to \mathbb{T}$ can be defined to determine the mode θ corresponding to a given mode $\sigma \in \mathbb{S}$ and timer deadline mode $l \in \mathbb{L}$. In the same manner we can define the mappings $\mu : \mathbb{T} \to \mathbb{L}$ and $\nu : \mathbb{T} \to \mathbb{S}$ to determine the timer deadline mode l and flow mode σ corresponding to a given mode $\theta \in \mathbb{T}$, respectively. These mappings satisfy the following equalities

$$\theta = T(\mu(\theta), \nu(\theta)), \tag{7a}$$
$$l = \mu(T(l, \sigma)), \tag{7b}$$
$$\sigma = \nu(T(l, \sigma)), \tag{7c}$$

for all $\theta \in \mathbb{T}$, $l \in \mathbb{L}$ and $\sigma \in \mathbb{S}$.

We now have the following propostion.

Proposition IV.1. *Consider system* (6) *with given sets $\mathbb{H}$, $\mathbb{L}$ and $\mathbb{S}$ and corresponding matrices A_σ, B_σ, C_σ and D_σ, $\sigma \in \mathbb{S}$, jump maps ϕ, L, and ψ and the mappings T, μ and ν satisfying* (7) *for $\theta \in \mathbb{T}$, $l \in \mathbb{L}$ and $\sigma \in \mathbb{S}$. Consider now system* (1) *with time-scaled matrices*

$$\begin{aligned} A_\theta &= h_{\mu(\theta)} A_{\nu(\theta)}, \qquad B_\theta = \sqrt{h_{\mu(\theta)}} B_{\nu(\theta)}, \\ C_\theta &= \sqrt{h_{\mu(\theta)}} C_{\nu(\theta)}, \quad D_\theta = D_{\nu(\theta)}, \end{aligned} \tag{8}$$

and the jump maps ϕ_θ and Ψ_θ defined as

$$\Phi_\theta(\xi) := \phi(\xi, \mu(\theta), \nu(\theta)), \tag{9a}$$
$$\Psi_\theta(\xi) := T(L(\xi, \mu(\theta), \nu(\theta)), \psi(\xi, \mu(\theta), \nu(\theta)). \tag{9b}$$

Then the following statement holds:

- *The system* (6) *is internally stable and contractive if and only if system* (1) *with* (8) *and* (9) *is internally stable and contractive.*

Exploiting the result of Proposition IV.1 allows to analyse the internal stability and contractivity properties of systems of the form (6) through a corresponding system of the form (1). In the next sections we will illustrate that PETC systems with varying delays and self-triggered control systems can be written in the form (6), and, therefore, analysis of the internal stability and contractivity properties of these systems can be performed through systems of the form (1).

B. Periodic-Event Triggered Control with Varying Delays

In this subsection, we will show that the PETC setup of Fig. 1 can be written as a system of the form (6).

The plant $\mathcal{P}$ in the PETC setup [22] of Fig. 1 is given by

$$\mathcal{P} : \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} x_p(t) = A_p x_p(t) + B_{pu} u(t) + B_{pw} w(t) \\ \quad y(t) = C_y x_p(t) + D_y u(t) \\ \quad z(t) = C_z x_p(t) + D_z u(t) + D_{zw} w(t), \end{cases} \tag{10}$$

where $x_p(t) \in \mathbb{R}^{n_{x_p}}$ denotes the state of the plant $\mathcal{P}$, $u(t) \in \mathbb{R}^{n_u}$ the control input, $w(t) \in \mathbb{R}^{n_w}$ the disturbance, $y(t) \in \mathbb{R}^{n_y}$ the measured output, and $z(t) \in \mathbb{R}^{n_z}$ the performance output at time $t \in \mathbb{R}_{\geq 0}$. This plant is controlled in an event-triggered feedback fashion using

$$\mathcal{C} : \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} x_c(t) = A_c x_c(t) + B_c \hat{y}(t) \\ \quad u(t) = C_u x_c(t) + D_u \hat{y}(t), \end{cases} \tag{11}$$

where $x_c(t) \in \mathbb{R}^{n_{x_c}}$ denotes the state of the controller $\mathcal{C}$, and $\hat{y}(t) \in \mathbb{R}^{n_y}$ represents the most recently received measurement of the output y available at the controller at time $t \in \mathbb{R}_{\geq 0}$. In

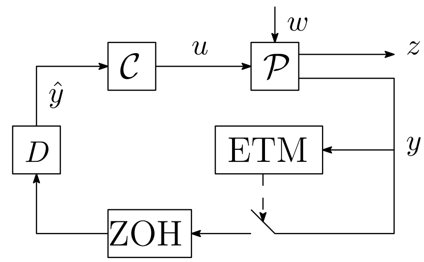

Figure 1. Schematic representation of an event-triggered control setup with communication delay (block D).

particular, $\hat{y}(t)$ is given for $t \in (\bar{t}_m + d_m, \bar{t}_{m+1} + d_{m+1}]$, $m \in \mathbb{N}$, by

$$\hat{y}(t) = \begin{cases} y(\bar{t}_m), & \text{when } g(\zeta(\bar{t}_m)) > 0, \\ \hat{y}(\bar{t}_m), & \text{when } g(\zeta(\bar{t}_m)) \leq 0, \end{cases} \tag{12}$$

where $\zeta := \begin{bmatrix} y^\top & \hat{y}^\top \end{bmatrix}^\top \in \mathbb{R}^{n_y}$ is the information available at the event-triggering mechanism (ETM) and $\bar{t}_m$, $m \in \mathbb{N}$, are the sampling times, which are periodic in the sense that $\bar{t}_m = m\bar{h}$, $m \in \mathbb{N}$, with $\bar{h}$ the sampling period. A communication delay is present and denoted by $d_m \in \{\tau_1, \tau_2, ..., \tau_{n_d}\}$, with $n_d \in \mathbb{N}$ the number of possible delays. The number of delays could also be countable, however for ease of exposition we take it finite here. Each delay $\tau_i, i \in \{1, 2, ..., n_d\}$, satisfies the small delay assumption, i.e., $\tau_i < \bar{h}, i \in \{1, 2, ..., n_d\}$. Note that (12) expresses that based on $\zeta(\bar{t}_m)$ the ETM decides at each time $\bar{t}_m$, $m \in \mathbb{N}$, using the relation g in (12), whether the measured output is transmitted to the controller or not. An example of a triggering condition is $|\hat{y}(\bar{t}_m) - y(\bar{t}_m)| > \sigma |y(\bar{t}_m)|$ for some $\sigma \in (0, 1)$ which translates in terms of (12) to

$$g(\zeta) = |\hat{y} - y|^2 - \sigma^2 |y|^2$$

although many other devices for g are possible as well (possibly involving auxiliary variables) are possible as well. Note that even though the system has a constant sampling period $\bar{h}$, the delay causes unequal inter-jump times (related to a transmission on $\bar{t}_m$ and an update of $\bar{y}$ on $\bar{t}_m + d_m$, $m \in \mathbb{N}$).

To show that the system can be written in the form (6), each sample will contain two jumps, one at t_m to model the transmission of the ETM, and one at $t_m + d_m$ to update the measurement output to the controller. Hence, we can define the timer-deadlines as:

$$\begin{aligned} &h_i := d_i, \; i = 1, ..., n_d \text{ and} \\ &h_i = \bar{h} - d_i, \; i = n_d + 1, ..., 2n_d \end{aligned} \tag{13}$$

with corresponding set $\mathbb{L} := \{1, 2, ..., 2n_d\}$.

In order to write the PETC system in the form (6), define a mode $\sigma \in \{1, 2\}$ where $\sigma = 1$ corresponds to a transmission and $\sigma = 2$ corresponds to an update. The jump maps L, ψ and ϕ can now be defined as

$$L(\sigma, l) \in \begin{cases} \{1, 2, ..., n_d\}, & \text{when } \sigma = 1, \\ \{l + n_d\}, & \text{when } \sigma = 2, \end{cases} \tag{14}$$

$$\psi(\sigma) \in \begin{cases} \{2\}, & \text{when } \sigma = 1, \\ \{1\}, & \text{when } \sigma = 2, \end{cases} \tag{15}$$

$$\phi(\xi,\sigma) = \begin{cases} \phi_t(\xi), & \text{when } \sigma = 1, \\ \phi_u(\xi), & \text{when } \sigma = 2, \end{cases} \tag{16}$$

The jump map ψ ensures the alternation between the two modes, the jump map L ensures an inter jump time corresponding to a delay, or the remainder of a sample, depending on the mode, the jump map ϕ models the update, where ϕ_t models the jumps at transmission times $\bar{t}_m$, $m \in \mathbb{N}$, and ϕ_u models the jumps at update times $\bar{t}_m + d_m$, $m \in \mathbb{N}$, (when transmitted data arrives at the controller). For this sequence of inter-jump times the initialisation $\tau(0) = 0$, $\sigma(0) = 1$ corresponds to the start of the flow, before a transmission. The maps ϕ_u and ϕ_t will be defined below after introducing a few auxiliary variables.

We first introduce the memory variable $s \in \mathbb{R}^{n_y}$ to store, at time $\bar{t}_m$, the value of the transmitted data $y(\bar{t}_m)$ to be used during an update at time $\bar{t}_m + d_m$, $m \in \mathbb{N}$. This leads to the overall state $\xi := \begin{bmatrix} x_p^\top & x_c^\top & \hat{y}^\top & s^\top \end{bmatrix}^\top \in \mathbb{R}^{n_\xi}$ with $n_\xi = n_{x_p} + n_{x_c} + 2n_y$. We also, define the matrix $Y \in \mathbb{R}^{2n_y \times n_\xi}$ as

$$Y := \begin{bmatrix} C_y & D_yC_u & D_yD_u & O \\ O & O & I & O \end{bmatrix} \tag{17}$$

such that $\zeta = Y\xi$.

Combining (10), (11), and (12) we can write the system in the form of (6) with

$$A = \begin{bmatrix} A_p & B_{pu}C_u & B_{pu}D_u & O \\ O & A_c & B_c & O \\ O & O & O & O \\ O & O & O & O \end{bmatrix}, \quad B = \begin{bmatrix} B_{pw} \\ O \\ O \\ O \end{bmatrix}, \tag{18}$$
$$C = \begin{bmatrix} C_z & D_zC_u & D_zD_u & O \end{bmatrix}, \qquad D = D_{zw}.$$

The update map ϕ_u is a linear map given for $\xi \in \mathbb{R}^{n_\xi}$ by

$$\phi_u(\xi) = \{J_0\xi\}, \text{ with } J_0 = \begin{bmatrix} I_{n_{x_p}} & O & O & O \\ O & I_{n_{x_c}} & O & O \\ O & O & O & I_{n_y} \\ O & O & O & I_{n_y} \end{bmatrix} \tag{19}$$

and the transmission map ϕ_t is a piecewise linear (PWL) map given for $\xi \in \mathbb{R}^{n_\xi}$ by

$$\phi_t(\xi) = \begin{cases} \{J_1\xi\}, & \text{when } g(Y\xi) > 0, \\ \{J_2\xi\}, & \text{when } g(Y\xi) \leq 0, \end{cases} \tag{20}$$

$$J_1 = \begin{bmatrix} I_{n_{x_p}} & O & O & O \\ O & I_{n_{x_c}} & O & O \\ O & O & I_{n_y} & O \\ C_y & D_yC_u & D_yD_u & O \end{bmatrix} \text{ and } J_2 = I_{n_\xi}.$$

Hence, the PETC system with varying delay is now written in the form (6) with data given by (14), (16), (18), (19), (20). Note that the addition of both the time-varying inter-jump times and jump map in (6), enables to model the varying transmission delay, which was not possible using the setup in [2]. The applications mentioned in [2] can also be put in this framework with the inclusion of delays in a similar fashion.

C. Self-Triggered Control

Similar to PETC above we will show that the STC setup of Fig. 2 can be written in a system of the form (6).

The plant and controller $\mathcal{C}$ are defined by (10) and (11), respectively. The most recent received measurement of the output y at the controller is denoted by $\hat{y}(t) \in \mathbb{R}^{n_y}, t \in \mathbb{R}_{\geq 0}$. The self-triggering mechanism (STM) decides, based on $\hat{y}(t_k)$ (or any other information that might be locally available), the next update time t_{k+1} for which the value of $\hat{y}$ is updated, using $t_{k+1} = t_k + L_s(\hat{y}(t_k))$, $k \in \mathbb{N}_{\geq 1}$, where the map $L_s : \mathbb{R}^{n_y} \to \{\bar{h}_1, ..., \bar{h}_{n_h}\}$. There are several posibilities for L_s, one concrete example being given in [17], where a STC is introduced with map L_s given by

$$L_s(\hat{y}) = \arg \min_{i \in \{1,2...,n_h\}} \hat{y}^\top R_i \hat{y} \tag{21}$$

for some given positive definite matrices $R_i, i \in \{1, 2, ..., n_h\}$. In particular $\hat{y}(t)$ is given by

$$\hat{y}(t) = y(t_k) \text{ for } t \in (t_k, t_{k+1}],\ k \in \mathbb{N}. \tag{22}$$

In order to write the STC system in the form (6), the overall state is defined as $\xi := \begin{bmatrix} x_p^\top & x_c^\top & \hat{y}^\top \end{bmatrix}^\top \in \mathbb{R}^{n_\xi}$ with $n_\xi = n_{x_p} + n_{x_c} + n_y$, and the matrix $Y \in \mathbb{R}^{2n_y \times n_\xi}$ is defined as

$$Y := \begin{bmatrix} O & O & I \end{bmatrix} \tag{23}$$

such that $\hat{y} = Y\xi$.

Combining (10), (11), (22), the STC system can be written in the form of (6) with

$$A = \begin{bmatrix} A_p & B_pC_u & B_pD_u \\ O & A_c & B_c \\ O & O & O \end{bmatrix}, \quad B = \begin{bmatrix} B_{pw} \\ 0 \\ 0 \end{bmatrix} \tag{24}$$
$$C = \begin{bmatrix} C_z & D_zC_u & D_zD_u \end{bmatrix}, \quad D = D_{zw}$$

Note that there is no dependency on the mode variable σ, hence, this variable is omitted. The update map is a linear map $\phi : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_\xi}$

$$\phi(\xi) = J\xi \tag{25}$$

with

$$J = \begin{bmatrix} I_{n_{x_p}} & O & O \\ O & I_{n_{x_c}} & O \\ C_z & D_zC_u & D_zD_u \end{bmatrix}.$$

Finally, the map $L(\xi)$ is defined as $L(\xi) := L_s(Y\xi)$ with L_s as in (21). Hence, the STC system is now written in the form (6).

V. Internal Stability and $\mathcal{L}_2$-gain Analysis

In this section we will analyse the $\mathcal{L}_2$-gain and the internal stability of (1) using ideas from lifting [3], [18]–[21]. As already indicated, in lifting $\tau(0)$ has to be a fixed value and natural candidates are $\tau(0) = 0$ or $\tau(0) = 1$.

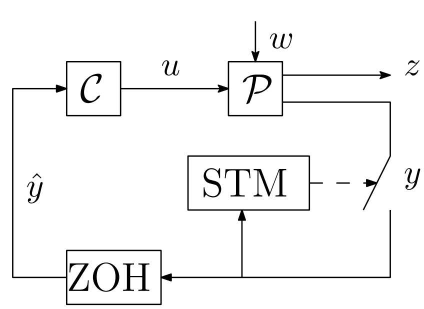

Figure 2. Schematic representation of a self-triggered control setup.

A. Lifting

To study b-internal stability and b-contractivity the lifting operator $W : \mathcal{L}_{2,e}[0,\infty) \to \ell(\mathcal{K})$ with $\mathcal{K} = \mathcal{L}_2[0,1]$ given for $w \in \mathcal{L}_{2,e}[0,\infty)$ by $W(w) = \tilde{w} = (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2, ...)$ with

$$\tilde{w}_k(s) = w(k+s) \text{ for } s \in [0,1] \tag{26}$$

for $k \in \mathbb{N}$. Obviously, W is a linear isomorphism mapping $\mathcal{L}_{2,e}[0,\infty)$ into $\ell(\mathcal{K})$ and moreover, W is isometric as a mapping from $\mathcal{L}_2[0,\infty)$ to $\ell_2(\mathcal{K})$. Using the lifting operator W, and by assuming $\tau(0) = 0$, in line with b-internal stability and b-contractivity, the model in (1) can be rewritten as

$$\xi^-_{k+1} = \hat{A}_{\theta_k}\xi_k + \hat{B}_{\theta_k}\tilde{w}_k \tag{27a}$$
$$\xi_{k+1} \in \Phi_{\theta_k}(\xi^-_{k+1}) \tag{27b}$$
$$\theta_{k+1} \in \Psi_{\theta_k}(\xi^-_{k+1}) \tag{27c}$$
$$\tilde{z}_k = \hat{C}_{\theta_k}\xi_k + \hat{D}_{\theta_k}\tilde{w}_k \tag{27d}$$

in which ξ_0 and θ_0 are given, and $\tilde{w} = (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2, ...) = W(w) \in \ell_2(\mathcal{K})$ and $\tilde{z} = (\tilde{z}_0, \tilde{z}_1, \tilde{z}_2, ...) = W(z) \in \ell(\mathcal{K})$. Assuming all signals are left-continuous we use the following notations for a signal $\alpha(t)$, $\alpha^-_k = \alpha(k^-) = \lim_{s\uparrow k}\alpha(s) = \alpha(k)$ and $\alpha_k = \alpha(k^+) = \lim_{s\downarrow k}\alpha(s)$. Moreover,

$$\hat{A}_\theta : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_\xi} \qquad \hat{B}_\theta : \mathcal{K} \to \mathbb{R}^{n_\xi}$$
$$\hat{C}_\theta : \mathbb{R}^{n_\xi} \to \mathcal{K} \qquad \hat{D}_\theta : \mathcal{K} \to \mathcal{K},$$

are given for $x \in \mathbb{R}^{n_\xi}$ and $\omega \in \mathcal{K}$ by

$$\hat{A}_\theta x = e^{A_\theta}x, \tag{28a}$$
$$\hat{B}_\theta\omega = \int_0^1 e^{A_\theta(1-s)}B_\sigma\omega(s)ds \tag{28b}$$
$$(\hat{C}_\theta x)(\varepsilon) = Ce^{A_\theta\varepsilon}x \tag{28c}$$
$$(\hat{D}_\theta\omega)(\varepsilon) = \int_0^\varepsilon C_\theta e^{A_\theta(\varepsilon-s)}B_\theta\omega(s)ds + D_\theta w(\varepsilon), \tag{28d}$$

where $\varepsilon \in [0,1]$, for all $\theta \in \mathbb{T}$. The set of maximal solutions from (27) starting from ξ_0, θ_0 and driven by $\tilde{w} \in \ell_2(\mathcal{K})$, will be denoted with $\mathcal{S}_b(\xi_0, \theta_0, \tilde{w})$. Note that for each solution in the set $\mathcal{S}(\xi_0, 0, \theta_0, w)$ there is a corresponding solution in the set $\mathcal{S}_b(\xi_0, \theta_0, \tilde{w})$ and vice versa.

In the same manner as for b-internal stability and b-contractivity, the lifting operator W can be used to study e-internal stability and e-contractivity. Assuming $\tau(0) = 1$ in line with e-internal stability and e-contractivity and using the operators (28) system (1) can be rewritten as

$$\xi'_{k+1} = \hat{A}_{\theta'_{k+1}}\xi'^+_k + \hat{B}_{\theta'_{k+1}}\tilde{w}'_k \tag{29a}$$
$$\xi'^+_k \in \Phi_{\theta'_k}(\xi'_k) \tag{29b}$$
$$\theta'_{k+1} = \theta'^+_k \in \Psi_{\theta'_k}(\xi'_k) \tag{29c}$$
$$\tilde{z}'_k = \hat{C}_{\theta'_{k+1}}\xi'^+_k + \hat{D}_{\theta'_{k+1}}\tilde{w}'_k \tag{29d}$$

in which ξ_0 and θ_0 are given, and $\tilde{w} = (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2, ...) = W(w) \in \ell_2(\mathcal{K})$ and $\tilde{z} = (\tilde{z}_0, \tilde{z}_1, \tilde{z}_2, ...) = W(z) \in \ell(\mathcal{K})$. For this system we use the following notations for a signal $\alpha'(t)$, $\alpha'_k = \alpha'(k^-) = \lim_{s\uparrow k}\alpha'(s) = \alpha'(k)$ and $\alpha'^+_k = \alpha'(k^+) = \lim_{s\downarrow k}\alpha'(s)$.

The set of maximal solutions from (29) starting from ξ'_0, θ'_0 and driven by $\tilde{w}' \in \ell_2(\mathcal{K})$, will be denoted with $\mathcal{S}'_e(\xi'_0, \theta'_0, \tilde{w}')$. Once more, note that for each solution in the set $\mathcal{S}(\xi'_0, 1, \theta'_0, w')$ there is a corresponding solution in the set $\mathcal{S}'_e(\xi'_0, \theta'_0, \tilde{w}')$ and vice versa.

By writing the solutions of (1) explicitly, and then comparing to the formulas (28) and using that W is an isometric isomorphism, it follows that (29) is contractive if and only if (1) is e-contractive. Moreover, by extending Propostition IV.1 of [2], we can establish the following proposition:

Proposition V.1. *The following statements hold:*

- *The hybrid system* (1) *is (b-)internally stable and (b-)contractive if and only if* (27) *is internally stable and contractive.*
- *The hybrid system* (1) *is e-internally stable and e-contractive if and only if* (29) *is internally stable and contractive.*
- *Moreover, in case* (1) *is internally stable, it also holds that* $\lim_{t\to\infty}\xi(t) = 0$ *and* $\|\xi\|_{\mathcal{L}_\infty} \leq \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_\infty}))$ *for all* $w \in \mathcal{L}_2$, $\xi(0) = \xi_0$ *and* $\tau(0) \in [0,1]$.

B. Main Result

The following result is an extension of the main result of [2]. Note that a necessary condition for (1) to be e-contractive is that the induced gains $\|\hat{D}_{\theta_k}\|_{\mathcal{L}_2[0,1]} < 1$ for all $k > 0$, and a necessary condition for b-contractivity of (1) is that the induced gains $\|\hat{D}_{\theta_k}\|_{\mathcal{L}_2[0,1]} < 1$ for all $k \geq 0$. Note that for e-contractivity there is no bound $\|\hat{D}_{\theta_0}\|_{\mathcal{L}_2[0,1]} < 1$, as the section $t \in [0,1]$ does not play any role. From this it is easy to come up with an example of an system which is e-contractive, but is not b-contractive

Example V.2. Consider the sequence of modes $\{\theta_k\}_{k\in\mathbb{N}}$ such that $\|\hat{D}_{\theta_k}\|_{\mathcal{L}_2[0,1]} < 1, k \in \mathbb{N}_{\geq 1}$, and $\|\hat{D}_{\theta_0}\|_{\mathcal{L}_2[0,1]} \geq 1$ holds. From the necessary conditions mentioned above, it is clear that this system can be e-contractive, (e.g. if it is satsified the condition of Theorem V.3 below) but not b-contractive.

Theorem V.3. *Consider system* (1) *and its b-lifted version* (27) *with* $\|\hat{D}_{\theta_k}\|_{\mathcal{K}} < 1$ *for all* $\theta_k \in \mathbb{T}$ *and* $k \geq 0$ *and its e-lifted version* (29) *with* $\|\hat{D}_{\theta_k}\|_{\mathcal{K}} < 1$ *for all* $\theta_k \in \mathbb{T}$ *and* $k > 0$. *Define the discrete-time nonlinear systems*

$$\bar{\xi}^-_{k+1} = \underline{A}_{\bar{\theta}_k}\bar{\xi}_k + \underline{B}_{\bar{\theta}_k}v_k \tag{30a}$$
$$\bar{\xi}_{k+1} \in \phi_{\bar{\theta}_k}(\bar{\xi}^-_{k+1}) \tag{30b}$$
$$\bar{\theta}_{k+1} \in \Psi_{\bar{\theta}_k}(\bar{\xi}^-_{k+1}) \tag{30c}$$
$$r_k = \underline{C}_{\bar{\theta}_k}\bar{\xi}_k \tag{30d}$$

and

$$\bar{\xi}'_{k+1} = \underline{A}_{\bar{\theta}'_{k+1}}\bar{\xi}'^+_k + \underline{B}_{\bar{\theta}'_{k+1}}v'_k \tag{31a}$$
$$\bar{\xi}'^+_k \in \phi_{\bar{\theta}'_{k+1}}(\bar{\xi}'_k) \tag{31b}$$
$$\bar{\theta}'_{k+1} = \bar{\theta}'^+_k \in \Psi_{\bar{\theta}'_{k+1}}(\bar{\xi}'_k) \tag{31c}$$
$$r'_k = \underline{C}_{\bar{\theta}'_{k+1}}\bar{\xi}'^+_k \text{ with,} \tag{31d}$$

$$\underline{A}_\theta = \hat{A}_\theta + \hat{B}_\theta\hat{D}^*_\theta(I - \hat{D}_\theta\hat{D}^*_\theta)^{-1}\hat{C}_\theta \tag{32a}$$

and $\underline{B}_\theta \in \mathbb{R}^{n_\xi \times n_v}$ *and* $\underline{C}_\theta \in \mathbb{R}^{n_r \times n_\xi}$ *are chosen such that*

$$\begin{aligned} \underline{B}_\theta \underline{B}_\theta^\top &= \bar{B}_\theta \bar{B}_\theta^\top = \hat{B}_\theta (I - \hat{D}_\theta^* \hat{D}_\theta)^{-1} \hat{B}_\theta^* \text{ and} \\ \underline{C}_\theta^\top \underline{C}_\theta &= \bar{C}_\theta^\top \bar{C}_\theta = \hat{C}_\theta^* (I - \hat{D}_\theta \hat{D}_\theta^*)^{-1} \hat{C}_\theta. \end{aligned} \tag{32b}$$

- *The system* (1) *is (b-)internally stable and (b-)contractive if and only if* (30) *is internally stable and contractive.*
- *The system* (1) *is e-internally stable and e-contractive if and only if* (31) *is internally stable and contractive.*

This main result can be exploited to analyse systems such as PETC and STC using a discrete-time non-linear system, of the form (30) or (31). A worked out example for the case of PETC with varying delays can be found in [4].

VI. Conclusion

In this paper, we studied the internal stability and $\mathcal{L}_2$-gain of hybrid systems with mode-dependent linear flow dynamics, periodic flow times, and time-varying nonlinear set-valued jump maps. This class of hybrid systems is relevant for various applications such as PETC, STC, NCS (that possibly include communication delays). We introduced new notions of internal stability and contractivity depending on the initial timer conditions. It is concluded that all three internal stability notions are equivalent. However, the notion of contractivity from the end of the flow (e-contractivity) as adopted in [2], [3] appears to be a weaker notion than the other two notions (b-contractivity and contractivity). We established that the internal stability and contractivity properties of these hybrid systems are equivalent to the internal stability and contractivity of an appropriate time-varying discrete-time nonlinear system.

Appendix

Proof of Proposition III.3. $(a) \Rightarrow (b)$, $(a) \Rightarrow (c)$ are immediate from $\{0\} \subseteq [0,1]$, $\{1\} \subseteq [0,1]$ respectively.

$(b) \Rightarrow (a)$ To prove this statement we assume b-internal stability as described in Definition III.2 for some $\mathcal{K}$-function β'. Let $\xi_0 \in \mathbb{R}^{n_\xi}$, $\tau_0 \in [0,1]$, and $w \in \mathcal{L}_2$ be given and consider $(\xi, \tau, \theta) \in \mathcal{S}(\xi_0, \tau_0, \theta_0, w)$. Consider now $w' \in \mathcal{L}_2$ defined as

$$w'(t) = \begin{cases} 0 & t \in [0, \tau_0) \\ w(t - \tau_0) & t \geq \tau_0. \end{cases} \tag{33}$$

Let $(\xi', \tau', \theta') \in \mathcal{S}(\xi_0', 0, \theta_0', w')$ for some arbitrary $\xi_0' \in \mathbb{R}^{n_\xi}$ (for now) and $\theta_0' = \theta_0$. Note that $\|w\|_{\mathcal{L}_2} = \|w'\|_{\mathcal{L}_2}$. For all $t \in [0, \tau_0]$ the response (ξ', τ', θ') satisfies

$$(\xi'(t), \tau'(t), \theta'(t)) = (e^{A_{\theta_0} t} \xi_0', \tau'(t + \tau_0), \theta_0'). \tag{34}$$

Using this, by taking $\xi_0' = e^{-A_{\theta_0} \tau_0} \xi_0$ we know that there exists a $(\xi', \tau', \theta_0') \in \mathcal{S}(\xi_0', 0, \theta_0', w')$ such that

$$(\xi(t), \tau(t), \theta(t)) = (\xi'(t + \tau_0), \tau'(t + \tau_0), \theta'(t + \tau_0)), \tag{35}$$

for $t \in \mathbb{R}_{\geq 0}$. Hence, $\|\xi\|_{\mathcal{L}_2} \leq \|\xi'\|_{\mathcal{L}_2}$. Using Definition III.2 for b-internal stability of the response (ξ', τ', θ') gives

$$\begin{aligned} \|\xi\|_{\mathcal{L}_2} \leq \|\xi'\|_{\mathcal{L}_2} &\leq \beta'(\max(|\xi_0'|, \|w'\|_{\mathcal{L}_2})) \\ &= \beta'(\max(|\xi_0'|, \|w\|_{\mathcal{L}_2})). \end{aligned} \tag{36}$$

Using that

$$\max_{\tau_0 \in [0,1]} \|e^{-A_{\theta_0} \tau_0}\| \leq c \tag{37}$$

for some $c > 1$, we get that $|\xi_0'| \leq c|\xi_0|$. Hence, from (36) we obtain for $\mathcal{K}$-function β given by $\beta(r) = \beta'(cr)$, $r \geq 0$, that

$$\|\xi\|_{\mathcal{L}_2} \leq \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_2})), \tag{38}$$

which establishes internal stability of system (1), as (4) holds.

$(c) \Rightarrow (b)$ To prove this statement we assume e-internal stability as described in Definition III.2 for some $\mathcal{K}$-function β'. Let $\xi_0 \in \mathbb{R}^{n_\xi}$, $w \in \mathcal{L}_2$ and consider $(\xi, \tau, \theta) \in \mathcal{S}(\xi_0, 0, \theta_0, w)$. Consider now $w', \bar{w} \in \mathcal{L}_2$ defined as

$$w'(t) = w(1 + t),\ t \in \mathbb{R}_{\geq 0} \tag{39}$$

and

$$\bar{w}(t) = \begin{cases} w(t), & t \in [0,1] \\ 0, & t \in (1, \infty). \end{cases} \tag{40}$$

Note that we have $\|w\|_{\mathcal{L}_2}^2 = \|\bar{w}\|_{\mathcal{L}_2}^2 + \|w'\|_{\mathcal{L}_2}^2$. In line with the definitions of w' and $\bar{w}$ we define ξ' and $\bar{\xi}$ such that

$$\|\xi\|_{\mathcal{L}_2}^2 = \|\bar{\xi}\|_{\mathcal{L}_2}^2 + \|\xi'\|_{\mathcal{L}_2}^2. \tag{41}$$

Part 1: First we establish that both $|\xi_0'|$ and $\|\bar{\xi}\|_{\mathcal{L}_2}$ are bounded by $|\xi_0|$ and $\|w\|_{\mathcal{L}_2}$ where

$$\xi_0' = e^{A_{\theta_0}} \xi_0 + N_{\theta_0}(\bar{w}), \tag{42}$$

with $N_\theta : \mathcal{L}_2 \to \mathbb{R}^{n_\xi}$ given for $\omega \in \mathcal{L}_2$ by

$$N_\theta \omega = \int_0^1 e^{A_\theta (1 - \tau)} B_\theta \omega(\tau) \, \mathrm{d}\tau. \tag{43}$$

Note that $N_\theta, \theta \in \mathbb{T}$ are bounded linear operators and $\|e^{A_\theta}\| < c_1$, $\forall \theta \in \mathbb{T}$ for some $c_1 > 0$, this gives

$$|\xi_0'| \leq c_1 |\xi_0| + c_2 \|\bar{w}\|_{\mathcal{L}_2} \leq c_1 |\xi_0| + c_2 \|w\|_{\mathcal{L}_2} \tag{44}$$

for some $c_2 > 0$. Next we define the operators $\hat{M}_\theta : \mathbb{R}^{n_\xi} \to \mathcal{L}_2[0,1]$ and $\hat{N}_\theta : \mathcal{L}_2[0,1] \to \mathcal{L}_2[0,1]$ given for $x \in \mathbb{R}^{n_\xi}$ and $\omega \in \mathcal{L}_2[0,1]$ by

$$(\hat{M}_\theta x)(s) = e^{A_\theta s} x, \quad (\hat{N}_\theta \omega)(\theta) = \int_0^s e^{A_\theta (s - \tau)} B_\theta \omega(\tau) \, \mathrm{d}\tau \tag{45}$$

with $s \in [0,1]$, such that

$$\bar{\xi} = \hat{M}_{\theta_0} \xi_0 + \hat{N}_{\theta_0} \bar{w}. \tag{46}$$

Note that $\hat{M}_\theta$ and $\hat{N}_\theta$ are bounded linear operators for all $\theta \in \mathbb{T}$. Hence,

$$\begin{aligned} \|\bar{\xi}\|_{\mathcal{L}_2[0,1)} &\leq c_3 |\xi_0| + c_4 \|\bar{w}\|_{\mathcal{L}_2[0,1)} \\ &\leq c_3 |\xi_0| + c_4 \|w\|_{\mathcal{L}_2} \end{aligned} \tag{47}$$

for some $c_3, c_4 > 0$.

Part 2: Using the results of Part 1 we will show that the responses (ξ, τ, θ) satisfies (4). Note that for this choice of ξ_0', and w' there is a $(\xi', \tau', \theta') \in \mathcal{S}(\xi_0', 1, \theta_0', w')$ such that

$$(\xi(t + 1), \tau(t + 1), \theta(t + 1)) = (\xi'(t), \tau'(t), \theta'(t)), \tag{48}$$

for $t \in \mathbb{R}_{\geq 0}$. From e-internal stability we get

$$\|\xi'\|_{\mathcal{L}_2} \leq \beta'(\max(|\xi_0'|, \|w'\|_{\mathcal{L}_2})). \tag{49}$$

Combining this with (41), (44) and (47) gives

$$\begin{aligned}\|\xi\|^2_{\mathcal{L}_2} &= \|\bar{\xi}\|^2_{\mathcal{L}_2} + \|\xi'\|^2_{\mathcal{L}_2} \\ &\leq \beta_3(\max(|\xi_0|, \|w\|_{\mathcal{L}_2})) + c_5|\xi_0|^2 + c_6\|w\|^2_{\mathcal{L}_2}\end{aligned} \tag{50}$$

for some $c_5, c_6 > 0$. This establishes internal stability of system (1) as for some $\mathcal{K}$-function β we have

$$\|\xi\|_{\mathcal{L}_2} \leq \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_2})). \tag{51}$$

□

Proof of Proposition III.4. $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ are immediate from $\{0\} \subseteq [0,1]$ and, $\{1\} \subseteq [0,1]$, respectively.

$(ii) \Rightarrow (i)$ To prove this statement we assume b-contractivity as described in Definition III.1 for some $\mathcal{K}$-function β' and some $\bar{\gamma}' < 1$. Let $\xi_0 \in \mathbb{R}^{n_\xi}$, $\tau_0 \in [0,1]$, $\theta_0 \in \mathbb{T}$ and $w \in \mathcal{L}_2$ be given and consider $(\xi, \tau, \theta, z) \in \mathcal{S}_p(\xi_0, \tau_0, \theta_0, w)$. Consider now $w' \in \mathcal{L}_2$ as in (33), which gives $\|w\|_{\mathcal{L}_2} = \|w'\|_{\mathcal{L}_2}$. Let $(\xi', \tau', \theta', z') \in \mathcal{S}_p(\xi_0', 0, \theta_0', w')$, with $\xi_0' \in \mathbb{R}^{n_\xi}$ and $\theta_0' \in \mathbb{T}$. For all $t \in [0, \tau_0]$ the response (ξ', θ', z') satisfies (34). Using this, and by taking $\xi_0' = e^{-A_{\theta_0}\tau_0}\xi_0$ as in proof of Proposition III.3, there is a $(\xi', \tau', \theta', z') \in \mathcal{S}_p(\xi_0', 0, \theta_0', w')$ such that

$$\begin{aligned}&(\xi(t), \tau(t), \theta(t), z(t)) = \\ &\qquad (\xi'(t+\tau_0), \tau'(t+\tau_0), \theta'(t+\tau_0)), z'(t+\tau_0),\end{aligned} \tag{52}$$

for $t \in \mathbb{R}_{\geq 0}$. Hence, $\|z\|_{\mathcal{L}_2} \leq \|z'\|_{\mathcal{L}_2}$. Using b-contractivity for the response $(\xi', \tau', \theta', z')$ gives

$$\begin{aligned}\|z\|_{\mathcal{L}_2} \leq \|z'\|_{\mathcal{L}_2} &\leq \beta'(|\xi_0'|) + \bar{\gamma}'\|w'\|_{\mathcal{L}_2} \\ &= \beta'(|\xi_0'|) + \bar{\gamma}'\|w\|_{\mathcal{L}_2}.\end{aligned} \tag{53}$$

Using again (37), we get that $|\xi_0'| \leq c|\xi_0|$. Hence, from (53) we obtain

$$\|z\|_{\mathcal{L}_2} \leq \beta'(c|\xi_0|) + \bar{\gamma}'\|w\|_{\mathcal{L}_2}, \tag{54}$$

which establishes the contractivity of system (1), as (3) holds with $\beta(r) = \beta'(cr)$, $r \in \mathbb{R}_{\geq 0}$, which is also a $\mathcal{K}$-function, and $\bar{\gamma} = \bar{\gamma}'$.

$(iii) \Rightarrow (ii)$ To prove this statement we assume e-contractivity as described in Definition III.1 for some $\mathcal{K}$-function β' and some $\bar{\gamma}' < 1$ and condition (1) and (2) are satisfied. Consider the response $(\xi, \tau, \theta, z) \in \mathcal{S}_p(\xi_0, 0, \theta_0, w)$ with $\xi_0 \in \mathbb{R}^{n_\xi}$, Consider the initial conditions ξ_0' and θ_0' for which condition (1) is satisfied and define M as value of t for which (5) is satisfied. Now define the the input w' as

$$w'(t) = \begin{cases} 0 & t \in [0, M) \\ w(t-M) & t \geq M \end{cases} \tag{55}$$

and let $(\xi', \tau', \theta', z') \in \mathcal{S}_p(\xi_0', 1, \theta_0', w')$ be the response for which

$$(\xi'(t'^+), \tau'(t'^+), \theta'(t'^+), z'(t'^+)) = (\xi(t^+), \tau(t^+), \theta(t^+)) \tag{56}$$

is satisfied with $t'^+ = (t-M)^+$ (this response exists due to condition (2)). Note that $\|z\|_{\mathcal{L}_2} \leq \|z'\|_{\mathcal{L}_2}$ and $\|w\|_{\mathcal{L}_2} = \|w'\|_{\mathcal{L}_2}$. Using e-contractivity for the response $(\xi', \tau', \theta', z')$ we have

$$\begin{aligned}\|z\|_{\mathcal{L}_2} \leq \|z'\|_{\mathcal{L}_2} &\leq \beta'(c|\xi_0'|) + \bar{\gamma}'\|w'\|_{\mathcal{L}_2} \\ &= \beta'(c|\xi_0'|) + \bar{\gamma}'\|w\|_{\mathcal{L}_2}.\end{aligned} \tag{57}$$

Using now that $|\xi_0'| \leq \alpha(|\xi_0|)$ for some $\mathcal{K}$-function α leads to

$$\|z\|_{\mathcal{L}_2} \leq \beta(|\xi_0|) + \bar{\gamma}\|w\|_{\mathcal{L}_2}, \tag{58}$$

which establishes (3) and thus contractivity of system (1) for $\mathcal{K}$-function $\beta(r) = \beta'(\alpha(r)), r \in \mathbb{R}_{\geq 0}$, and $\bar{\gamma} = \gamma' \in [0,1)$.

□

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