On Integration of Event-based Estimation and Robust MPC in a Feedback Loop

Joris Sijs
TNO, Dept. of Science & Industry
Delft, The Netherlands
joris.sijs@tno.nl

Mircea Lazar
Eindhoven University of Technology
Dept. of Electrical Engineering
Eindhoven, The Netherlands
m.lazar@tue.nl

W.P.M.H. (Maurice) Heemels
Eindhoven University of Technology
Dept. of Mechanical Engineering
Eindhoven, The Netherlands
m.heemels@tue.nl

ABSTRACT

The main purpose of event-based control, if compared to periodic control, is to minimize data transfer or processing power in networked control systems. Current methods have an (implicit) dependency between triggering the events and the control algorithm. To decouple these two, we introduce an event-based state estimator in between the sensor and the controller. The event-based estimator is used to obtain a state estimate with a bounded covariance matrix in the estimation error at every synchronous time instant, under the assumption that the set in the measurement-space that is used for event generation is bounded. The estimation error is then translated into explicit polytopic bounds that are fed into a robust MPC algorithm. We prove that the resulting MPC closed-loop system is input-to-state stable (ISS) to the estimation error. Moreover, whenever the network requirements are satisfied, the controller could explicitly request for an additional measurement in case there is a desire for a better disturbance rejection.

Categories and Subject Descriptors

G.1.0 [**Numerical analysis**]: General—*Stability (and instability)*

General Terms

Theory

Keywords

Event-based estimation, Event-based control, Predictive control, Robust control, Networked control systems

1. INTRODUCTION

Event-based control has emerged recently as a viable alternative to classical, periodic control, with many relevant applications in networked control systems (NCS). A recent overview of the main pros and cons of event-based control can be found in [1]. The main motivations for event-based control are the limitations imposed by NCS, such as limited bandwidth and computational power, which led to the objective of reducing data transfer or energy consumption. Basically, it was proposed that measurements should be sent to the controller only when an event occurs, of which "Send-on-Delta" (or Lebesque sampling) [2, 3] and "Integral sampling" [4] are some examples. Subsequent studies on control that are based on the event sampling method "Send-on-Delta" were presented in [5–10]. The conclusion that can be drawn from these works is that when measurements are sent only at event instants, i.e. dictated by NCS requirements such as minimizing data transfer, it is difficult to guarantee (practical) stability of the closed-loop system.

The natural solution that emerged for solving this problem was to include the controller in the event-triggering decision process. Various alternatives are presented in [5, 9, 11–14] and the references therein. The generic procedure within this framework is to define a specific criterion for triggering events as a function of the state vector. This function can either be related to guaranteeing closed-loop robust stability, see, e.g., [13], or to improving disturbance rejection, see, e.g., [14]. One of the concerns regarding this framework is that data transfer or energy consumption might be compromised. Another relevant aspect is the fact that controllers are designed for a specific type of event sampling method or, the sampling method is designed specifically for the controller. This implies that both functionalities, i.e. event sampling and control, of the process depend heavily on each other and changing one requires a re-design of the other to guarantee the same properties for the closed-loop system.

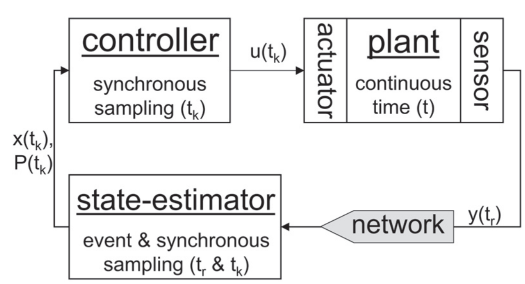


Figure 1: Schematic representation of the feedback loop.

In this paper we investigate the possibility of designing an event-based control system where closed-loop robust stability is decoupled from event generation. To that extent, an event-based state estimator (EBSE) is introduced in the feedback loop, as depicted in Figure 1. The purpose of the EBSE is to deliver a state estimation to the controller synchronously in time, while it receives measurements only at events. Such an EBSE with a synchronous update was recently developed in [15] for autonomous systems. The first contribution of this work is to extend the estimation algorithm of [15] to systems with control inputs. It is shown that under certain assumptions, the EBSE has a bounded covariance matrix. This is possible because the state is updated both when an event occurs, at which a measurement sample is received, as well as at sampling instants synchronous in time, without receiving a measurement sample. In the latter case the update is based on the knowledge that the monitored variable, i.e., the measurement, is within a bounded set that is used to define the event.

The controller that uses the state estimate is based on a new robust MPC algorithm, which forms the second contribution of this paper. This MPC scheme achieves input-to-state stability with respect to the estimation error. Moreover, the MPC algorithm offers the possibility to optimize on-line the closed-loop trajectory-dependent ISS gain, which enhances disturbance rejection. The controller is chosen to run synchronously in time. Therefore, this setup provides most benefits in situations where the sensors are connected to the controller via a (wireless) network link but the controller itself is wired to the actuator/plant. Such a setup is often seen in applications where there are more limitations regarding sensing as to actuation. To integrate the EBSE and MPC in a feedback loop, we develop an efficient method for translating, at each synchronous time instant, the bounds on the covariance matrix of the EBSE into a polytope where the estimation error lies. The latter bound is then fed to the MPC algorithm that uses it to optimize the closed-loop ISS gain. Obviously, if the EBSE receives more real measurements, the resulting bounds on the estimation error will be smaller, which will ultimately improve the trade-off between event generation and closed-loop performance.

The remainder of the paper is structured as follows. Preliminaries are presented in Section 2, while the EBSE is described in Section 3. Section 4 presents the MPC algorithm. Section 5 discusses several issues related to integration of the EBSE and the robust MPC controller in a feedback loop. An example illustrates the effectiveness of the proposed event-based control scheme in Section 6. Conclusions are summarized in Section 7.

2. PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integers and the set of non-negative integers, respectively. For any $\mathscr{C} \subset \mathbb{R}$, let $\mathbb{Z}_{\mathscr{C}} := \{c \in \mathbb{Z} | c \in \mathscr{C}\}$. For a set $\mathscr{S} \subseteq \mathbb{R}^n$, we denote by $\partial\mathscr{S}$ the boundary, by $\text{int}(\mathscr{S})$ the interior and by $\text{cl}(\mathscr{S})$ the closure of $\mathscr{S}$. For two arbitrary sets $\mathscr{S} \subseteq \mathbb{R}^n$ and $\mathscr{P} \subseteq \mathbb{R}^n$, let $\mathscr{S} \oplus \mathscr{P} := \{x+y \,|\, x \in \mathscr{S}, y \in \mathscr{P}\}$ denote their Minkowski sum. A polyhedron (or a polyhedral set) in $\mathbb{R}^n$ is a set obtained as the intersection of a finite number of open and/or closed half-spaces. Given $(n+1)$ *affinely independent* points $(\theta_0, \dots, \theta_n)$ of $\mathbb{R}^n$, i.e. $(1\ \theta_0^\top)^\top, \dots, (1\ \theta_n^\top)^\top$ are linearly independent in $\mathbb{R}^{n+1}$, we define the corresponding simplex S as

$$S := \text{Co}(\theta_0, \dots, \theta_n) := \left\{ x \in \mathbb{R}^n \middle| x = \sum_{l=0}^{n} \mu_l \theta_l,\ \sum_{l=0}^{n} \mu_l = 1 \right. \\ \left. \mu_l \in \mathbb{R}_+ \text{ for } l \in \mathbb{Z}_{[0,n]} \right\},$$

where $\text{Co}(\cdot)$ denotes the convex hull.

The notation $\underline{0}$ is used to denote either the null-vector or the null-matrix. Its size will be clear from the context. The transpose, inverse, determinant and trace of a matrix $A \in \mathbb{R}^{n \times n}$ are denoted as $A^\top$, A^{-1}, $|A|$ and $\text{tr}(A)$, respectively. The i^{th}, minimum and maximum eigenvalue of a square matrix A are denoted as $\lambda_i(A)$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively. The Hölder p-norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\|_p := (|[x]_1|^p + \ldots + |[x]_n|^p)^{\frac{1}{p}}$ for $p \in \mathbb{Z}_{[1,\infty)}$ and $\|x\|_\infty := \max_{i=1,\dots,n} |[x]_i|$, where $[x]_i$, $i \in \mathbb{Z}_{[1,n]}$, is the i-th element of x. For brevity, let $\|\cdot\|$ denote an arbitrary p-norm. For a matrix $Z \in \mathbb{R}^{m \times n}$ let $\|Z\| := \sup_{x \neq 0} \frac{\|Zx\|}{\|x\|}$ denote its corresponding induced matrix norm. It is well known that $\|Z\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |Z^{\{ij\}}|$, where $Z^{\{ij\}}$ is the ij-th entry of Z. Let $\mathbf{z} := \{z(l)\}_{l \in \mathbb{Z}_+}$ with $z(l) \in \mathbb{R}^o$ for all $l \in \mathbb{Z}_+$ denote an arbitrary sequence. Define $\|\mathbf{z}\| := \sup\{\|z(l)\| \mid l \in \mathbb{Z}_+\}$ and $\mathbf{z}_{[k]} := \{z(l)\}_{l \in \mathbb{Z}_{[0,k]}}$.

The Gaussian function (shortly noted as Gaussian) is defined as $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \to \mathbb{R}_+$,

$$G(x, \mu, P) = \frac{1}{\sqrt{(2\pi)^n |P|}} e^{(x-\mu)^T P^{-1} (x-\mu)}. \tag{1}$$

By definition it follows that if $x \in \mathbb{R}^n$ is a random variable with a probability density function (PDF) $p(x) = G(x, \mu, P)$, then the expectation and covariance of x are given by $E[x] = \mu$ and $cov(x) = P$, respectively.

For a bounded Borel set [16] $Y \subset \mathbb{R}^n$, the set PDF is defined as $\Lambda_Y : \mathbb{R}^n \to \{0, \nu\}$, with $\nu \in \mathbb{R}$ the Lebesque measure [17] of Y, i.e.,

$$\Lambda_Y(x) = \begin{cases} 0 & \text{if} \quad x \notin Y, \\ \nu^{-1} & \text{if} \quad x \in Y. \end{cases} \tag{2}$$

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathscr{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathscr{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathscr{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k \to \infty} \beta(s, k) = 0$.

Considered the following discrete time system,

$$x(t_{k+1}) \in \Phi(x(t_k), w(t_k)), \quad t_k = k\tau_s, \tag{3}$$

where $x(t_k)$ is the state and $w(t_k) \in \mathbb{R}^n$ is the unknown disturbance at time instant $t_k = k\tau_s$, $k \in \mathbb{Z}_+$ and for some $\tau_s \in \mathbb{R}_+$. The mapping $\Phi : \mathbb{R}^n \times \mathbb{R}^n \hookrightarrow \mathbb{R}^n$ is an arbitrary compact and non-empty set-valued function. For zero input in (3) we assume that $\Phi(0,0) = \{0\}$. Suppose $w(t_k)$ takes a value in a bounded set $\mathbb{W} \subset \mathbb{R}^n$ for all $t_k \in \mathbb{R}_+$.

Definition 2.1 We call a set $\mathscr{P} \subseteq \mathbb{R}^n$ *robustly positively invariant (RPI)* for system (3) with respect to $\mathbb{W}$ if for all $x \in \mathscr{P}$ it holds that $\Phi(x, w) \subseteq \mathscr{P}$ for all $w \in \mathbb{W}$.

Definition 2.2 Let $\mathbb{X}$ with $0 \in \text{int}(\mathbb{X})$ and $\mathbb{W}$ be subsets of $\mathbb{R}^n$. We call system (3) *ISS in* $\mathbb{X}$ *for inputs in* $\mathbb{W}$ if there exist a $\mathscr{KL}$-function $\beta(\cdot,\cdot)$ and a $\mathscr{K}$-function $\gamma(\cdot)$ such that, for each $x(t_0) \in \mathbb{X}$ and all $\mathbf{w} = \{w(t_l)\}_{l \in \mathbb{Z}_+}$ with $w(t_l) \in \mathbb{W}$ for all $l \in \mathbb{Z}_+$, it holds that all corresponding state trajectories of (3) satisfy the following inequality: $\|x(t_k)\| \le \beta(\|x(t_0)\|, k) + \gamma(\|\mathbf{w}_{[t_{k-1}]}\|)$, $\forall k \in \mathbb{Z}_{\ge 1}$.

We call $\gamma(\cdot)$ an ISS gain of system (3).

Theorem 2.3 *Let* $\mathbb{W}$ *be a subset of* $\mathbb{R}^n$ *and let* $\mathbb{X}$ *be a RPI set for* (3) *with respect to* $\mathbb{W}$, *with* $0 \in \text{int}(\mathbb{X})$. *Furthermore, let* $\alpha_1(s) := as^\delta$, $\alpha_2(s) := bs^\delta$, $\alpha_3(s) := cs^\delta$ *for some* $a, b, c, \delta \in \mathbb{R}_{>0}$, $\sigma \in \mathscr{K}$ *and let* $V : \mathbb{R}^n \to \mathbb{R}_+$ *be a function such that:*

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|), \tag{4a}$$

$$V(x^+) - V(x) \le -\alpha_3(\|x\|) + \sigma(\|w\|) \tag{4b}$$

for all $x \in \mathbb{X}$, $w \in \mathbb{W}$ and all $x^+ \in \Phi(x,w)$. Then the system (3) is ISS in $\mathbb{X}$ for inputs in $\mathbb{W}$ with

$$\beta(s,k) := \alpha_1^{-1}(2\rho^k \alpha_2(s)), \quad \gamma(s) := \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right),$$
$$\rho := 1 - \frac{c}{b} \in [0,1). \tag{5}$$

The proof of Theorem 2.3 can be found in [18]. We call a function $V(\cdot)$ that satisfies the hypothesis of Theorem 2.3 an *ISS Lyapunov function*.

3. EVENT-BASED STATE-ESTIMATION

In this section we will present the extension of the EBSE, as recently developed in [15], to systems with control inputs. Therefore, let us assume that a dynamical system with state vector $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, process noise $q \in \mathbb{R}^n$, measurement vector $y \in \mathbb{R}^l$ and measurement noise $v \in \mathbb{R}^l$ is given. This process is described by a generic discrete-time state-space model, i.e.,

$$x(t) = A_\tau x(t-\tau) + B_\tau u(t-\tau) + q(t,\tau), \tag{6a}$$
$$y(t) = Cx(t) + Du(t) + v(t). \tag{6b}$$

with $A_\tau \in \mathbb{R}^{n\times n}$ and $B_\tau \in \mathbb{R}^{n\times m}$, for all $\tau \in \mathbb{R}_+$, $C \in \mathbb{R}^{l\times n}$ and $D \in \mathbb{R}^{l\times m}$. It is assumed that $u(s)$ remains constant for all $t-\tau \le s < t$. Basically, the above system description (6a) could be perceived as a discretized version of a continuous-time plant $\dot{x}(t) = Ax(t) + Bu(t)$. In this case the matrices A_τ and B_τ would then be defined with the time difference τ of two sequential sample instants, i.e.,

$$A_\tau := e^{A\tau} \quad \text{and} \quad B_\tau := \int_0^\tau e^{A\eta} d\eta B.$$

However, we allow for the more general description (6). We assume that the process- as well as the measurement-noise have Gaussian PDFs with zero mean, for some $Q_\tau \in \mathbb{R}^{n\times n}$, $\tau \in \mathbb{R}_+$ and $R_v \in \mathbb{R}^{l\times l}$, i.e.,

$$p(q(t,\tau)) := G(q(t,\tau), \underline{0}, Q_\tau) \quad \text{and} \quad p(v(t)) := G(v(t), \underline{0}, R_v).$$

The sensor uses an event sampling method which is based on y. Its sample instants are indexed by r, i.e. $y(t_r)$ denotes a measurement taken at the event instant t_r. As proposed in [15], $H_r \subset \mathbb{R}^{l+1}$ is a set, determined at the event instant t_{r-1}, in the *time-measurement*-space that induces the event instants. An example of this set, in case the measurement-space is 2D, is graphically depicted in Figure 2. To be precise, given that t_{r-1} was the latest event instant, the next event instant t_r is defined as:

$$t_r := \inf\left\{ t \in \mathbb{R}_+ \mid t > t_{r-1} \text{ and } \begin{pmatrix} y(t) \\ t \end{pmatrix} \notin H_r \right\}. \tag{7}$$

To prevent that more than one sample action occurs at t_{r-1}, it should hold that $(y^\top(t_{r-1}), t_{r-1})^\top \in \text{int}(H_r)$.

To illustrate the event triggering, let us present two examples of how to choose the set H_r. In the first example the events are triggered by applying the sampling method "Send-on-Delta" [2, 3]. A new measurement sample $y(t_r)$ is generated when $|y(t) - y(t_{r-1})| > \Delta$. Notice that this is equivalent with (7) in case

$$H_r := \{(y^\top, t)^\top \big| \, |y - y(t_{r-1})| \le \Delta\}.$$

The second example of a method for triggering the events is taken from [14, 19]. Therein, a sampling method is described which is similar to "Send-on-Delta", although $y(t_{r-1})$ is replaced with the current predicted measurement $C\hat{x}(t)$. Notice that in this case $H_r := \{(y^\top, t)^\top \big| \, |y - C\hat{x}(t)| \le \Delta\}$.

As the sensor uses an event-sampling method on $y(t)$, the EBSE receives $y(t_r)$ to perform a state-update. However, typically the event instants t_r do not occur at the same time as the synchronous instants at which the controller needs to calculate a new control-input. Hence, the EBSE has to keep track of the state at both the event instants as well as the synchronous instants. Let us define $\mathbb{T}_r(t)$ and $\mathbb{T}_c(t)$ as the set of time instants that correspond to all event instants and synchronous instants, respectively. Therefore, if $\tau_s \in \mathbb{R}_+$ denotes the controller's sampling time, we have that

$$\mathbb{T}_r := \{t_r \mid r \in \mathbb{Z}_+\} \quad \text{and} \quad \mathbb{T}_c := \{k\tau_s \mid k \in \mathbb{Z}_+\},$$

where the event instants t_r are generated by (7). Notice that it could happen that an event instant coincides with a synchronous instant. Therefore $\mathbb{T}_r \cap \mathbb{T}_c$ might be non-empty. The EBSE calculates an estimate of the state-vector and an error-covariance matrix at each sample instant $t \in \mathbb{T}$, with $\mathbb{T} := \mathbb{T}_r \cup \mathbb{T}_c$. At an event instant, i.e. $t \in \mathbb{T}_r$, the EBSE receives a new measurement $y(t_r)$ with which a state-update can be performed. At the synchronous instants $t \in \mathbb{T}_c \backslash \mathbb{T}_r$, the EBSE does not receive a measurement. Standard estimators would perform a state-*prediction* using the process-model. However, from (7) we observe that if no measurement y was received at $t > t_{r-1}$, still it is known that $(y(t)^\top, t)^\top \in H_r$. The estimator can exploit this information to perform a state-*update* not only at the event instants but also at the synchronous instants $t \in \mathbb{T}_c \backslash \mathbb{T}_r$. Next, we describe how this is implemented. Let us define $H_{r|t} \subset \mathbb{R}^l$ as a section of H_r at the time instant $t \in (t_{r-1}, t_r)$, which is graphically depicted in Figure 2, and formally defined as:

$$H_{r|t} := \left\{ y \in \mathbb{R}^l \,\middle|\, \begin{pmatrix} y \\ t \end{pmatrix} \in H_r \right\}.$$

Therefore, the following two conditions hold for any $t \in \mathbb{T}$:

$$y(t) \in \begin{cases} \{y(t)\} & \text{if} \quad t \in \mathbb{T}_r, \\ H_{r|t} & \text{if} \quad t \in \mathbb{T}_c \backslash \mathbb{T}_r. \end{cases} \tag{8}$$

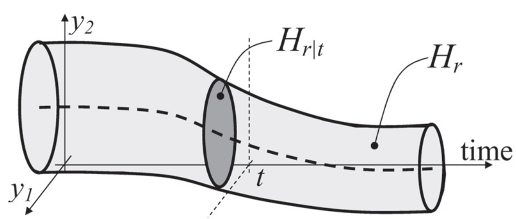


Figure 2: An example of H_r defining a set in the *time-measurement*-space and $H_{r|t}$ defining a section in the *measurement*-space at a certain time-instant $t \in (t_{r-1}, t_r)$.

The estimator must first determine a PDF of the measurement $y(t)$. Therefore if $\delta(\cdot)$ denotes the Dirac-pulse and $\Lambda_Y(\cdot)$ denotes the set PDF as defined in (2), then from equation (8) it follows that:

$$p(y(t)) = \begin{cases} \delta(y(t)) & \text{if} \quad t \in \mathbb{T}_r, \\ \Lambda_{H_{r|t}}(y(t)) & \text{if} \quad t \in \mathbb{T}_c \backslash \mathbb{T}_r. \end{cases} \tag{9}$$

In [20] is was shown that any PDF can be approximated as a sum of Gaussians. Therefore, let us assume that $p(y(t))$ of (9) is approximated by $\sum_{i=1}^N \frac{1}{N} G(y(t), \hat{y}_i(t), R_H(t))$, for some $N \in \mathbb{Z}_+$, $R_H(t) \in \mathbb{R}^{l\times l}$ and $\hat{y}_i(t) \in \mathbb{R}^l$ for all $i \in \mathbb{Z}_{[1,N]}$. Notice that in case $t \in \mathbb{T}_r$ it follows that $N = 1$, $\hat{y}_1(t) = y(t)$ and $R_H(t)$ can be taken arbitrary small to approximate the Dirac pulse. Let $\hat{x}(t)$ denote the estimated state-vector of the EBSE and let $P(t)$ denote the error-covariance matrix both at $t \in \mathbb{T}$. Furthermore, let $R(t) := R_v + R_H(t)$. Then, the set of equations of the EBSE, in standard Kalman filter form, yields:

Step 1: prediction

$$\begin{aligned} \hat{x}^-(t) &= A_\tau \hat{x}(t-\tau) + B_\tau u(t-\tau), \\ P^-(t) &= A_\tau P(t-\tau) A_\tau^\top + Q_\tau, \end{aligned} \tag{10a}$$

Step 2: measurement-update, $\forall i \in \mathbb{Z}_{[1,N]}$

$$\begin{aligned} K(t) &= P^-(t)C^\top (CP^-(t)C^\top + R(t))^{-1}C, \\ P_i(t) &= (I - K(t)C)P^-(t), \\ \hat{x}_i(t) &= \hat{x}^-(t) + K(t)(\hat{y}_i(t) - Du(t) - C\hat{x}^-(t)), \\ \beta_i(t) &= G(y_i(t), C\hat{x}^-(t) + Du(t), CP^-(t)C^\top + R(t)), \end{aligned} \tag{10b}$$

Step 3: state-approximation

$$\begin{aligned} \hat{x}(t) &= \sum_{i=1}^{N} \frac{\beta_i(t)}{\sum_{i=1}^{N}\beta_i(t)} \hat{x}_i(t), \\ P(t) &= \sum_{i=1}^{N} \frac{\beta_i(t)}{\sum_{i=1}^{N}\beta_i(t)} \left(P_i(t) + (\hat{x}(t) - \hat{x}_i(t))(\hat{x}(t) - \hat{x}_i(t))^\top \right). \end{aligned} \tag{10c}$$

The main reason for the approximation of (10c) is to limit the amount of processing demand of the EBSE. Next, we present a brief account of the numerical complexity of the proposed EBSE in comparison with the original Kalman filter [21], i.e.,

- Prediction: $O((m+1)n) + O(2n^2) + O(2n^3)$;
- Update: $O(l^3) + O(l^2(1+2n)) + O(n^2(2l+1)) + O(2n^3) + O(nm + l(n+1)) + O(N(2l^2 + 3l + nl + n))$;
- Approximation: $O(n + (2+3n)N) + O((4N+1)n^2)$.

For clarity, let us neglect terms that are of second order or less. Then the computational complexity of the EBSE is

$$O(4n^3 + l^3 + 2l^2 n + 2ln^2 + N(2l^2 + nl + 4n^2),$$

which is still proportional to the complexity of the original Kalman filter, i.e., $O(4n^3 + l^3 + 2l^2 n + 2ln^2)$.

Remark 3.1 The set of equations of the EBSE is based on a Sum-of-Gaussians approach. The main reason for choosing the Sum-of-Gaussians approach is that it enables an asymptotic bound on $P(t)$. This property is important as it enables a guarantee, in a probabilistic sense, that the estimation error is bounded. □

Next, we recall the main result of [15, 22], where it was proven that all the eigenvalues of $P(t)$, i.e. $\lambda_i(P(t))$, are asymptotically bounded. To that extent, let us define $\bar{H} \subset \mathbb{R}^l$ as a bounded set such that $H_{r|t} \subseteq \bar{H}$ for all $t \in \mathbb{T}$. This further implies that each set $H_{r|t}$ is bounded for all $t \in \mathbb{T}$. With the set $\bar{H}$ we can now determine a covariance-matrix $R \in \mathbb{R}^{l\times l}$, such that $R \succeq R(t)$ for all $t \in \mathbb{T}$. Notice that $R = R_v + R_{\bar{H}}$, in which $R_{\bar{H}}$ can be derived from $\Lambda_{\bar{H}}(y)$ by approximating this PDF as a single Gaussian. Similarly, there exist a $\omega \in \mathbb{R}_+$, such that the Euclidean distance between any two elements of $H_{r|t}$ is less than $(\omega+1)\lambda_{\min}^{-1}(R(t))$, for all $t \in \mathbb{T}_c \backslash \mathbb{T}_r$. The next preliminary result [23] states the standard conditions for the existence of a bounded asymptotic covariance matrix $\Sigma_\infty \in \mathbb{R}^{n\times n}$ for a scaled synchronous Kalman filter. The update of the covariance matrix for this type of Kalman filter, denoted with $\Sigma[k] \in \mathbb{R}^{n\times n}$ at the synchronous instant $t = k\tau_s$, yields:

$$\Sigma[k] = \omega \left((A_{\tau_s}\Sigma([k-1]A_{\tau_s} + Q_{\tau_s})^{-1} + C^\top R^{-1} C \right)^{-1}, \quad \forall k \in \mathbb{Z}_+.$$

Proposition 3.2 [23] *Let Σ_∞ be defined as the solution of $\Sigma_\infty^{-1} = \left(\omega A_{\tau_s}\Sigma_\infty A_{\tau_s}^\top + Q_{\tau_s}\right)^{-1} + C^\top R^{-1} C$. If Σ_∞ exists, (A_{τ_s}, C) is an observable pair and $\lambda_i(\bar{A}) \leq 1$, for all $i \in \mathbb{Z}_{[1,n]}$, where $\bar{A} := \sqrt{\omega}\Big(A_{\tau_s} - A_{\tau_s}\bar{\Sigma}C^\top \left(C\bar{\Sigma}C^\top + R\right)^{-1} C\Big)$ and $\bar{\Sigma} := A_{\tau_s}\Sigma_\infty A_{\tau_s}^\top + Q_{\tau_s}$, then it holds that $\lim_{k\to\infty}\Sigma[k] = \Sigma_\infty$.*

Now we can state the main result on the asymptotic bound of $P(t)$.

Theorem 3.3 *Let the scalars a_{τ_s} and b_{τ_s} be defined as follows: $a_{\tau_s} := \sup_{\tau\in[0,\tau_s]} \sigma_{max}(A_\tau)$ and $b_{\tau_s} := \sup_{\tau\in[0,\tau_s]} \sigma_{max}(B_\tau)$. If the hypothesis of Proposition 3.2 holds, then we have that*

$$\lim_{t\to\infty} \lambda_{max}(P(t)) \leq a_{\tau_s}^2 \lambda_{max}(P_\infty) + b_{\tau_s}^2 \lambda_{max}(Q_{\tau_s}).$$

This result guarantees a bound on the covariance of the estimation error at all time instants. The proof of the above theorem is obtained *mutatis mutandis* from the proof given in [15, 22] for the case of an autonomous process. Indeed, the difference in the EBSE algorithm (10) corresponding to the system with a control input versus the autonomous system is given by the term $B_\tau u(t-\tau)$ in (10a). However, as this additional term is present in both $\hat{x}_i(t)$ and $\hat{x}(t)$, it cancels out in $\hat{x}(t) - \hat{x}_i(t)$ and therefore, it is not present in the expression of $P(t)$. As such, the proof given in [15, 22] applies to a system with control input as well.

In Section 5 we will show how $P(t)$ is used to determine, with a certain probability, a bound on the estimation error at every synchronous instant. Knowledge of this bound is used to design the robust MPC algorithm, as it is explained in the next section.

4. ROBUST MPC ALGORITHM

In this section we start from the fact that a state estimate $\hat{x}(t_k)$, provided by the EBSE at each time instant $t_k = k\tau_s$, $k \in \mathbb{Z}_+$, is available for the controller at all synchronous instants. Moreover, we assume that the corresponding estimation error $w(t_k) := \hat{x}(t_k) - x(t_k)$ satisfies $w(t_k) \in \mathbb{W}(t_k)$ at all t_k, where $\mathbb{W}(t_k)$ is a known polytope (closed and bounded polyhedron). An efficient procedure for determining $\mathbb{W}(t_k)$ from $P(t_k)$, $k \in \mathbb{Z}_+$, will be presented in Section 5.

As the controller samples synchronously in time, the process-model of (6) can be rewritten from an asynchronous model into a synchronous one. Hence, consider the following discrete-time model of the system used for controller synthesis:

$$\begin{aligned} x(t_{k+1}) :&= Ax(t_k) + Bu(\hat{x}(t_k)) \\ &= Ax(t_k) + Bu(x(t_k) + w(t_k)),\ k \in \mathbb{Z}_+, \end{aligned} \tag{11}$$

where $x(t_k) \in \mathbb{X} \subseteq \mathbb{R}^n$ is the real state, $\hat{x}(t_k) \in \mathbb{X} \subseteq \mathbb{R}^n$ is the estimated state, $u(t_k) \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control action and $w(t_k) \in \mathbb{W}(t_k) \subset \mathbb{R}^n$ is an unknown estimation error at the discrete-time instant t_k. The matrices $A = A_{\tau_s}$ and $B = B_{\tau_s}$ correspond to the discretized model of system (6). From the above equations it can be seen that at any synchronous time instant t_k there exists a $w(t_k) \in \mathbb{W}(t_k)$ such that $x(t_k) = \hat{x}(t_k) - w(t_k) \in \{\hat{x}(t_k)\} \oplus \mathbb{W}(t_k)$ and, any $w(t_k)$ can be obtained as a convex combination of the vertices of $\mathbb{W}(t_k)$. We assume that $0 \in \text{int}(\mathbb{X})$ and $0 \in \text{int}(\mathbb{U})$ and indicate that $0 \in \text{int}(\mathbb{W}(t_k))$ for all $k \in \mathbb{Z}_+$, as it will be shown in Section 5. For simplicity and clarity of exposition, the index t_k is omitted throughout a part of this section, i.e. $\hat{x}$, x, $\mathbb{W}$, etc. will denote $\hat{x}(t_k)$, $x(t_k)$, $\mathbb{W}(t_k)$ and so on.

Our goal is now to design a control algorithm that finds a control action $u(\hat{x}) \in \mathbb{U}$ such that for all $w \in \mathbb{W}$

$$V(Ax + Bu(\hat{x})) - V(x) + \alpha_3(\|x\|) - \sigma(\|w\|) \leq 0. \tag{12}$$

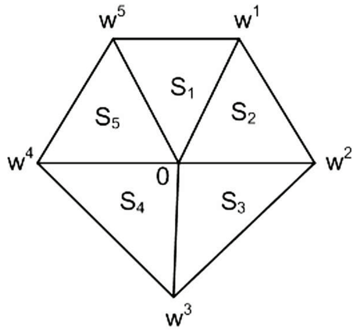


Figure 3: An example of the set $\mathbb{W}$.

Satisfaction of the above inequality for some $\sigma \in \mathscr{K}$ would guarantee ISS of the corresponding closed-loop system, see Theorem 2.3. Moreover, to improve disturbance rejection, we will adopt the idea of optimizing the closed-loop ISS gain by minimizing the gain of the function σ, which was recently proposed in [24]. Therein, the case of additive disturbances was considered. In what follows we extend the results of [24] to estimation errors, which act as a disturbance on the measurement that is fed to the controller.

The first relevant observation is that an optimization problem based directly on the constraint (12) is not finite dimensional in w. However, we demonstrate that by considering *continuous and convex*[1] Lyapunov functions and bounded polyhedral sets $\mathbb{X},\mathbb{U},\mathbb{W}$ (with non-empty interiors containing the origin) a solution to inequality (12) can be obtained via a finite set of inequalities that only depend on the vertices of $\mathbb{W}$.

Let w^e, $e = 1,...,E$, be the vertices of $\mathbb{W}$ (notice that $E > n$, as $\mathbb{W}$ is assumed to have a non-empty interior). Next, consider a finite set of simplices $S_1,\dots,S_M$ with each simplex S_i equal to the convex hull of a subset of the vertices of $\mathbb{W}$ and the origin, and such that $\cup_{i=1}^M S_i = \mathbb{W}$, $\mathrm{int}(S_i)\cap \mathrm{int}(S_j) = \emptyset$ for $i \neq j$, $\mathrm{int}(S_i) \neq \emptyset$ for all i. More precisely, $S_i = \mathrm{Co}\{0, w^{e_{i,1}},\dots,w^{e_{i,l}}\}$ and

$$\{w^{e_{i,1}},\dots,w^{e_{i,l}}\} \subseteq \{w^1,\dots,w^E\}$$

(i.e. $\{e_{i,1},\dots,e_{i,l}\} \subseteq \{1,\dots,E\}$) with $w^{e_{i,1}},\dots,w^{e_{i,l}}$ linearly independent. For an illustrative example see Figure 3: the polyhedron $\mathbb{W}$ consists of $S_1,S_2,\dots,S_5$, where, for instance, the simplex S_3 is generated by $0,w^{e_{3,1}},w^{e_{3,2}}$, with $e_{3,1} = 2$ and $e_{3,2} = 3$. For each simplex S_i we define the matrix $W_i := [w^{e_{i,1}}\ \dots\ w^{e_{i,l}}] \in \mathbb{R}^{l\times l}$, which is invertible. Let $\gamma_e \in \mathbb{R}_+$ be variables associated with each vertex w^e, which are both depending on the time instant t_k.

Next, suppose that both x and $\hat{x}$ are known. Notice that the assumption that x is known is only used here to show how one can transform (12) into a finite dimensional problem. The dependence on x will be removed later, leading to a main stability result and an MPC algorithm that only use the estimated state $\hat{x}$, see Problem 4.2. Let $\alpha_3 \in \mathscr{K}_\infty$ and consider the following set of constraints:

$$V(A\hat{x}+Bu(\hat{x})) - V(x) + \alpha_3(\|x\|) \le 0, \tag{13a}$$
$$V(A(\hat{x}-w^e)+Bu(\hat{x})) - V(x) + \alpha_3(\|x\|) - \gamma_e \le 0 \tag{13b}$$

for all $e = 1,\dots,E$.

Theorem 4.1 *Let V be a continuous and convex Lyapunov function. If for $\alpha_3 \in \mathscr{K}_\infty$, $\hat{x}$ and x there exist $u(\hat{x})$ and $\{\gamma_e\}_{e=1,\dots,E}$, such that* (13a) *and* (13b) *hold, then* (12) *holds for the same $u(\hat{x})$, with $\sigma(s) := \eta s$ and*

$$\eta := \max_{i=1,\dots,M} \|\bar{\gamma}_i W_i^{-1}\|, \tag{14}$$

where $\bar{\gamma}_i := [\gamma_{e_{i,1}}\ \dots\ \gamma_{e_{i,l}}] \in \mathbb{R}^{1\times l}$ and $\|\cdot\|$ is the corresponding induced matrix norm.

Proof: For any $w \in \mathbb{W} = \bigcup_{i=1}^M S_i$ there exists an i such that $w \in S_i = \mathrm{Co}\{0,w^{e_{i,1}},\dots,w^{e_{i,l}}\}$, which means that there exist non-negative $\mu_0,\mu_1,\dots,\mu_l$ with $\sum_{j=0,1,\dots,l}\mu_j = 1$ such that

$$w = \sum_{j=1,\dots,l} \mu_j w^{e_{i,j}} + \mu_0 0 = \sum_{j=1,\dots,l} \mu_j w^{e_{i,j}}.$$

In matrix notation we have that $w = W_i[\mu_1\ \dots\ \mu_l]^\top$ and thus

$$[\mu_1\ \dots\ \mu_l]^\top = W_i^{-1}w.$$

By multiplying each inequality in (13b) corresponding to the index $e_{i,j}$ and the inequality (13a) with $\mu_j \ge 0$, $j = 0,1,\dots,l$, summing up and using $\sum_{j=0,1,\dots,l}\mu_j = 1$ yields:

$$\mu_0 V(A\hat{x}+Bu(\hat{x})) + \sum_{j=1,\dots,l} \mu_j V(A(\hat{x}-w^{e_{i,j}})+Bu(\hat{x})) \\ - V(x) + \alpha_3(\|x\|) - \sum_{j=1,\dots,l}\mu_j\gamma_{e_{i,j}} \le 0.$$

Furthermore, using $\sum_{j=0,1,\dots,l}\mu_j = 1$ and convexity of V yields

$$V(A(\hat{x}-\sum_{j=1,\dots,l}\mu_j w^{e_{i,j}})+Bu(\hat{x})) - V(x) \\ + \alpha_3(\|x\|) - \sum_{j=1,\dots,l}\mu_j\gamma_{e_{i,j}} \le 0,$$

or equivalently

$$V(A(\hat{x}-w)+Bu(\hat{x})) - V(x) + \alpha_3(\|x\|) - \bar{\gamma}_i[\mu_1\ \dots\ \mu_l]^\top \le 0.$$

Using that $[\mu_1\ \dots\ \mu_l]^\top = W_i^{-1}w$ and $x = \hat{x}-w$ we obtain (12) for $\sigma(s) = \eta s$ and $\eta \ge 0$ as in (14). □

Based on the result of Theorem 4.1 we are now able to formulate a finite dimensional optimization problem that results in closed-loop ISS with respect to the estimation error $w(t_k)$ and moreover, in optimization of the closed-loop ISS gain. This will be achieved only based on the estimate $\hat{x}(t_k)$ and the set $\mathbb{W}(t_k)$.

Let $\bar{\gamma} := [\gamma_1,\dots,\gamma_E]^\top$ and let $J : \mathbb{R}^E \to \mathbb{R}_+$ be a function that satisfies $\alpha_4(\|\bar{\gamma}\|) \le J(\gamma_1,\dots,\gamma_E) \le \alpha_5(\|\bar{\gamma}\|)$ for some $\alpha_4,\alpha_5 \in \mathscr{K}_\infty$. Define next:

$$V_{\min}(t_k) := \min_{x\in\{\hat{x}(t_k)\}\oplus\mathbb{W}(t_k)} V(x) \tag{15}$$

and

$$\alpha_{3,\max}(t_k) := \max_{x\in\{\hat{x}(t_k)\}\oplus\mathbb{W}(t_k)} \alpha_3(\|x\|). \tag{16}$$

Problem 4.2 Let $\alpha_3 \in \mathscr{K}_\infty$, a cost J and a Lyapunov function V be given. At time $k \in \mathbb{Z}_+$ let an estimate of the state $\hat{x}(t_k)$ be known and minimize $J(\gamma_1(t_k),\dots,\gamma_E(t_k))$ over $u(t_k),\gamma_1(t_k),\dots,\gamma_E(t_k)$, subject to the constraints

$$u(t_k) \in \mathbb{U},\ \gamma_e(t_k) \ge 0 \tag{17a}$$
$$Az+Bu(t_k) \in \mathbb{X}, \forall z \in \{\hat{x}(t_k)\}\oplus\mathbb{W}(t_k) \tag{17b}$$
$$V(A\hat{x}(t_k)+Bu(t_k)) - V_{\min}(t_k) + \alpha_{3,\max}(t_k) \le 0, \tag{17c}$$
$$V(A(\hat{x}(t_k)-w_e(t_k))+Bu(t_k)) - V_{\min}(t_k) \\ + \alpha_{3,\max}(t_k) - \gamma_e(t_k) \le 0 \tag{17d}$$

for all $e = 1,\dots,E$. □

[1]This includes quadratic functions, $V(x) = x^\top Px$ with $P \succ 0$, and functions based on norms, $V(x) = \|Px\|$ with P a full-column rank matrix.

Let $\pi(\hat{x}(t_k)) := \{u(t_k) \in \mathbb{R}^m \mid (17) \text{ holds}\}$ and let

$$x(t_{k+1}) \in \phi_{\text{cl}}(x(t_k), \pi(\hat{x}(t_k))) := \{Ax(t_k) + Bu \mid u \in \pi(\hat{x}(t_k))\}$$

denote the difference inclusion corresponding to system (11) in "closed-loop" with the set of feasible solutions obtained by solving Problem 4.2 at each $k \in \mathbb{Z}_+$.

Theorem 4.3 *Let $\alpha_1, \alpha_2, \alpha_3 \in \mathscr{K}_\infty$ of the form specified in Theorem 2.3, a continuous and convex Lyapunov function V and a cost J be given. Let $\overline{\mathbb{W}}$ be a bounded set such that $\mathbb{W}(t_k) \subseteq \overline{\mathbb{W}}$ for all t_k. Suppose that Problem 4.2 is feasible for all states $\hat{x}$ in $\mathbb{X} \oplus \overline{\mathbb{W}}$. Then the difference inclusion*

$$x(t_{k+1}) \in \phi_{cl}(x(t_k), \pi(\hat{x}(t_k))), \quad k \in \mathbb{Z}_+ \tag{18}$$

is ISS in $\mathbb{X}$ for inputs in $\overline{\mathbb{W}}$.

Proof: As $x(t_k) \in \{\hat{x}(t_k)\} \oplus \mathbb{W}(t_k)$ for all $k \in \mathbb{Z}_+$ we have from (17b) that $x(t_{k+1}) = Ax(t_k) + Bu(t_k) \in \mathbb{X}$, i.e. the real state satisfies state constraints and, $\hat{x}(t_{k+1}) \in \mathbb{X} \oplus \mathbb{W}(t_{k+1}) \subseteq \mathbb{X} \oplus \overline{\mathbb{W}}$, i.e. Problem 4.2 remains feasible at $k+1$ and $\mathbb{X}$ is an RPI set w.r.t. the estimation error. Then, from the definitions (15) and (16), and from (17c), (17d) we have that

$$\begin{aligned} &V(A\hat{x}(t_k) + Bu(t_k)) - V(x(t_k)) + \alpha_3(\|x(t_k)\|) \leq 0, \\ &V(A(\hat{x}(t_k) - w_e(t_k)) + Bu(t_k)) - V(x(t_k)) \\ &\qquad\qquad + \alpha_3(\|x(t_k)\|) - \gamma_e(t_k) \leq 0, \end{aligned}$$

for all $x(t_k) \in \{\hat{x}(t_k)\} \oplus \mathbb{W}(t_k)$, as $V(x(t_k)) \geq V_{\min}(t_k)$ and also $\alpha_3(\|x(t_k)\|) \leq \alpha_{3,\max}(t_k)$. Then, from Theorem 4.1 we have that (4b) holds with $\sigma(s) := \eta(t_k)s$ and $\eta(t_k)$ as in (14), i.e.

$$V(Ax(t_k) + Bu(t_k)) - V(x(t_k)) + \alpha_3(\|x(t_k)\|) - \sigma(\|w(t_k)\|) \leq 0,$$

for all $x(t_k) \in \{\hat{x}(t_k)\} \oplus \mathbb{W}(t_k)$. Next let

$$\gamma^* := \max_{x \in \text{cl}(\mathbb{X}), u \in \text{cl}(\mathbb{U})} \{V(Ax + Bu) - V(x) + \alpha_3(\|x\|)\}.$$

Since $\mathbb{X}$, $\mathbb{U}$ and $\overline{\mathbb{W}}$ are assumed to be bounded sets, γ^* exists, and inequality (17d) is always satisfied for $\gamma_e(t_k) = \gamma^*$ for all $e = 1, \ldots, E$, $k \in \mathbb{Z}_+$, irrespective of x, u and the vertices of $\mathbb{W}(t_k) \subseteq \overline{\mathbb{W}}$. This in turn, via (14) ensures the existence of a positive η^* such that $\eta(t_k) \leq \eta^*$ for all t_k and for all $w(t_k) \in \overline{\mathbb{W}}$. Hence, we proved that inequality (12) holds, and thus, the continuous and convex Lyapunov function V is a ISS Lyapunov function. Then, due to RPI of $\mathbb{X}$, ISS in $\mathbb{X}$ for inputs in $\overline{\mathbb{W}}$ follows directly from Theorem 2.3. □

Note that in Theorem 4.3 we used a worst case evaluation of $\gamma_e(k)$ to prove ISS, which corresponds to a worst case evaluation of the set $\mathbb{W}(t_k)$. However, in reality the gain $\eta(t_k)$ of the function σ can be much smaller for $k \geq k_0$, for some $k_0 \in \mathbb{Z}_+$. This is achieved via the minimization of the cost $J(\cdot)$, which produces small values of $\gamma_e(t_k)$, $e = 1, \ldots, E$. This in turn, via (14), will result in a small $\eta(t_k)$. Furthermore, this will ultimately yield a smaller ISS gain for the closed-loop system, due to the relation (5). Hence, Problem 4.2, although it inherently guarantees a constant ISS gain, as shown in the proof of Theorem 4.3, it provides freedom to optimize the ISS gain of the closed-loop system, by minimizing the variables $\gamma_1(t_k), \ldots, \gamma_E(t_k)$ via the cost $J(\cdot)$.

Remark 4.4 The relations (5) and (14) yield an explicit expression of the gain of γ at every time instant. This can be used to set-up an event triggering mechanism as follows: at every $k \in \mathbb{Z}_+$, if NCS requirements allow event generation and $\eta(t_k) \geq \eta_{\text{bound}}$ trigger event. In this way, a healthy trade-off between minimization of data transmission and performance can be achieved. □

A cost on the future states can be added to Problem 4.2. As ISS is guaranteed for any feasible solution, optimization of the cost can still be used to improve performance, without requiring that the global optimum is attained in real-time. An example of such a cost will be given next.

In what follows, we will indicate certain ingredients which allow the implementation of Problem 4.2 via linear programming. For this, we restrict our attention to Lyapunov functions defined using the infinity norm, i.e.,

$$V(x) = \|P_V x\|_\infty, \tag{19}$$

where $P_V \in \mathbb{R}^{p \times n}$ is a full-column rank matrix. Note that this type of function satisfies (4a), for $\alpha_1(s) := \frac{\nu(P_V)}{\sqrt{p}}s$ (where $\nu(P_V) > 0$ is the smallest singular value of P_V) and for $\alpha_2(s) := \|P_V\|_\infty s$. Letting $\alpha_3(s) := cs$ for some $c \in \mathbb{R}_+$ we directly obtain from (16) that $\alpha_{3,\max}(t_k)$ is equal to the maximum of α_3 over the vertices of the set $\{\hat{x}(t_k)\} \oplus \mathbb{W}(t_k)$. Similarly, it is sufficient to impose (17b) only for the vertices of $\{\hat{x}(t_k)\} \oplus \mathbb{W}(t_k)$, i.e., for $\{w^e(t_k) + \hat{x}(t_k)\}_{e \in \mathbb{Z}_{[1,E]}}$.

By definition of the infinity norm, for $\|x\|_\infty \leq c$ to be satisfied for some vector $x \in \mathbb{R}^n$ and constant $c \in \mathbb{R}_+$, it is necessary and sufficient to require that $\pm[x]_i \leq c$ for all $i \in \mathbb{Z}_{[1,n]}$. So, for (17c)-(17d) to be satisfied, it is necessary and sufficient to require that

$$\begin{aligned} &\pm[P_V(A\hat{x}(t_k) + Bu(t_k))]_i - V_{\min}(t_k) + \alpha_{3,\max}(t_k) \leq 0, \\ &\pm[P_V(A(\hat{x}(t_k) - w_e(t_k)) + Bu(t_k))]_i - V_{\min}(t_k) \\ &\qquad\qquad + \alpha_{3,\max}(t_k) - \gamma_e(t_k) \leq 0 \end{aligned} \tag{20}$$

for all $i \in \mathbb{Z}_{[1,p]}$ and $e \in \mathbb{Z}_{[1,E]}$. Moreover, by choosing an infinity-norm based cost function

$$\begin{aligned} &J(x(t_k), u(t_k), \gamma_i(t_k)) := \|P_J(A(\hat{x}(t_k) - w(t_k)) + Bu(t_k))\|_\infty \\ &\quad + \|Q_J(\hat{x}(t_k) - w(t_k))\|_\infty + \|R_J u(t_k)\|_\infty + \sum_{i=1}^{E} \|\Gamma_i \gamma_i(t_k)\|_\infty, \end{aligned} \tag{21}$$

with full-column rank matrices $P_J \in \mathbb{R}^{p_j \times n}, Q_J \in \mathbb{R}^{q_j \times n}, R_J \in \mathbb{R}^{r_j \times m}$ and $\Gamma_i \in \mathbb{R}_+$, we can reformulate the optimization of the cost J subject to the constraints (17) as the linear program

$$\min_{u(t_k), \gamma_1(t_k), \ldots, \gamma_E(t_k), \varepsilon_1, \varepsilon_2} \varepsilon_1 + \varepsilon_2 + \sum_{i=1}^{E} \Gamma_i \gamma_i(t_k) \tag{22}$$

subject to (17a), (17b), (20) and

$$\begin{aligned} &A(w_e(t_k) + \hat{x}(t_k)) + Bu(t_k) \in \mathbb{X}, \quad \forall e \in \mathbb{Z}_{[1,E]}, \\ &\pm[P_J(A(\hat{x}(t_k) - w_{e_1}(t_k)) + Bu(t_k))]_i + \|Q_J(\hat{x}(t_k) - w_{e_2}(t_k))\|_\infty \leq \varepsilon_1 \\ &\forall (e_1, e_2) \in \mathbb{Z}_{[1,E]} \times \mathbb{Z}_{[1,E]}, \forall i \in \mathbb{Z}_{[1,p_j]}, \\ &\pm[R_J u(t_k)]_i \leq \varepsilon_2, \quad \forall i \in \mathbb{Z}_{[1,r_j]}. \end{aligned}$$

The only thing left for implementing Problem 4.2 is to compute $V_{\min}(k)$. Using the same reasoning as above, it can be shown that the optimization problem (15) can be formulated as a linear program. As such, finding a solution to Problem 4.2 amounts to solving 2 linear programs and calculating the maximum over a finite set of real numbers, which can be performed efficiently.

This completes the design procedure of the robust MPC and we continue with the integration of the EBSE and MPC.

5. INTEGRATION OF EBSE AND MPC

In this section we provide a method for designing a set $\mathbb{W}(t_k)$ based on $P(t_k)$. To that extent, we will use the fact that $P(t_k)$ is a model for $cov(x(t_k) - \hat{x}(t_k))$, i.e., the covariance matrix of $w(t_k)$. Hence, $G(w(t_k), \underline{0}, P(t_k))$ is a model for the true PDF $p(w(t_k))$. The first step in this design is defining ellipsoidal sets that are based on $P(t_k)$. Each ellipsoidal set represents a model of the probability that the true estimation-error, i.e., $w(t_k)$, is within that set. The second step is to define $\mathbb{W}(t_k)$ as an over-approximation of a certain ellipsoidal set that can be computed efficiently and such that $w(t_k) \in \mathbb{W}(t_k)$ has a high probability. As all variables of this section are at the time-instant t_k, we will omit t_k from every time-dependent variable, i.e. $w(t_k)$ and $P(t_k)$ become w and P, respectively. Let us start by describing the sub-level-sets of a Gaussian. For any Gaussian $G(w, \underline{0}, P)$ one can define the sub-level set $\varepsilon(P,c) \subset \mathbb{R}^n$, for some $c \in \mathbb{R}_+$, as follows:

$$\varepsilon(P,c) := \left\{ w \in \mathbb{R}^n \middle| w^\top P^{-1} w \leq c \right\}. \tag{23}$$

The shape of this sub-level-set is an ellipsoid, or an ellipse in the 2D case as it is graphically depicted in Figure 4 (for $c = 1$). This figure illustrates the relation of the ellipsoid with the eigenvalues of the corresponding covariance matrix P. In this case the relation between eigenvalues of P yields $\lambda_1(P) > \lambda_2(P)$.

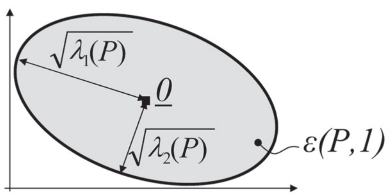


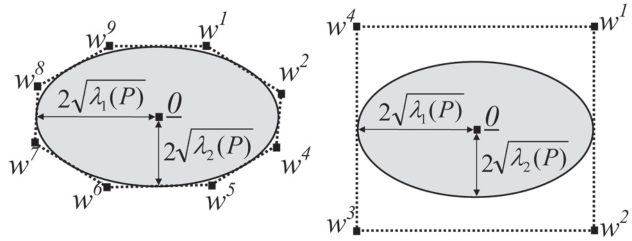
Figure 4: Graphical representation of the Gaussian $G(w, \underline{0}, P)$ as the sub-level-set $\varepsilon(P, 1)$. The direction of each arrow is defined as the eigenvector of the corresponding eigenvalue $\lambda_i(P)$.

Applying some basics of probability theory one can calculate the probability, depending on c, that the random vector w is within the set $\varepsilon(w,P,c)$. Some examples of this probability, denoted with $Pr(\cdot)$, for different values of c are:

$$Pr(w \in \varepsilon(P,1)) \approx 0.68,$$
$$Pr(w \in \varepsilon(P,4)) \approx 0.95,$$
$$Pr(w \in \varepsilon(P,9)) \approx 0.997.$$

Notice that $\varepsilon(P,c)$ defines an ellipsoid. As such, to use the MPC law as proposed in the previous section, we need to obtain a polytopic over-approximation of this set. This over-approximated set can then be used as the set $\mathbb{W}$ where the estimation error is bounded. Recall that the vertices of $\mathbb{W}$ are used in the controller to determine the control action. There is no optimal method to calculate this set, as it amounts to the ancient problem of "squaring the circle". See for example the recent results in [25] and the references therein. What can be stated is that a trade-off should be made between the size of the set $\mathbb{W}$ on the one hand, indicating the worst case estimation error bound, and the computational complexity of obtaining $\mathbb{W}$ on the other hand, which should be kept reasonable for on-line calculation.

If for example one chooses that $Pr(w \in \mathbb{W}) = 0.95$, then $\mathbb{W}$ can be taken as a tight over-approximation of $\varepsilon(P,4)$, as illustrated in Figure 5(a). However, to obtain this tight over-approximation, knowledge of all the eigenvalues of P and vertex computation is required, which is computationally expensive. If the real-time properties of the resulting algorithm allow this tight over-approximation, then Figure 5(a) can be considered. However, here we want to have the least processing-time, i.e. computational complexity. Therefore, we aim at describing the vertices of $\mathbb{W}$ by a single parameter that depends on the maximum eigenvalue of P. An example of such a set is shown in Figure 5(b). Therein, $\mathbb{W}$ is fully determined by the scalar $d = \sqrt{c\lambda_{\max}(P)} = 2\sqrt{\lambda_{\max}(P)}$, which represents an upper bound on the infinity norm of w.



(a) Accurate over-approximate. (b) Fast over-approximation.

Figure 5: Examples of over-approximating $\varepsilon(P,4)$ by $\mathbb{W}$.

Notice that the vertices of $\mathbb{W}$ as shown in Figure 5(b) are explicitly defined as all possible realizations of the vectors $(w^1, \ldots, w^E)^\top$ when $[w^i]_j \in \{-d, d\}$ for all $j \in \mathbb{Z}_{[1,n]}$ and $i \in \mathbb{Z}_{[1,E]}$. The next result provides an expression for $d \in \mathbb{R}_+$, which is calculated at each synchronous sample instant t_k, such that $\varepsilon(P,c) \subseteq \mathbb{W}$, for a given $c \in \mathbb{R}_+$.

Lemma 5.1 *Suppose a random vector $w \in \mathbb{R}^n$ is given of which its PDF is a Gaussian with zero mean and a covariance-matrix $P \in \mathbb{R}^{n\times n}$. Let us define $\mathbb{W} := \{w \mid \|w\|_\infty \leq d\}$ with $d := \sqrt{c * \lambda_{\max}(P)}$. Then for any $c \in \mathbb{R}_+$, it holds that $\varepsilon(P,c) \subset \mathbb{W}$.*

PROOF. Let $w \in \varepsilon(P,c)$. Then it holds that

$$\lambda_{\min}(P^{-1})\|w\|_\infty^2 \leq \lambda_{\min}(P^{-1})\|w\|_2^2 \leq w^\top P^{-1} w \leq c,$$

from which $\|w\|_\infty \leq \sqrt{c(\lambda_{\min}(P^{-1}))^{-1}}$ follows. Applying the fact that $\lambda_{\min}(P^{-1}) = (\lambda_{\max}(P))^{-1}$ gives that

$$\|w\|_\infty \leq \sqrt{c * \lambda_{\max}(P)}.$$

Hence, $w \in \mathbb{W}$ and as $w \in \varepsilon(P,c)$ was arbitrary, the proof is complete. □

The above results shows that in order to compute the vertices of the set $\mathbb{W}$ at each t_k, one only needs to calculate the current $\lambda_{\max}(P)$. The computational efficiency of the procedure for obtaining the set $\mathbb{W}$ can be further improved by using the known fact that $\lambda_{\max}(P) \leq \mathrm{tr}(P)$, at the cost of a more conservative error bound. To increase the probability that $w(t_k) \in \mathbb{W}(t_k)$, one can use Lemma 5.1 to observe that:

$$Pr(w(t_k) \in \varepsilon(P(t_k),c)) \leq Pr(w(t_k) \in \mathbb{W}(t_k)).$$

Therefore, by choosing $c = 9$ yields:

$$\begin{aligned} 0.997 &\approx Pr(w(t_k) \in \varepsilon(P(t_k),9)) \\ &\leq Pr(w(t_k) \in \mathbb{W}(t_k)). \end{aligned} \tag{24}$$

Remark 5.2 As the covariance matrix $P(t)$ is bounded for all $t \in \mathbb{T}$, shown in [15, 22], it follows that $\lambda_{\max}(P(t))$ is also bounded for all $t \in \mathbb{T}$. Then, by the definition of the set $\mathbb{W}(t_k)$, it holds that $\mathbb{W}(t_k) \subseteq \overline{\mathbb{W}}$ for all t_k and

$$\overline{\mathbb{W}} := \left\{ w \in \mathbb{R}^n \mid \|w\|_\infty \leq \sup_t \sqrt{c\lambda_{\max}(P(t))} \right\}.$$

□

This completes the overall design of the feedback loop that consists of *(i)* the EBSE algorithm, which provides an estimate of the state $\hat{x}(t_k)$ at all t_k, *(ii)* the algorithm for computation of the vertices of $\mathbb{W}(t_k)$ at all t_k such that $w(t_k) = \hat{x}(t_k) - x(t_k) \in \mathbb{W}(t_k)$ has a high probability and *(iii)* the robust MPC algorithm, which provides inherent ISS w.r.t. to estimation errors $w(t_k)$ and optimized ISS w.r.t. to $w(t_k) \in \mathbb{W}(t_k)$.

The only aspect left to be treated is concerned with the fact that the EBSE is a stochastic estimator, while the robust MPC algorithm is a deterministic controller. Due to the stochastic nature of the estimator, one does not have a guarantee that $w(t_k) \in \mathbb{W}(t_k)$ for all t_k and, as such, that the real state is bounded. Instead, one has the information that $w(t_k) \in \mathbb{W}(t_k)$ for all t_k only with a certain (high) probability, which implies that the ISS property of the MPC scheme applies to the overall integrated closed-loop system in a probabilistic sense only. If for some $t_k \in \mathbb{R}_+$, $w(t_k) \notin \mathbb{W}(t_k)$ but it is still bounded, it would be desirable to still have an ISS guarantee (i.e., independent of the bound on $w(t_k)$) for the closed-loop system. A possible solution could be to use a uniformly continuous control Lyapunov function to establish inherent ISS to the estimation error, which forms the object of future research.

6. ILLUSTRATIVE EXAMPLE

In this section we illustrate the effectiveness of the developed EBSE and robust MPC scheme. The case study is a 1D object-tracking system. The states $x(t)$ of the object are position and speed while the measurement-vector $y(t)$ is position. The control input $u(t)$ is defined as the object's acceleration. The states and control input are subject to the constraints $x(t) \in \mathbb{X} = [-5,5] \times [-5,5]$ and $u(t) \in \mathbb{U} = [-2,2]$. Both the process-noise as well as the measurement-noise are chosen to have a zero-mean Gaussian PDF with $Q_\tau = 3\tau \cdot 10^{-4} I$ and $R_v = 1 \cdot 10^{-4}$. As the process is a double integrator, the process model becomes:

$$\begin{aligned} x(t+\tau) &= \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} \frac{\tau^2}{2} \\ \tau \end{pmatrix} u(t) + q(t,\tau), \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + v(t). \end{aligned} \tag{25}$$

The sampling time of the controller is $\tau_s = 0.7[s]$. For simplicity, we use "Send-on-Delta" as the sampling method with $\Delta = 0.1[m]$. This means that in case the object drove an additional 0.1 meter with respect to its last sampled position, a new measurement of the position is taken. Therefore, the set which defines event sampling becomes $H_{r|t} = [y(t_{r-1}) - \Delta, y(t_{r-1}) + \Delta]$. The PDF $\Lambda_{H_{r|t}}(y(t))$, for all $t \in \mathbb{T}$, of the EBSE is approximated as a sum of 5 Gaussians that are equidistantly distributed along $[y(t_{r-1}) - \Delta, y(t_{r-1}) + \Delta]$. Therefore, we set

$$N = 5, \quad y_i(t) = y(t_{r-1}) - \left(\frac{N - 2(i-1) - 1}{N}\right)\Delta, \quad \forall i \in \mathbb{Z}_{[1,N]},$$

$$R_H(t) = \left(\frac{2\Delta}{N}\right)^2 \left(0.25 - 0.05 e^{-\frac{4(N-1)}{15}} - 0.08 e^{-\frac{4(N-1)}{180}}\right).$$

Next, let us design the parameters of the robust MPC. The technique of [24] was used to compute the weight $P_V \in \mathbb{R}^{2\times 2}$ of the Lyapunov function $V(x) = \|P_V x\|_\infty$ for $\alpha_3(s) := 0.01s$, yielding

$$P_V = \begin{pmatrix} 2.7429 & 0.7121 \\ 0.1989 & 4.0173 \end{pmatrix}.$$

Following the procedure described in Section 5, the set $\mathbb{W}(t_k)$ will have 4 vertices at all $k \in \mathbb{Z}_+$. As such, to optimize robustness, 4 optimization variables $\gamma_1(t_k), \cdots, \gamma_4(t_k)$ were introduced, each one assigned to a vertex of the set $\mathbb{W}(t_k)$. The MPC cost was chosen as $J(x(t_k), u(t_k), \gamma_1(t_k), \ldots, \gamma_4(t_k))$ with $P_J = 0.4I$, $Q_J = 0.2I$, $R_J = 0.1$ and $\Gamma_i = 4$ for all $i \in \mathbb{Z}_{[1,4]}$. The resulting linear program has 11 optimization variables and 108 constraints. In this cost-function, the variable $V_{\min}(t_k)$ of equation (15) is used. Its value is also calculated via solving a linear program with 3 optimization variables and 5 constraints. During the simulations, the worst case computational time required by the CPU over 100 runs was 20 $[ms]$ for the controller and 5 $[ms]$ for the EBSE, which shows the potential for controlling fast linear systems.

In the simulation scenario we tested the closed-loop system response for $x(t_0) = [3,1]^\top$ with the origin as reference. The initial state estimates of the EBSE were chosen as $\hat{x}(t_0) = [3.5, 1.2]^\top$ and $P(t_0) = I$. The evolution of the *true* state is graphically depicted in Figure 6. Figure 7 presents the control input $u(t_k)$. The evolution of the different values for $\gamma_1(t_k), \cdots, \gamma_4(t_k)$ is shown in Figure 8. Figure 9 presents the absolute error between the estimated and true state, i.e. position $|[w(t_k)]_1|$ and speed $|[w(t_k)]_2|$, and the modeled error bound which is chosen to be $d = \sqrt{9\lambda_{\max}(P(t_k))}$ and defines $\mathbb{W}(t_k)$.

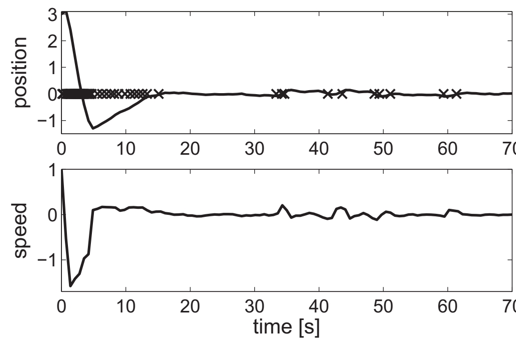


Figure 6: Evolution of the true state: position, i.e. $x_1(t_k)$ and speed, i.e. $x_2(t_k)$.

The symbols "×" in the plot of the true object position of Figure 6 denote the instants when an event occurs, i.e., whenever the object drove an additional 0.1 $[m]$ with respect to its previous sampled position measurement. Notice that the number of events increases when position is changing fast. Therefore, a large amount of samples are generated in the first 5 seconds. After 20 seconds, both the position and speed of the true state are zero and no event occurs anymore.

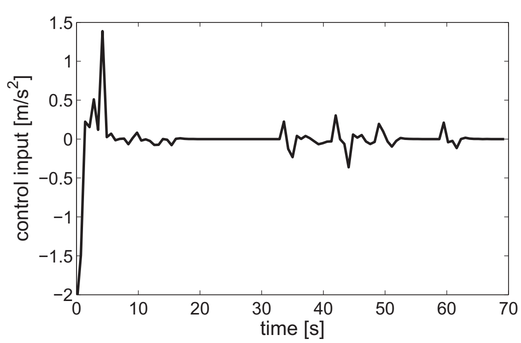


Figure 7: Evolution of the the control input $u(t_k)$.

Figure 7 shows that the input constraints are fulfilled at all times, and sometime they are active.

Notice that when the state is close to zero, due to the optimization

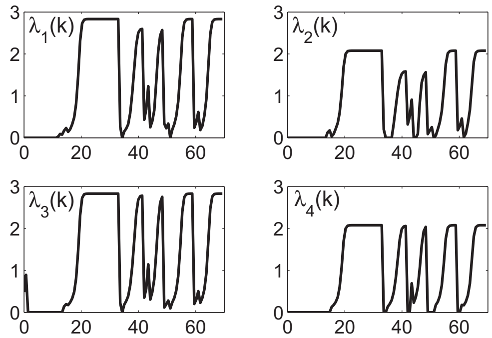


Figure 8: Evolution of the 4 **optimization variables** $\gamma_i(t_k)$ **.**

of the cost J, this pushes the control input to zero. As such, the optimization variables $\gamma_i(t_k)$ of Figure 8 must satisfy $V(w^e(t_k)) - \gamma_e(t_k) \leq 0$, $e = 1,\ldots,E$. This explains the non-zero value of $\gamma_i(t_k)$ when the state reaches the equilibrium, for example in between 20 and 30 seconds.

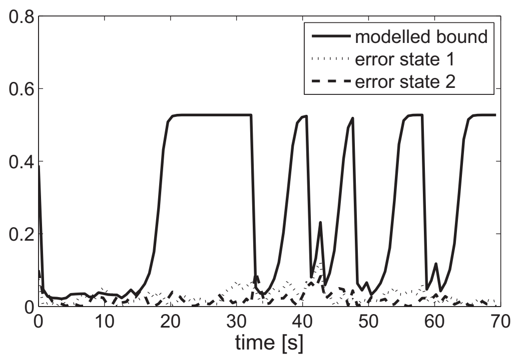


Figure 9: Evolution of the true estimation error compared to the bounds on the modeled one.

Figure 9 shows that the true estimation-error remained within the limits of the modeled one at almost all sample instances. Furthermore, after 20 seconds no new measurement is received anymore for some longer time period. Standard state-estimators would predict the state in that case, causing $\lambda_{\max}(P(t_k))$ to diverge. Due to the EBSE, which makes use of the bounded set $H_{r|t}$, $\lambda_{\max}(P(t_k))$ converges to a constant, although no events are generated anymore. This confirms that boundedness of the covariance matrix of the EBSE is independent of the number of events for a bounded set $H_{r|t}$, although the actual value of the bound is influenced by the choice of $H_{r|t}$. Notice the variety in the modeled estimation-error at 33, 41, 49 and 60 seconds. This is caused by the fact that an event occurs at these instances. Hence, the state-update is based on a received measurement rather then a bounded set, which reduces its uncertainty. A final remark is to be made on the fact that at around 10 seconds the true estimation-error exceeds the modeled one. This means that at this time instant $w(t_k) \notin \mathbb{W}(t_k)$. Nevertheless, the trajectory of the closed-loop system remains bounded with reasonable robust performance, which encourages us to further analyze the ISS of the integrated closed-loop system. In light of the solution proposed at the end of the previous section, it is worth to mention that the CLF used in the example is globally Lipschitz.

For future research it would also be interesting to compare the results obtained with the developed robust MPC scheme with a stochastic MPC set-up, such as the algorithm presented in [26].

7. CONCLUSIONS

In this paper an event-based control system was designed with the property that control actions can take place synchronously in time but data transfer between the plant and the controller is kept low. This was achieved by introducing an event-based state estimator in the feedback loop. The event-based estimator was used to obtain a state estimate with a bounded covariance matrix in the estimation error at every synchronous time instant, under the assumption that the set used for event generation is bounded in the measurement-space. This covariance matrix was then used to estimate explicit polytopic bounds on the estimation-error that were fed into a robust MPC algorithm. We proved that the resulting MPC controller achieves ISS to the estimation error and, moreover, it optimizes the closed-loop trajectory-dependent ISS gain. We provided justification of our main ideas on all the parts of the overall "output-based controller" (e.g., a bounded covariance matrix of ESBE-plant interconnection and ISS of MPC-plant loop) that show that in principle such an event-based controller should work. Several aspects of the integration of the stochastic event-based estimator and the deterministic MPC algorithm were discussed. The formal proof of closed-loop properties of the EBSE-MPC-plant interconnection or its variations is a topic of future research, although simulations provide convincing and promising evidence of the potential of the proposed methods.

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9. REFERENCES

[1] K. J. Åström, “Event based control,” in *Analysis and Design of Nonlinear Control Systems: In Honor of Alberto Isidori*. eds. A. Astolfi and L. Marconi, Springer-Verlag, 2008, pp. 127–147.

[2] K. J. Åström and B. M. Bernhardsson, “Comparison of Riemann and Lebesque sampling for first order stochastic systems,” in *41st IEEE Conf. on Dec. and Contr.*, Las Vegas, USA, 2002, pp. 2011 – 2016.

[3] M. Miskowicz, “Send-on-delta concept: An event-based data-reporting strategy,” *Sensors*, vol. 6, pp. 49–63, 2006.

[4] ——, “Asymptotic Effectiveness of the Event-Based Sampling according to the Integral Criterion,” *Sensors*, vol. 7, pp. 16–37, 2007.

[5] R. W. Brockett and D. Liberzon, “Quantized feedback stabilization of linear systems,” *IEEE Transactions on Automatic Control*, vol. 47, no. 7, pp. 1279–1289, 2000.

[6] E. Kofman and J. H. Braslavsky, “Level crossing sampling in feedback stabilization under data-rate constraints,” in *IEEE Conference on Decision and Control*, San Diego, California, 2006, pp. 4423–4428.

[7] W. P. M. H. Heemels, J. H. Sandee, and P. P. J. van den Bosch, “Analysis of event-driven controllers for linear systems,” *International Journal of Control*, vol. 81, no. 4, 2008.

[8] T. Henningsson, E. Johannesson, and A. Cervin, “Sporadic event-based control of first-order linear stochastic systems,” *Automatica*, vol. 44, no. 11, pp. 2890–2895, 2008.

[9] D. V. Dimarogonas and K. H. Johansson, “Event-triggered control for multi-agent systems,” in *IEEE Conference on Decision and Control*, Shanghai, China, 2009.

[10] W. P. M. H. Heemels, R. J. A. Gorter, A. van Zijl, P. P. J. van den Bosch, S. Weiland, W. H. A. Hendrix, and M. R. Vonder, “Asycnhronous measurement and control: A case study on motor synchronization,” *Control Engineering Practice*, vol. 7, pp. 1467–1482, 1999.

[11] P. Tabuada, “Event-Triggered Real-Time Scheduling for Stabilizing Control Tasks,” *IEEE Transactions on Automatic Control*, vol. 52, pp. 1680–1685, 2007.

[12] M. Velasco, P. Marti, and C. Lozoya, “On the Timing of Discrete Events in Event-Driven Control Systems,” in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, vol. 44981. St. Louis, USA: Springer Verlag, 2008, pp. 670–673.

[13] X. Wang and M. Lemmon, “Self-triggered Feedback Control Systems with Finite-Gain L_2 Stability,” *IEEE Transactions on Automatic Control*, vol. 45, pp. 452–467, 2009.

[14] D. Lehmann and J. Lunze, “Event-based control: A state-feedback approach,” in *European Control Conference*, Budapest, Hungary, 2009, pp. 1717–1721.

[15] J. Sijs and M. Lazar, “On event-based state estimation,” in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, vol. 5469. Springer Verlag, 2009, pp. 336–350.

[16] L. Aggoun and R. Elliot, *Measure Theory and Filtering*. Cambridge University Press, 2004.

[17] H. L. Lebesque, “Integrale, longueur, aire,” Ph.D. dissertation, University of Nancy, 1902.

[18] C. M. Kellett and A. R. Teel, “On the robustness of $\mathscr{KL}$-stability for difference inclusions: Smooth discrete-time Lyapunov functions,” *SIAM Journal on Control and Optimization*, vol. 44, no. 3, pp. 777–800, 2005.

[19] D. Bernardini and A. Bemporad, “Energy-Aware Robust Model Predictive Control with Feedback from Multiple Noisy Wireless Sensors,” in *Proceedings of the European Control Conference*, Budapest, Hungary, 2009.

[20] H. W. Sorenson and D. L. Alspach, “Recursive Bayesian estimation using Gaussian sums,” *Automatica*, vol. 7, pp. 465–479, 1971.

[21] R. E. Kalman, “A new approach to linear filtering and prediction problems,” *Transaction of the ASME Journal of Basic Engineering*, vol. 82, no. D, pp. 35–42, 1960.

[22] J. Sijs and M. Lazar, “Event based state estimation with time synchronous updates,” Eindhoven Universtiy of Technology,” Available at: http://www.cs.ele.tue.nl/MLazar/Home.html.

[23] S. W. Chan, G. C. Goodwin, and K. S. Sin, “Convergence Properties of the Riccati Difference Equation in Optimal Filtering of Nonstabilizable Systems,” *IEEE Transactions on Automatic Control*, vol. 29, pp. 110–118, 1984.

[24] M. Lazar and W. P. M. H. Heemels, “Optimized input-to-state stabilization of discrete-time nonlinear systems with bounded inputs,” in *American Control Conference*, Seattle, Washington, 2008, pp. 2310–2315.

[25] A. Alessio, M. Lazar, A. Bemporad, and W. P. M. H. Heemels, “Squaring the circle: An algorithm for obtaining polyhedral invariant sets from ellipsoidal ones,” *Automatica*, vol. 43, no. 12, pp. 2096–2103, 2007.

[26] D. Bernardini and A. Bemporad, “Scenario-based model predictive control of stochastic constrained linear systems,” in *48th IEEE Conference on Decision and Control*, Shanghai, China, 2009.