2017 IEEE 56th Annual Conference on Decision and Control (CDC)
December 12-15, 2017, Melbourne, Australia

$\mathcal{L}_2$-gain Analysis of Periodic Event-triggered Systems with Varying Delays using Lifting Techniques

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***Abstract*— In this paper we study the stability and $\mathcal{L}_2$-gain properties of periodic event-triggered control (PETC) systems including time-varying delays. We introduce a general framework that captures these PETC systems and encompasses a class of hybrid systems that exhibit linear flow, aperiodic time-triggered jumps (possibly with different deadlines) and arbitrary nonlinear time-varying jump maps. New notions on the stability and contractivity ($\mathcal{L}_2$-gain strictly smaller than 1) from the beginning of the flow and from the end of the flow are introduced and formal relationships are deduced between these notions, revealing that some are stronger than others. Inspired by ideas from lifting, it is shown that the internal stability and contractivity in $\mathcal{L}_2$-sense of a continuous-time hybrid system in the framework is equivalent to the stability and contractivity in ℓ_2-sense of an appropriate time-varying discrete-time nonlinear system. These results recover existing works in the literature as special cases and indicate that analysing different discrete-time nonlinear systems (of the same level of complexity) than in existing works yield stronger conclusions on the $\mathcal{L}_2$-gain. At the end of the paper we indicate several extensions of the framework, which even include the possibility of the inter-jump times depending on the state, such that, for instance, self-triggered control systems can also be included allowing their stability and contractivity analysis. A numerical example is presented showing how stability and contractivity analyses are carried out for PETC systems with delays.**

I. Introduction

In this paper, we consider hybrid systems that can be written in the framework

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\\ k\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\\ 0\end{bmatrix}, \text{ when } \tau \in [0, h_k] \tag{1a}$$

$$\begin{bmatrix}\xi^+\\ \tau^+\\ k^+\end{bmatrix} \in \phi_k(\xi) \times \{0\} \times \{k+1\}, \text{ when } \tau = h_k \tag{1b}$$

$$z = C\xi + Dw, \tag{1c}$$

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W.P.M.H. Heemels is partially supported by the Innovational Research Incentives Scheme under the VICI grant "Wireless control systems: A new frontier in automation" (No. 11382) awarded by NWO (The Netherlands Organisation for Scientific Research) and STW (Dutch Technology Foundation). G.E. Dullerud is partially supported by AFOSR grant FA9550-15-1-0059. A.R. Teel is partially supported by AFOSR grant number FA9550-15-1-0155, and NSF grant number ECCS-1508757.

The states of this hybrid system consist of $\xi \in \mathbb{R}^{n_\xi}$, a timer variable $\tau \in \mathbb{R}_{\geq 0}$, and a counter variable $k \in \mathbb{N}$. The variable $w \in \mathbb{R}^{n_w}$ denotes the disturbance input and $z \in \mathbb{R}^{n_z}$ the performance output. Moreover, A, B, C, D are constant real matrices of appropriate dimensions and $\{h_k\}_{k\in\mathbb{N}}$ with $h_k \in \mathbb{R}_{>0}$ are given inter-jump times. When τ reaches h_k the state of the system undergoes a jump. The timer deadlines $h_k \in \mathbb{R}_{>0}$, for all $k \in \mathbb{N}$, and the set-valued jump map $\phi_k : \mathbb{R}^{n_\xi} \rightrightarrows \mathbb{R}^{n_\xi}$, for all $k \in \mathbb{N}$ are time-varying. Here ϕ_k, $k \in \mathbb{N}$, is an arbitrary nonlinear set-valued and possibly discontinuous map with $\phi_k(0) = \{0\}$, $k \in \mathbb{N}$.

Interestingly, the class of hybrid systems captured by (1) is a generalization and unification of those found in [1], [2], [3]. In [1], [2] a class of hybrid systems is discussed with periodic jumps and time-invariant possibly nonlinear jump maps. This class of hybrid systems is included in (1), for $h_k = \bar{h}$ and $\phi_k = \phi$, $k \in \mathbb{N}$. The class of hybrid systems discussed in [3] uses linear jump maps, i.e., ϕ_k are all linear, although the jumps are aperiodic, as in (1).

The hybrid systems as in [1] are relevant for the closed-loop systems arising from PETC [4], networked control systems with constant transmission intervals and shared communication networks requiring medium access protocols [5], reset control systems [6], [7], [8] with periodically verified reset conditions, and sampled-data saturated controls [9], see also [1]. The time-varying jump maps ϕ_k and time-varying deadlines h_k in (1) allow to model all the mentioned applications including also communication delays, which were not considered before. Note that towards the end of the paper we consider an even more general setup than (1), which allows for the inclusion of k-dependent matrices A_k, B_k, C_k, D_k, and deadlines h_k depending on the state ξ or certain discrete dynamics. This opens up the inclusion of new applications such as self-triggered control [10], [11], and the aforementioned applications with *varying* transmission delays.

Initially for systems as in (1), and later for the extended framework, we analyse the stability and contractivity ($\mathcal{L}_2$-gain strictly smaller than 1) exploiting ideas from lifting [3], [12], [13], [14], [15]. This will lead to novel necessary and sufficient conditions of stability and contractivity of (1) (and its extensions) in terms of the stability and contractivity in ℓ_2-sense of an appropriate time-varying discrete-time nonlinear system. To arrive at the time-varying discrete-time nonlinear system some restrictions on the initial conditions of (1) are required related to taking, e.g., $\tau(0) = 0$ or $\tau(0) = h_0$. In fact, such restrictions will lead to three different notions of stability and contractivity, which will be introduced formally in this paper. We study the relationships between these notions, which reveals that so-called contractivity from the beginning of the flow (corresponding to $\tau(0) = 0$) is stronger than stability and contractivity from the end of the flow ($\tau(0) = h_0$). This also leads to different discrete-time nonlinear systems after including lifting as the ones used in [1], [2], [3]. Interestingly, these new results not only lead to stronger conclusions but also apply to a significantly larger classes of systems than in [1], [2], [3] encompassing new applications in areas such as PETC systems with (varying) delays and self-triggered control for which no necessary and sufficient conditions on stability and contractivity were available in the literature before. Through a numerical example of a PETC system with delays, we will show how the results can be used to perform the stability and contractivity analyses.

The remainder of this paper is organized as follows. In Section II we show how PETC systems with a delay can be captured in (1). In Section III we introduce the preliminaries. In Section IV we introduce new notions of stability and contractivity and derive formal relationships between them. In the main result (Theorem V.3) we connect the corresponding internal stability and contractivity properties of (1) to the internal stability and contractivity of an appropriate discrete-time system. In Section VI we introduce an extension of the framework (1) allowing to include systems with varying delays and self-triggered control applications. In Section VII we show how these results can be used to analyse the stability and contractivity for PETC systems with varying delays. In Section VIII we present a numerical example. Conclusions are given in Section IX.

II. Periodic Event-Triggered Control Systems with Delays

In this paper, we consider the PETC setup of Fig. 1, in which the plant $\mathcal{P}$ is given by

$$\mathcal{P}: \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}x_p(t) = A_p x_p(t) + B_{pu}u(t) + B_{pw}w(t) \\ \quad y(t) = C_y x_p(t) + D_y u(t) \\ \quad z(t) = C_z x_p(t) + D_z u(t) + D_{zw}w(t), \end{cases} \tag{2}$$

where $x_p(t) \in \mathbb{R}^{n_{x_p}}$ denotes the state of the plant $\mathcal{P}$, $u(t) \in \mathbb{R}^{n_u}$ the control input, $w(t) \in \mathbb{R}^{n_w}$ the disturbance, $y(t) \in \mathbb{R}^{n_y}$ the measured output, and $z(t) \in \mathbb{R}^{n_z}$ the performance output at time $t \in \mathbb{R}_{\geq 0}$. This plant is controlled in an event-triggered feedback fashion using

$$\mathcal{C}: \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}x_c(t) = A_c x_c(t) + B_c \hat{y}(t) \\ \quad u(t) = C_u x_c(t) + D_u \hat{y}(t), \end{cases} \tag{3}$$

where $x_c(t) \in \mathbb{R}^{n_{x_c}}$ denotes the state of the controller $\mathcal{C}$, and $\hat{y}(t) \in \mathbb{R}^{n_y}$ represents the most recently received measurement of the output y available at the controller at time $t \in \mathbb{R}_{\geq 0}$. In particular, $\hat{y}(t)$ is given for $t \in (\bar{t}_m + \tau_d, \bar{t}_{m+1} + \tau_d]$, $m \in \mathbb{N}$, by

$$\hat{y}(t) = \begin{cases} y(\bar{t}_m), & \text{when } \zeta(\bar{t}_m)^\top \hat{Q}\zeta(\bar{t}_m) > 0, \\ \hat{y}(\bar{t}_m), & \text{when } \zeta(\bar{t}_m)^\top \hat{Q}\zeta(\bar{t}_m) \leq 0, \end{cases} \tag{4}$$

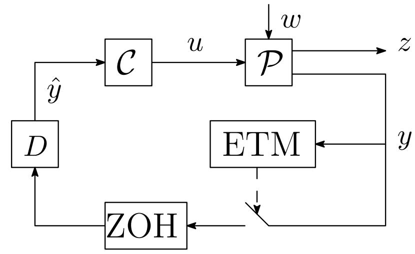

Fig. 1. Schematic representation of an event-triggered control setup with communication delay (block D).

where $\zeta := \begin{bmatrix} y^\top & \hat{y}^\top \end{bmatrix}^\top \in \mathbb{R}^{n_y}$ is the information available at the event-triggering mechanism (ETM) and $\bar{t}_m$, $m \in \mathbb{N}$, are the sampling times, which are periodic in the sense that $\bar{t}_m = m\bar{h}$, $m \in \mathbb{N}$, with $\bar{h}$ the sampling period. The communication delay is denoted by τ_d and satisfies the small delay assumption, $\tau_d < \bar{h}$. Note that (4) expresses that based on $\zeta(\bar{t}_m)$ the ETM decides at each time $\bar{t}_m$, $m \in \mathbb{N}$, using the quadratic relation in (4), whether the measured output is transmitted to the controller or not. An example of a triggering condition is $|\hat{y}(\bar{t}_m) - y(\bar{t}_m)| > \sigma|y(\bar{t}_m)|$ for some $\sigma \in (0,1)$ which translates in terms of (4) to

$$\hat{Q} = \begin{bmatrix} (1-\sigma^2)I & -I \\ -I & I \end{bmatrix}.$$

Although the system has a constant sampling period $\bar{h}$, the delay causes unequal inter-jump times (related to a transmission on $\bar{t}_m$ and an update on $\bar{t}_m + \tau_d$). In terms of (1), the inter-jump times are defined as

$$h_k = \begin{cases} \tau_d & \text{when } k \text{ is odd} \\ \bar{h} - \tau_d & \text{when } k \text{ is even,} \end{cases} \tag{5}$$

and map

$$\phi_k = \begin{cases} \phi_u & \text{when } k \text{ is odd} \\ \phi_t & \text{when } k \text{ is even,} \end{cases} \tag{6}$$

where ϕ_u models the jumps at update times $\bar{t}_m + \tau_d$, $m \in \mathbb{N}$ (when transmitted data arrives at the controller), and ϕ_t models the jumps at transmission times $\bar{t}_m$, $m \in \mathbb{N}$. For this sequence of inter-times the initial conditions are $\tau(0) = 0$. The maps ϕ_u and ϕ_t will be defined below after introducing a few auxiliary variables.

In order to write the PETC system in the form (1) we first introduce the memory variable $s \in \mathbb{R}^{n_y}$ to store, at time $\bar{t}_m$ the value of transmitted data $y(\bar{t}_m)$ to be used during an update at time $\bar{t}_m + \tau_d$, $m \in \mathbb{N}$, this leads to the overall state $\xi := \begin{bmatrix} x_p^\top & x_c^\top & \hat{y}^\top & s^\top \end{bmatrix}^\top \in \mathbb{R}^{n_\xi}$ with $n_\xi = n_{x_p} + n_{x_c} + 2n_y$. We also, define the matrix $Y \in \mathbb{R}^{2n_y \times n_\xi}$ as

$$Y := \begin{bmatrix} C_y & D_y C_u & D_y D_u & O \\ O & O & I & O \end{bmatrix} \tag{7}$$

such that $\zeta = Y\xi$.

Combining (2), (3), and (4) we can write the system in the form of (1) with

$$A = \begin{bmatrix} A_p & B_{pu}C_u & B_{pu}D_u & O \\ O & A_c & B_c & O \\ O & O & O & O \\ O & O & O & O \end{bmatrix}, \quad B = \begin{bmatrix} B_{pw} \\ O \\ O \\ O \end{bmatrix}, \tag{8}$$
$$C = \begin{bmatrix} C_z & D_zC_u & D_zD_u & O \end{bmatrix}, \qquad D = D_{zw}.$$

The update map ϕ_u is a linear map given for $\xi \in \mathbb{R}^{n_\xi}$ by

$$\phi_u(\xi) = \{J_0\xi\}, \text{ with } J_0 = \begin{bmatrix} I_{n_{x_p}} & O & O & O \\ O & I_{n_{x_c}} & O & O \\ O & O & O & I_{n_y} \\ O & O & O & I_{n_y} \end{bmatrix} \tag{9}$$

and the transmission map ϕ_t is a piecewise linear (PWL) map given for $\xi \in \mathbb{R}^{n_\xi}$ by

$$\phi_t(\xi) = \begin{cases} \{J_1\xi\}, & \text{when } \xi^\top Q\xi > 0 \\ \{J_2\xi\}, & \text{when } \xi^\top Q\xi \le 0 \end{cases} \tag{10}$$

with $Q = Y^\top \hat{Q} Y$,

$$J_1 = \begin{bmatrix} I_{n_{x_p}} & O & O & O \\ O & I_{n_{x_c}} & O & O \\ O & O & I_{n_y} & O \\ C_y & D_yC_u & D_yD_u & O \end{bmatrix} \text{ and } J_2 = I_{n_\xi}.$$

Hence, the PETC system with delay is now written in the form (1) with data given by (5), (6), (8), (9), (10). Note that the addition of both the time-varying inter-jump times and jump map in (1) compared to [1] enables to model the transmission delay. The applications mentioned in [1], [2] can also be put in this framework with the inclusion of delays in a similar fashion.

III. Preliminaries

We recall a few necessary preliminaries, mostly taken from [1]. As usual, we denote by $\mathbb{R}^n$ the standard n-dimensional Euclidean space with inner product $\langle x, y\rangle = x^\top y$ and norm $|x| = \sqrt{x^\top y}$ for $x, y \in \mathbb{R}^n$. $\mathcal{L}_2^n[0,\infty)$ denotes the set of square-integrable functions defined on $\mathbb{R}_{>0} := [0,\infty)$ and taking values in $\mathbb{R}^n$ with $\mathcal{L}_2$-norm $\|x\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty |x(t)|^2 \, \mathrm{d}t}$ and inner product $\langle x, y\rangle_{\mathcal{L}_2} = \int_0^\infty x^\top y \, \mathrm{d}t$ for $x, y \in \mathcal{L}_2[0,\infty)$. If n is clear from the context we also write $\mathcal{L}_2$. We also use square-integrable functions on subsets $[a,b]$ of $\mathbb{R}_{\ge 0}$ and then we write $\mathcal{L}_2^n[a,b]$ (or $\mathcal{L}_2[a,b]$ if n is clear form the context) with inner product and norm defined analogously. The set $\mathcal{L}_{2,e}[0,\infty)$ consists of all locally square-integrable functions, i.e., all functions x defined on $\mathbb{R}_{\ge 0}$, such that for each bounded domain $[a,b] \subset \mathbb{R}_{\ge 0}$ the restriction $x|_{[a,b]}$ is contained in $\mathcal{L}_2^n[a,b]$. We also will use the set of essentially bounded functions defined on $\mathbb{R}_{\ge 0}$ or $[a,b] \subset \mathbb{R}_{\ge 0}$, which are denoted by $\mathcal{L}_\infty^2([0,\infty))$ or $\mathcal{L}_\infty^2([a,b])$, respectively, with the norm given by the essential supremum denoted by $\|x\|_{\mathcal{L}_\infty}$ for an essentially bounded function x. A function $\beta : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is called a $\mathcal{K}$-function if it is continuous, strictly increasing and $\beta(0) = 0$.

For X, Y Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$, respectively, a linear operator $U : X \to Y$ is called isometric if $\langle Ux_1, Ux_2\rangle_Y = \langle x_1, x_2\rangle_X$ for all $x_1, x_2 \in X$. The (Hilbert) adjoint operator is denoted by $U^* : X \to Y$ and satisfies $\langle Ux, y\rangle_Y = \langle x, U^*y\rangle_X$ for all $x \in X$ and $y \in Y$. Note that U being isometric is equivalent to $U^*U = I$ (or $U^*U = I$). The operator U is called an isomorphism if it is an invertible mapping, i.e., if it is one-to-one. The induced norm of U (provided it is finite) is denoted by $\|U\|_{X,Y} = \sup_{x \in X\setminus\{0\}} \frac{\|Ux\|_Y}{\|x\|_X}$. If the induced norm is finite we say U is a bounded linear operator. If $X = Y$ we write $\|U\|_X$ and if X, Y are clear form the context we use the notation $\|U\|$.

To an infinite sequence of Hilbert spaces $\{X_k\}_{k\in\mathbb{N}}$, we can associate a Hilbert space $\ell_2(\{X_k\}_{k\in\mathbb{N}})$ consisting of infinite sequences $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, ...)$, with $\tilde{x}_k \in X_k$, $k \in \mathbb{N}$, satisfying $\sum_{i=0}^\infty \|\tilde{x}_i\|_{X_i}^2 < \infty$, and the inner product $\langle \tilde{x}, \tilde{y}\rangle_{\ell_2(\{X_k\}_{k\in\mathbb{N}})} = \sum_{i=0}^\infty \langle \tilde{x}_i, \tilde{y}_i\rangle_{X_i}$. In case $X_k = V$ for all $k \in \mathbb{N}$, we also write $\ell_2(V)$ for short. We denote $\ell_2(\mathbb{R}^n)$ by ℓ_2 when $n \in \mathbb{N}_{\ge 1}$ is clear from the context. We also use the notation $\ell(\{X_k\}_{k\in\mathbb{N}})$ to denote the set of all infinite sequences $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, ...)$ with $x_k \in X_k$, $k \in \mathbb{N}$.

Consider the discrete-time system of the form

$$\begin{bmatrix} \xi_{k+1} \\ r_k \end{bmatrix} \in \psi(\xi_k, v_k) \tag{11}$$

with $v_k \in V$, $r_k \in R$, $\xi_k \in \mathbb{R}^{n_\xi}$, $k \in \mathbb{N}$, with V and R Hilbert spaces and $\psi : \mathbb{R}^{n_\xi} \times V \rightrightarrows \mathbb{R}^{n_\xi} \times R$.

Definition III.1 The discrete-time system (11) is said to have an ℓ_2-gain from v to r smaller than γ if there exist a $\gamma_0 \in [0,\gamma)$ and a $\mathcal{K}$-function β such that, for any $v \in \ell_2(V)$ and any intial state $\xi_0 \in \mathbb{R}^{n_\xi}$, the corresponding solutions to (11) satisfy $\|r\|_{l_2(R)} \le \beta(\|\xi_0\|) + \gamma_0\|v\|_{l_2(V)}$. The terminology γ-contractivity is used if this property holds. Moreover, 1-contractivity is also called contractivity (in ℓ_2-sense).

Definition III.2 The discrete-time system (11) is said to be internally stable if there is a $\mathcal{K}$-function β such that, for any $v \in \ell_2(V)$ and any initial state $\xi_0 \in \mathbb{R}^{n_\xi}$, the corresponding solutions ξ to (11) satisfy $\|\xi\|_{\ell_2} \le \beta(\max(|\xi_0|, \|v\|_{\ell_2(V)}))$.

IV. Stability and Contractivity Notions

In this paper, we will focus on both internal stability and the question if the $\mathcal{L}_2$-gain of (1) is smaller than 1, called contractivity. Note that by proper scaling of C and D in (1), it can be determined from contractivity properties if the $\mathcal{L}_2$-gain is smaller than any other value of $\gamma \in \mathbb{R}_{>0}$ as well.

In fact, we define three notions of internal stability and contractivity, depending on restrictions regarding the set of initial states, especially on the initial condition of τ. We will use (τ_0, ξ_0) for the initial value of (τ, ξ) and write $\mathcal{S}(\tau_0, \xi_0, w)$ for the set of maximal solutions starting at $k(0) = 0$ with $\tau(0) = \tau_0$ and $\xi(0) = \xi_0$ driven by $w \in \mathcal{L}_2$, where we assume all signals are left-continuous. See [16] for the definition of maximal solutions.

Definition IV.1 The hybrid system (1) is said to be contractive, if there exist a $\gamma_0 \in [0,1)$ and a $\mathcal{K}$-function β such that, for any $w \in \mathcal{L}_2$, $\xi_0 \in \mathbb{R}^{n_\xi}$, $\tau_0 \in [0, h_0]$ any $(\xi, z) \in \mathcal{S}(\tau_0, \xi_0, w)$ satisfies

$$\|z\|_{\mathcal{L}_2} \le \beta(|\xi_0|) + \gamma_0\|w\|_{\mathcal{L}_2}. \tag{12}$$

If this property holds for $\tau_0 \in \{0\}$, or $\tau_0 \in \{h_0\}$ the system is said to be contractive from the beginning of the flow (b-contractive) or contractive from the end of the flow (e-contractive), respectively.

Definition IV.2 The hybrid system (1) is said to be internally stable if there exist a $\mathcal{K}$-function β such that for any $w \in \mathcal{L}_2$, $\xi_0 \in \mathbb{R}^{n_\xi}$, $\tau_0 \in [0, h_0]$ any $(\xi, z) \in \mathcal{S}(\tau_0, \xi_0, w)$ satisfies

$$\|\xi\|_{\mathcal{L}_2} \leq \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_2})). \tag{13}$$

If this property holds for $\tau_0 \in \{0\}$, or $\tau_0 \in \{h_0\}$ the system is said to be internally stable from the beginning of the flow (b-internally stable) or internally stable from the end of the flow (e-internally stable), respectively.

In the context of lifting (see Section V below) it is important to work with a fixed initial time $\tau(0)$, which suggests the consideration of b-contractivity or e-contractivity, although from a system theoretic point of view one would be interested in contractivity as this gives the strongest guarantees on the system properties (as $\tau(0) \in [0, h_0]$ is arbitrary). Therefore, we study the relationships between these notions.

Proposition IV.3 *The following statements are equivalent:*

- *The hybrid system* (1) *is internally stable.*
- *The hybrid system* (1) *is b-internally stable*
- *The hybrid system* (1) *is e-internally stable*

Proposition IV.4 *Consider the following statements:*

(i) The hybrid system (1) *is contractive.*
(ii) The hybrid system (1) *is b-contractive.*
(iii) The hybrid system (1) *is e-contractive.*

Then it holds that

$$(i) \Leftrightarrow (ii) \Rightarrow (iii).$$

Moreover, it holds that (iii) *implies* (i)*, if the following two conditions hold:*

(1) There is a constant mapping ϕ *and* $\bar{h} > 0$ *such that for all* $k \in \mathbb{N}$, $\phi_k = \phi$ *and* $h_k = \bar{h}$,
(2) there is a $\mathcal{K}$*-function* α *such that for all* $\xi_0 \in \mathbb{R}^{n_\xi}$ *there is a* $\xi_0' \in \mathbb{R}^{n_\xi}$ *with* $\phi(\xi_0') = \xi_0$ *and* $|\xi_0'| \leq \alpha(|\xi_0|)$.

Using example V.2 we illustrates a situation for which a system is e-contractive but not b-contractive (showing (iii) does not imply (i) in general).

Remark IV.5 When condition (1) is satisfied system (1) is equal to the class of systems as discussed in [1], [2]. The authors of [1], [2] assume $\tau(0) = h$, and hence solely consider e-internal stability and e-contractivity, the weakest of the three notions.

V. Internal Stability and $\mathcal{L}_2$-Gain Analysis

In this section we will analyse the $\mathcal{L}_2$-gain and the internal stability of (1) using ideas from lifting [3], [12], [13], [14], [15]. As already indicated, in lifting we have to set $\tau(0)$ to a fixed value and natural candidates are $\tau(0) = 0$ or $\tau(0) = h_0$. Interestingly in the earlier works [1], [2], [3] always $\tau(0) = h_0$ was chosen (corresponding to e-contractivity and e-internal stability). The results in Proposition IV.4 hint upon the fact that it is better to use $\tau(0) = 0$ corresponding to b-contractivity and b-internal stability, as we know that these imply contractivity and internal stability (i.e., for all $\tau(0) \in [0, h_0]$) and the fact that the analysis is essentially not different for the b-versions compared to the e-versions. For completeness, we provide below necessary and sufficient conditions for all three notions of internal stability and contractivity.

A. Lifting

To study e-contractivity, we introduce the lifting operator $W_e : \mathcal{L}_{2,e}[0, \infty) \to \ell(\{\mathcal{L}_2([0, h_{k+1}])\}_{k\in\mathbb{N}})$ given for $w \in \mathcal{L}_{2,e}[0, \infty)$ by $W_e(w) = \tilde{w} = (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2, ...)$ with

$$\tilde{w}_k(s) = w(t_k + s) \text{ for } s \in [0, h_{k+1}] \tag{14}$$

for $k \in \mathbb{N}$, $t_0 = 0$ and $t_k = \sum_{i=1}^{k} h_i$, $k \in \mathbb{N}_{\geq 1}$. Using the lifting operator and assuming $\tau(0) = h_0$, in line with e-internal stability and e-contractivity, we can rewrite the model in (1) as

$$\xi_{k+1} = \hat{A}_{k+1}\xi_k^+ + \hat{B}_{k+1}\tilde{w}_k \tag{15a}$$
$$\xi_k^+ \in \phi_k(\xi_k) \tag{15b}$$
$$\tilde{z}_k = \hat{C}_{k+1}\xi_k^+ + \hat{D}_{k+1}\tilde{w}_k \tag{15c}$$

in which ξ_0 is given and $\xi_k = \xi(t_k^-) = \lim_{s\uparrow t_k} \xi(s), k \in \mathbb{N}_{\geq 1} = \xi(t_k)$, (assuming ξ is continuous from the left) for $k \in \mathbb{N}_{\geq 1}$ and $\xi_k^+ = \xi(t_k^+) = \lim_{s\downarrow t_k} \xi(s)$ for $k \in \mathbb{N}$, and $\tilde{w} = (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2, ...) = W_e(w)$ and $\tilde{z} = (\tilde{z}_0, \tilde{z}_1, \tilde{z}_2, ...) = W_e(z)$. Moreover,

$$\begin{aligned} &\hat{A}_k : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_\xi} && \hat{B}_k : \mathcal{L}_2[0, h_k] \to \mathbb{R}^{n_\xi} \\ &\hat{C}_k : \mathcal{L}_2[0, h_k] \to \mathbb{R}^{n_\xi} && \hat{D}_k : \mathcal{L}_2[0, h_k] \to \mathcal{L}_2[0, h_k], \end{aligned}$$

are given for $x \in \mathbb{R}^{n_\xi}$ and $\omega \in \mathcal{L}_2[0, h_k]$ by

$$\hat{A}_k x = e^{Ah_k}x,\ \hat{B}_k\omega = \int_0^{h_k} e^{A(h_k-s)}B\omega(s)ds \tag{16a}$$
$$(\hat{C}_k x)(\theta_k) = Ce^{A\theta_k}x \tag{16b}$$
$$(\hat{D}_k\omega)(\theta_k) = \int_0^{\theta_k} Ce^{A(\theta_k-s)}B\omega(s)ds + Dw(\theta_k), \tag{16c}$$

where $\theta_k \in [0, h_k]$, $k \in \mathbb{N}$.

In the same manner as for e-contractivity, we introduce the lifting operator $W_b : \mathcal{L}_{2,e}[0, \infty) \to \ell(\{\mathcal{L}_2([0, h_k])\}_{k\in\mathbb{N}})$ to study b-contractivity. This lifting operator is given for $w \in \mathcal{L}_{2,e}[0, \infty)$ by $W_b(w) = \tilde{w}' = (\tilde{w}_0', \tilde{w}_1', \tilde{w}_2', ...)$ with

$$\tilde{w}_k'(s) = w(t_k' + s) \text{ for } s \in [0, h_k] \tag{17}$$

for $k \in \mathbb{N}$, and $t_k' = \sum_{i=0}^{k-1} h_i$, $k \in \mathbb{N}$. Using W_b and now assuming $\tau(0) = 0$, in line with b-internal stability and b-contractivity, the model in (1) can be rewritten as

$$\xi_{k+1}' \in \phi_k(\hat{A}_k\xi_k' + \hat{B}_k\tilde{w}_k') \tag{18a}$$
$$\tilde{z}_k' = \hat{C}_k\xi_k' + \hat{D}_k\tilde{w}_k' \tag{18b}$$

in which ξ_0' is given, $\xi_k' = \xi(t_k'^+) = \lim_{s\downarrow t_k} \xi(s)$ for $k \in \mathbb{N}_{\geq 1}$, and $\tilde{w}' = (\tilde{w}_0', \tilde{w}_1', \tilde{w}_2', ...) = W_b(w)$ and $\tilde{z}' = (\tilde{z}_0', \tilde{z}_1', \tilde{z}_2', ...) = W_e(z)$.

By writing the solutions of (1) explicitly, and then comparing to the formulas (16) and using that W_e and W_b are isometric isomorphisms, it follows that (15) is contractive if and only if (1) is e-contractive and (18) is contractive if and only if (1) is b-contractive. Moreover, by extending a result in [1], we can establish the following proposition:

Proposition V.1 *The following statements hold:*

- *The hybrid system* (1) *is e-internally stable and e-contractive if and only if* (15) *is internally stable and contractive.*
- *The hybrid system* (1) *is (b-)internally stable and (b-)contractive if and only if* (18) *is internally stable and contractive.*
- *Moreover, in case* (1) *is internally stable, it also holds that* $\lim_{t\to\infty} \xi(t) = 0$ *and* $\|\xi\|_{\mathcal{L}_\infty} \leq \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_\infty}))$ *for all* $w \in \mathcal{L}_2$, $\xi(0) = \xi_0$ *and* $\tau(0) \in [0, h_0]$.

B. Main Result

The following result is an extension of the main result of [1]. Note that a necessary condition for (1) to be e-contractive is that the induced gains $\|\hat{D}_k\|_{\mathcal{L}_2[0,h_k]} < 1$ for all $k \in \mathbb{N}_{\geq 1}$, and a necessary condition for b-contractivity of (1) is that the induced gains $\|\hat{D}_k\|_{\mathcal{L}_2[0,h_k]} < 1$ for all $k \in \mathbb{N}$. Note that for e-contractivity there is no bound $\|\hat{D}_0\|_{\mathcal{L}_2[0,h_k]} < 1$, as the section $[0, h_0]$ does not play any role. From this it is easy to come up with an example of an system for which is e-contractive, but is not b-contractive

Example V.2 Consider the sequence of inter-jump times $\{h_k\}_{k\in\mathbb{N}}$ such that $\|\hat{D}_k\|_{\mathcal{L}_2[0,h_k]} < 1, k \in \mathbb{N}_{\geq 1}$, and $\|\hat{D}_0\|_{\mathcal{L}_2[0,h_k]} \geq 1$ holds. From the necessary conditions mentioned above, it is clear that this system can be e-contractive, but not b-contractive.

Theorem V.3 *Consider system* (1) *and its e-lifted version* (15) *with* $\|\hat{D}_k\|_{\mathcal{L}_2[0,h_k]} < 1$ *for all* $k \in \mathbb{N}_{\geq 1}$ *and b-lifted version* (18) *with* $\|\hat{D}_k\|_{\mathcal{L}_2[0,h_k]} < 1$ *for all* $k \in \mathbb{N}$. *Define the discrete-time nonlinear systems*

$$\begin{bmatrix} \bar{\xi}_{k+1} \\ r_k \end{bmatrix} \in \begin{bmatrix} \bar{A}_{k+1} \\ \bar{C}_{k+1} \end{bmatrix} \phi_k(\bar{\xi}_k) + \begin{bmatrix} \bar{B}_{k+1} \\ 0 \end{bmatrix} v_k \tag{19}$$

$$\begin{bmatrix} \bar{\xi}'_{k+1} \\ r'_k \end{bmatrix} \in \begin{bmatrix} \phi_k(\bar{A}_k\bar{\xi}'_k + \bar{B}_k v'_k) \\ \bar{C}_k\bar{\xi}'_k \end{bmatrix} \tag{20}$$

with

$$\bar{A}_k = \hat{A}_k + \hat{B}_k\hat{D}_k^*(I - \hat{D}_k\hat{D}_k^*)^{-1}\hat{C}_k \tag{21a}$$

and $B_k \in \mathbb{R}^{n_\xi \times n_v}$ *and* $C_k \in \mathbb{R}^{n_r \times n_\xi}$ *are chosen such that*

$$\begin{aligned} \bar{B}_k\bar{B}_k^\top &= \hat{B}_k(I - \hat{D}_k^*\hat{D}_k)^{-1}\hat{B}_k^* \text{ and} \\ \bar{C}_k^\top\bar{C}_k &= \hat{C}_k^*(I - \hat{D}_k\hat{D}_k^*)^{-1}\hat{C}_k \end{aligned} \tag{21b}$$

- *The system* (1) *is e-internally stable and e-contractive if and only if* (19) *is internally stable and contractive.*
- *The system* (1) *is (b-)internally stable and (b-)contractive if and only if* (20) *is internally stable and contractive.*

Remark V.4 The operators $\hat{B}_k\hat{D}_k^*(I-\hat{D}_k\hat{D}_k^*)^{-1}\hat{C}_k$, $\hat{B}_k(I-\hat{D}_k^*\hat{D}_k)^{-1}\hat{B}_k^*$, and $\hat{C}_k(I - \hat{D}_k\hat{D}_k^*)^{-1}\hat{C}_k$ need to be determined in order to explicitly compute the discrete-time systems (19) and (20). Assuming that $\|\hat{D}_k\|_{\mathcal{L}_2[0,h_k]} < 1$ is satisfied for all $k \in \mathbb{N}$ the procedures in [17] can be used to compute the matrices $(\bar{A}_k, \bar{B}_k, \bar{C}_k)$ explicitly. To verify $\|\hat{D}\|_{\mathcal{L}_2[0,h_k]} < 1$ one can use Lemma 3.2 in [17] or Theorem 13.5.1 in [12].

VI. Extended framework

In this section, we propose an extension to the framework (1). Using this extension it is possible to model, among others, the PETC system as discussed in Section II with *varying* delays. Moreover, the stability and contractivity analysis provided for (1) in the sections as above can be used for the extended framework *mutatis mutandis* thereby opening up many new applications for which an exact $\mathcal{L}_2$-gain analysis can be carried out. We envision self-triggered control [10], [11] to be one of these new applications.

A. Formulation of extended framework

The extended framework is formulated as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \xi \\ \tau \\ k \\ \ell \end{bmatrix} = \begin{bmatrix} A_k\xi + B_k w \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ when } \tau \in [0, \ell] \tag{22a}$$

$$\begin{bmatrix} \xi^+ \\ \tau^+ \\ k^+ \\ \ell^+ \end{bmatrix} \in \phi_k(\xi) \times \{0\} \times \{k+1\} \times L(k, \ell, \xi), \quad \text{when } \tau = \ell \tag{22b}$$

$$z = C_k\xi + D_k w. \tag{22c}$$

As an extension to (1), the inter-jump time ℓ is added to the state of the hybrid system. This variable is constant during the flow, and changes value at the jumps, according to the set-valued mapping $L : \mathbb{N}\times\mathbb{F}\times\mathbb{R}^{n_\xi} \rightrightarrows \mathbb{F}$, with $\mathbb{F}$ a countable subset of $\mathbb{R}_{>0}$. Moreover, the matrices A_k, B_k, C_k, and D_k are dependent on k, instead of constant matrices as in (1).

B. PETC with varying delays and transmission intervals

Using the framework (22) the PETC systems of Section II can be extended to PETC systems that have varying transmission intervals such that $\bar{h}_m \in \mathcal{H} := \{\tilde{h}_1, \tilde{h}_2, ..., \tilde{h}_{n_h}\}$, $m \in \mathbb{N}$, and varying delays such that $\tau_m \in \mathcal{D} := \{d_1, d_2, ..., d_{n_d}\}$, $m \in \mathbb{N}$. The set $\mathcal{H}$ contains the $n_h \in \mathbb{N}$ possible transmission intervals, $\tilde{h}_i \in \mathbb{R}_{\geq 0}, i \in \{1, 2, ..., n_h\}$. The set $\mathcal{D}$ contains the $n_d \in \mathbb{N}$ possible delays, $d_j \in \mathbb{R}_{\geq 0}$, $j \in \{1, 2, ..., n_d\}$, and satisfies the small delay assumption $d_j \leq \tilde{h}_i$, $i \in \{1, 2, ..., n_h\}, j \in \{1, 2, ..., n_d\}$

We can now define the map $L : \mathbb{N} \times \mathbb{F} \rightrightarrows \mathbb{F}$, which is dependent only on k and ℓ as

$$L(k, \ell) = \begin{cases} \{d_1, d_2, ..., d_{n_d}\} & \text{when } k \text{ is odd} \\ \{\tilde{h}_1 - \ell, ..., \tilde{h}_{n_h} - \ell\} & \text{when } k \text{ is even} \end{cases} \tag{23}$$

with $\mathbb{F} := \{d_1, ..., d_{n_d}, \tilde{h}_1 - d_1, ..., \tilde{h}_{n_h} - d_1, ..., \tilde{h}_1 - d_{n_d}, ..., \tilde{h}_{n_h} - d_{n_d}, \}$. Interpreting L reveals that when k is odd the inter-jump time is selected out all the possible delays, and when k is even the inter-jump time is defined as one of the possible transmission intervals minus the delay chosen at the previous jump. Moreover, the matrices A_k, B_k, C_k, D_k are constant, i.e. $A_k = A$, $B_k = B$, $C_k = C$, $D_k = D$, $k \in \mathbb{N}$, and ϕ_u and ϕ_t, are the same as in Section II.

For all of these systems fitting (24) we can carry out a stability and $\mathcal{L}_2$-gain analysis along the lines of the previous Sections IV and V. We will illustrate this in the next section for a PETC system with variable delays as in Section VI-B.

VII. Main results exploited for PETC systems with varying delays

For several important applications, including the PETC systems with delays mentioned in Section VI-B, the sequence of inter-jump times is defined by (23), with a corresponding linear mapping ϕ_u as in (9) and a PWL mapping ϕ_t as specified in (10). We will show that e-internal stability and e-contractivity analyses can be performed using an arbitrarily switching PWL discrete-time system in line with (19). Where in Section II the delay is constant, we now use the results from Section VI and consider varying delays, and constant transmission intervals, i.e., $\mathcal{D} := \{d_1, d_2, ..., d_{n_d}\}$ and $\mathcal{H} := \{\bar{h}\}$.

First we define the state $\chi_m := \bar{\xi}_{2m}(= \xi(m\bar{h})) \in \mathbb{R}^{n_\chi}$, with $n_\chi = n_\xi$, input $w_m := \begin{bmatrix} v_{2m}^\top & v_{2m+1}^\top \end{bmatrix}^\top$, and output $z_m := \begin{bmatrix} r_{2m}^\top & r_{2m+1}^\top \end{bmatrix}^\top$. Using this and applying the ideas from Section IV and V for the extended framework show that the e-internal stability and e-contractivity of (22) is equivalent to the internal stability and contractivity of (after an additional discrete-time lifting of the corresponding extended system (19)), leads to the switched PWL system

$$\chi_{m+1} = \begin{cases} A_{1l_m}\chi_m + B_{l_m} w_m & \text{when } \chi_m^\top Q \chi_m > 0 \\ A_{2l_m}\chi_m + B_{l_m} w_m & \text{when } \chi_m^\top Q \chi_m \leq 0 \end{cases} \tag{24a}$$

$$z_m = \begin{cases} C_{1l_m}\chi_m + D_{l_m} w_m & \text{when } \chi_m^\top Q \chi_m > 0 \\ C_{2l_m}\chi_m + D_{l_m} w_m & \text{when } \chi_m^\top Q \chi_m \leq 0, \end{cases} \tag{24b}$$

$l_m \in \{1, 2, ..., n_d\}$, $m \in \mathbb{N}$, (note that l_m switches arbitrarily) with $A_{il} = A_l^h J_0 A_l^\tau J_i$, $B_l = \begin{bmatrix} A_l^h J_0 B_l^\tau & B_l^h \end{bmatrix}$, $C_{il} = \begin{bmatrix} C_l^\tau J_i \\ C_l^h J_0 A_l^\tau J_i \end{bmatrix}$, $D_l = \begin{bmatrix} O & O \\ C_l^h J_0 B_l^\tau & O \end{bmatrix}$, $i = 1, 2$, $l \in \{1, 2, ..., n_d\}$. Note that here we denoted the operators corresponding to $\ell = d_l$, $l \in \{1, 2, ..., d_{n_d}\}$ with $(A_l^\tau, B_l^\tau, C_l^\tau)$, $l \in \{1, 2, ..., n_d\}$, and the operators corresponding to $\ell = \bar{h} - d_l$, $l \in \{1, 2, ..., n_d\}$ with (A_l^h, B_l^h, C_l^h), $l \in \{1, 2, ..., n_d\}$.

To guarantee the internal stability and contractivity of (24), an effective approach is to use versatile piecewise quadratic Lyapunov/storage functions [18] of the form

$$V(\chi) = \begin{cases} \chi^\top P_1^{pl} \chi & \text{with } p = \min\{q \in \{1, ..., N\} | \chi \in \Omega_q\} \\ & l \in \{1, 2, ..., n_d\} \text{ when } \chi^\top Q \chi > 0 \\ \chi^\top P_2^{pl} \chi & \text{with } p = \min\{q \in \{1, ..., N\} | \chi \in \Omega_q\} \\ & l \in \{1, 2, ..., n_d\} \text{ when } \chi^\top Q \chi \leq 0 \end{cases} \tag{25}$$

based on the regions

$$\Omega_p := \{\chi \in \mathbb{R}^{n_\chi} | X_p \chi \geq 0\}, \, p \in \{1, ..., N\} \tag{26}$$

in which the matrices X_p, $p \in \{1, ..., N\}$, are such that $\{\Omega_1, \Omega_2, ..., \Omega_N\}$ forms a partition of $\mathbb{R}^{n_\chi}$, i.e., $\bigcup_{p=1}^N \Omega_p = \mathbb{R}^{n_\chi}$ and the intersection of $\Omega_p \bigcap \Omega_q$ is of zero measure for all $p, q \in \{1, ..., N\}$ with $p \neq q$.

To establish contractivity of (24) we will use the dissipation inequality [19]

$$V(\chi_{m+1}) - V(\chi_m) \leq -r_m^\top r_m + v_m^\top v_m, \, m \in \mathbb{N} \tag{27}$$

and require that it holds along the trajectories of the system (24). This condition can be translated into sufficient LMI-based conditions using three S-procedure relaxations [20]:

Theorem VII.1 *If there exist symmetric matrices* $P_i^{pl} \in \mathbb{R}^{n_\chi \times n_\chi}$*, scalars* a_i^{pl}, $c_{ij}^{pql\kappa}$, $d_{ij}^{pql\kappa} \in \mathbb{R}_{>0}$*, and symmetric matrices* E_i^{pl}, $U_{ij}^{pql\kappa}$, $W_{ij}^{pql\kappa} \in \mathbb{R}_{\geq 0}^{n_\chi \times n_\chi}$, $i, j \in \{1, 2\}$, $p, q \in \{1, 2, ..., N\}$, $l, \kappa \in \{1, 2, ..., n_d\}$ *such that*

$$\begin{bmatrix} P_i^{pl} + (-1)^i a_i^{pl} Q - X_p E_i^{pl} X_p \end{bmatrix} \succ 0, \text{ and} \tag{28a}$$

$$\begin{aligned} &\begin{bmatrix} P_i^{pl} - C_{il}^\top C_{il} - A_{il}^\top P_j^{q\kappa} A_{il} & -C_{il}^\top D_l - A_{il}^\top P_j^{q\kappa} B_l \\ -D_l^\top C_{il} - B_l^\top P_j^{q\kappa} A_{il} & I - D_l^\top D_l - B_l^\top P_j^{q\kappa} B_l \end{bmatrix} \\ &+ \begin{bmatrix} (-1)^i c_{ij}^{pql\kappa} Q + (-1)^j d_{ij}^{pql\kappa} A_{il}^\top Q A_{il} & (-1)^j d_{ij}^{pql\kappa} A_{il}^\top Q B_l \\ (-1)^j d_{ij}^{pql\kappa} B_l^\top Q A_{il} & (-1)^j d_{ij}^{pql\kappa} B_l^\top Q B_l \end{bmatrix} \\ &- \begin{bmatrix} X_p^\top U_{ij}^{pql\kappa} X_p + A_{il}^\top X_q^\top W_{ij}^{pql\kappa} X_q A_{il} & A_{il}^\top X_q^\top W_{ij}^{pql\kappa} X_q B_l \\ B_l^\top X_q^\top W_{ij}^{pql\kappa} X_q A_{im} & B_l^\top X_q^\top W_{ij}^{pql\kappa} X_q B_l \end{bmatrix} \succ 0 \end{aligned} \tag{28b}$$

hold for all $i, j \in \{1, 2\}$, $p, q \in \{1, 2, ..., N\}$ *and* $l, \kappa \in \{1, ..., n_d\}$*, then the switched discrete-time PWL system* (24) *is internally stable and contractive.*

Due to our main results we have that the LMIs (28) also guarantee the internal stability and e-contractivity of the PETC systems as discussed in Section II with time-varying delays (assuming that $\|\hat{D}_k\|_{\mathcal{L}_2[0,h_k]} < 1$ for all $k \in \mathbb{N}_{\geq 1}$). In order to study internal-stability and b-contractivity similar LMIs to (28) can be proposed.

VIII. Numerical Example

In this example the plant

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_p(t) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} x_p(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t) \\ y(t) = x_p(t) \\ z(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x_p(t) \end{cases} \tag{29}$$

will be controlled using a PETC strategy with controller

$$u(t) = K\hat{y}(t) \tag{30}$$

and $\hat{y}$ specified by (4), in which $K = \begin{bmatrix} -0.45 & -3.25 \end{bmatrix}$. At sampling times $t_m = m\bar{h}$, $m \in \mathbb{N}$ ($\tau(0) = h_0$), with $\bar{h} = 0.40$, the output y will be transmitted when $\|K\hat{y}(t_m) - Ky(t_m)\| > \rho \|K\hat{y}(t_m)\|$ with $\rho \geq 0$. This corresponds to

$$Q = Y^\top \begin{bmatrix} (1-\rho^2) K^\top K & -K^\top K \\ -K^T K & K^\top K \end{bmatrix} Y \tag{31}$$

in the function ϕ_t given in (10).

To study the internal stability and the $\mathcal{L}_2$-gain (γ-contractivity) of the PETC system in the form (22) we determine the internal stability and contractivity of the discrete-time PWL system (24) for various scaled valued of C and D. To perform this analyis we follow the procedure based on the method discussed in Section VII and the piecewise quadratic Lyapunov/storage function (25). This results in Fig. 2 for different sets of (varying) delays, and no delay.

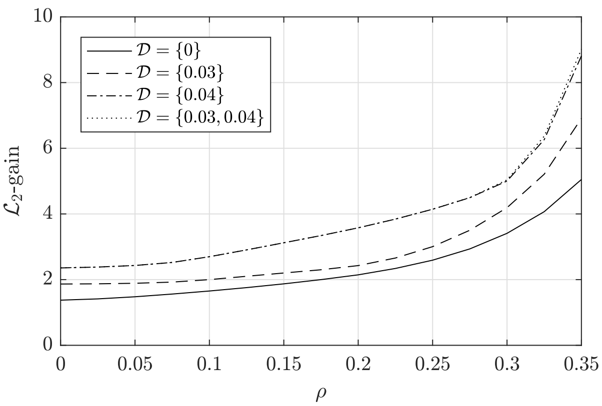

Fig. 2. Upper bound of the $\mathcal{L}_2$-gain as a function of the triggering parameter ρ for various sets $\mathcal{D}$.

IX. Conclusions

In this paper, we studied the internal stability and $\mathcal{L}_2$-gain of a hybrid system with linear flow dynamics, aperiodic flow times, and time-varying nonlinear set-valued jump maps. This class of hybrid systems is relevant for various applications that include communication delays. We introduced new notions of internal stability and contractivity depending on the initial conditions. It is concluded that all three internal stability notions are equivalent. However, the notion of contractivity from the end of the flow (e-contractivity) as adopted in [1], [2], [3] appears to be a weaker notion than the other two notions (b-contractivity and contractivity). In addition, we proposed an extension to the framework (1), which encompasses new applications in areas such as PETC systems with varying delays and self-triggered control. We established that the internal stability and contractivity properties of these hybrid systems (and their extensions) are equivalent to the internal stability and contractivity of an appropriate time-varying discrete-time nonlinear system. Through a numerical example, it was shown how stability and contractivity analyses can be carried out for PETC systems with varying delays.

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