

# Extended Projected Dynamical Systems with Applications to Hybrid Integrator-Gain Systems

B. Sharif M.F. Heertjes W.P.M.H. Heemels

**Abstract**—The class of projected dynamical systems (PDS) has proven to be a powerful framework for modeling dynamical systems of which the trajectories are constrained to a set by means of projection. However, PDS fall short in modeling systems in which the constraint set does not satisfy certain regularity conditions and only part of the dynamics can be projected. This poses limitations in terms of the phenomena that can be described in this framework especially in the context of systems and control. Motivated by hybrid integrator-gain systems (HIGS), which are recently proposed control elements in the literature that aim at overcoming fundamental limitations of linear time-invariant feedback control, a new class of discontinuous dynamical systems referred to as extended projected dynamical systems (ePDS) is introduced in this paper. Extended projected dynamical systems include PDS as a special case and are well-defined for a wider variety of constraint sets as well as partial projections of the dynamics. In this paper, the ePDS framework is connected to the classical PDS literature and is subsequently used to provide a formal mathematical description of a HIGS-controlled system, which was lacking in the literature so far. Based on the latter result, HIGS-controlled systems are shown to be well-posed, in the sense of global existence of solutions.

## I. INTRODUCTION

Nonlinear and hybrid control strategies have been demonstrated to be effective tools in dealing with the well-known and classical trade-off in linear control systems between (i) low frequency disturbance suppression by means of integral control and (ii) a desired transient performance. A famous example of such a nonlinear control technique is reset control initially proposed by Clegg [1] and later developed into the first order reset element (FORE) [2]. A reset controller is a linear time-invariant (LTI) system whose states, or part of the states, are reset to zero (or some other value) whenever its input and output signals meet certain conditions [3]. More specifically, the reset generally enforces that the input and output signals of the reset element have the same sign. As a result of this construction, a reset integrator exhibits less phase lag when compared to a linear integrator and consequently can be used to break free from the above mentioned performance trade-off in linear control. Successful applications of reset control and its superiority to its linear counterparts have been showcased in for example [3], [4].

In spite of the design advantages that reset control systems offer when compared to linear control methods, there are also some features that might limit the performance obtained by controllers designed with such techniques. More specifically, by resetting (part of) the states, reset controllers produce

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discontinuous control signals that can potentially excite high-frequency plant dynamics or amplify high-frequency noise, which can be highly undesirable. For these reasons, a novel hybrid integrator-gain system (HIGS) was presented in [5] that aims at overcoming the above mentioned limitation of reset controllers while offering the same promising features regarding phase by keeping the sign of its input and output equal. In particular, a HIGS element  $\mathcal{H}$  in its preferred mode of operation is described by an integrator element

$$\dot{x}_c(t) = \omega_h e(t), \quad u(t) = x_c(t), \quad (1)$$

where  $x_c(t) \in \mathbb{R}$ ,  $e(t) \in \mathbb{R}$  and  $u(t) \in \mathbb{R}$  denote the state, input and output of  $\mathcal{H}$ , respectively, at time  $t \in \mathbb{R}_{\geq 0}$  and  $\omega_h \in (0, \infty)$  is the integrator frequency. However, the integrator mode (1) can only be used as long as the input/output pair  $(e, u)$  remains inside the sector

$$\mathcal{F} := \{(e, u) \in \mathbb{R}^2 \mid eu \geq u^2\}. \quad (2)$$

At moments when the input/output pair  $(e, u)$  of  $\mathcal{H}$  tends to leave  $\mathcal{F}$ , which can happen when  $u = e$ , the vector field of  $\mathcal{H}$  is changed to  $\dot{x}_c = \dot{e}$  so that the trajectories remain on the sector boundary,  $u = x_c = e$ , until the integrator mode can be used again. This second mode of operation where  $\dot{x}_c = \dot{e}$ , is called the gain mode of the HIGS in [5]. As a consequence, the switching operation of the HIGS is such that its input/output pair  $(e, u)$  is always contained in  $\mathcal{F}$  and thus always have the same sign<sup>1</sup>. Moreover, the controller output  $u$  produced by the HIGS is a continuous signal. This explains why the HIGS offers similar advantages as reset control elements while overcoming their shortcomings related to discontinuous control signals, making HIGS a powerful control element for high-performance control. The strength of the HIGS in terms of improving the closed-loop performance of control systems (mainly in applications to high-precision mechatronics) as well as frequency and time domain tools for design/analysis of HIGS-controlled systems have been portrayed in [5]–[8].

Despite the engineering success of HIGS, up to now a formal mathematical description and well-posedness analysis of HIGS were not provided. Addressing these open issues may prove instrumental for gaining further insight into the operation of the HIGS and paving the way for further developments. This forms one of the objectives in this paper. When considering a HIGS element with state  $x_c$  in feedback interconnection with a linear physical plant with state  $x_p$ , as previously explained, the intended operation of the HIGS is to use integrator dynamics, as long as its

<sup>1</sup> $(e, u) \in \mathcal{F}$  does not just imply that  $e$  and  $u$  have the same sign since  $\mathcal{F}$  does not necessarily cover the whole first and third quadrants of the  $e-u$  space. However, the size of  $\mathcal{F}$  can be tuned by modifying the integrator frequency  $\omega_h$ .

input signal, being a function of  $x = (x_c^\top, x_p^\top)^\top$  (plus possible exogenous signals such as disturbance and references), and output signal, satisfy  $(x_c, x_p) \in S$  (where  $S$  is related to the sector  $\mathcal{F}$ ). At moments when the sector condition tends to be violated by following integrator dynamics, the vector field of the controller changes by switching to gain mode thereby enforcing the input/output signals to stay within the sector. Similar dynamics are observed in the literature in the class of projected dynamical systems (PDS), where the trajectories of the system are ensured to be within a given constraint set at all times by means of projection. However, the PDS literature (see, e.g., [9]) is limited to cases where the full state vector  $(x_c, x_p)$  is projected onto a constraint set, which is generally assumed to satisfy certain regularity conditions. More specifically, in [10] PDS are shown to be well-defined for convex constraint sets. Weaker conditions are required in [11] where Clarke regularity and prox-regularity (see [12]) of constraint sets are required for existence and uniqueness of solutions, respectively. In the case of HIGS, however, the constraint set  $S$  does not satisfy any of these regularity conditions. Moreover, in the context of control, one can only project the controller ( $x_c$ ) dynamics while the plant ( $x_p$ ) dynamics cannot be projected, as these represent states that abide the laws of physics. As a consequence, for the mathematical formalization of the HIGS (and related systems), the PDS framework should be generalized to accommodate for *partial* projections onto a wider range of constraint sets. Motivated by these arguments, as a first contribution in this paper, we introduce a new class of dynamical systems, which we refer to as extended projected dynamical systems (ePDS). This class of systems includes as a special case the classical PDS available in the literature. As a second contribution the introduced ePDS framework is used for formalizing HIGS. As it turns out, ePDS is indeed the right framework to consider the HIGS and connects well to the engineering “philosophy” underlying this new control element, as indicated in [5]–[7]. As a third contribution, this paper gives a proof of well-posedness in terms of global existence of Carathéodory solutions for the feedback interconnection of an LTI system with a HIGS element. It should be noted that well-posedness analysis of the HIGS is a challenging task as a close look at its underlying dynamics reveals that these dynamics lack the (continuity) properties typically used in the literature related to hybrid systems and differential inclusions such as [13], [14], for proving well-posedness of hybrid systems.

The remainder of this paper is organized as follows. Section II contains preliminary definitions and notation. In Section III the ePDS framework is introduced. In Section IV, our formulation presented in Section III is linked to alternative formulations encountered in the PDS literature. A HIGS-controlled system is formalized in the ePDS framework in Section V. Subsequently, proofs of local and global well-posedness of the HIGS-controlled system, are presented in Sections VI and VII, respectively. Section VIII contains concluding remarks and future directions of research.

## II. PRELIMINARIES

### A. Preliminary definitions and notation

A sequence of scalars  $(u^1, u^2, \dots, u^k)$  with  $k \in \mathbb{N}$ , is called lexicographically non-negative (non-positive), written

as  $(u^1, u^2, \dots, u^k) \geq_l 0$  ( $\leq_l 0$ ), if  $(u^1, u^2, \dots, u^k) = (0, 0, \dots, 0)$  or  $u^j > 0$  ( $< 0$ ) where  $j = \min\{p \in \{1, \dots, k\} \mid u^p \neq 0\}$ . A polyhedral set in  $\mathbb{R}^n$  is a set given by the intersection of a finite number of closed half-spaces. As a particular polyhedral set, consider  $G = [g_1 \ g_2 \ \dots \ g_m] \in \mathbb{R}^{n \times m}$ , where  $g_i \in \mathbb{R}^n$ ,  $i = 1, 2, \dots, m$  are the columns of  $G$ . Then the polyhedral set  $\text{pos}G$  is the convex cone consisting of all positive combinations of the columns of  $G$  given by,  $\text{pos}G = \{\sum_{i=1}^m \alpha_i g_i \mid \alpha_i \geq 0, i = 1, 2, \dots, m\}$ , [15]. For the column space of a matrix  $H \in \mathbb{R}^{n \times m}$  we write,  $\text{im}H = \{Hx \mid x \in \mathbb{R}^m\}$ .

*Definition 2.1:* A function  $w : I \rightarrow \mathbb{R}^{n_w}$ , with  $I \subseteq \mathbb{R}$  is called a Bohl function, denoted by  $w \in B_I$ , if there exist matrices  $H \in \mathbb{R}^{n_w \times n_F}$ ,  $F \in \mathbb{R}^{n_F \times n_F}$ , and a vector  $v \in \mathbb{R}^{n_F}$  such that  $w(t) = He^{Ft}v$  for all  $t \in I$ .

*Definition 2.2:* A function  $w : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_w}$  is called a piece-wise Bohl function, denoted by  $w \in PB$ , if there exists a sequence  $\{t_i\}_{i \in \mathbb{N}}$  with  $t_{i+1} > t_i > t_0 = 0$  for all  $i \in \mathbb{N}$  and  $t_i \rightarrow \infty$  when  $i \rightarrow \infty$  such that  $w : [t_i, t_{i+1}) \rightarrow \mathbb{R}^{n_w}$  is a Bohl function for each  $i \in \mathbb{N}$ .

*Definition 2.3:* The tangent cone to a set  $K \subset \mathbb{R}^n$  at a point  $x \in K$ , denoted by  $T_K(x)$ , is the set of all vectors  $w \in \mathbb{R}^n$  for which there exist sequences  $\{x_i\}_{i \in \mathbb{N}} \in K$  and  $\{\tau_i\}_{i \in \mathbb{N}}$ ,  $\tau_i > 0$  with  $x_i \rightarrow x$ ,  $\tau_i \downarrow 0$  and  $i \rightarrow \infty$ , such that

$$w = \lim_{i \rightarrow \infty} \frac{x_i - x}{\tau_i}. \quad (3)$$

When  $K$  is a closed convex cone, we have  $T_K(x) = \bigcup_{t > 0} \frac{K-x}{t}$  (see Remark 5.2.2. in III, [15]). In fact, due to convexity, it holds that for  $x \in K$ ,

$$\frac{K-x}{t_2} \subseteq \frac{K-x}{t_1} \text{ when } 0 < t_1 \leq t_2. \quad (4)$$

*Definition 2.4:* [15] The projection of a vector  $x \in \mathbb{R}^n$  onto a closed, non-empty set  $S \in \mathbb{R}^n$ , denoted by  $P_S(x)$ , is defined as

$$P_S(x) = \operatorname{argmin}_{s \in S} \|s - x\|. \quad (5)$$

### B. Projected dynamical systems

To introduce “classical” projected dynamical systems (PDS) [9], [14], consider a differential equation

$$\dot{x}(t) = f(x(t)), \quad (6)$$

in which  $x(t) \in \mathbb{R}^n$  denotes the state at time  $t \in \mathbb{R}_{\geq 0}$ . In PDS there is a restriction on the state  $x(t)$  in the sense that  $x(t)$  has to remain inside a set  $S \subseteq \mathbb{R}^n$ , which is ensured by redirecting the vector field at the boundary of  $S$ . Formally, a PDS is given for a continuous vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a set  $S \subseteq \mathbb{R}^n$  (with further additional conditions to make the definitions below meaningful) by

$$\dot{x}(t) = \Pi_S(x(t), f(x(t))), \quad (7)$$

with

$$\Pi_S(x, v) = \operatorname{argmin}_{w \in T_S(x)} \|w - v\|, \quad (8)$$

for  $x \in S$  and  $v \in \mathbb{R}^n$ . Based on this formulation,  $\Pi_S(x, v)$  can be interpreted as an operator that selects the vector closest to  $v$ , which lies in the set of admissible velocities at  $x$ . An equivalent characterization of  $\Pi_S(x, v)$ , (see Proposition 5.3.5 in III, [15]) is

$$\Pi_S(x, v) = \lim_{\delta \downarrow 0} \frac{P_S(x + \delta v) - x}{\delta}, \quad (9)$$

when  $S$  is a closed, convex and non-empty set. In fact, in several works on PDS (see, e.g., [16], [17]) the expression (9) is used to define PDS.

### III. EXTENDED PROJECTED DYNAMICAL SYSTEMS

Inspired by the philosophy behind the HIGS discussed in the introduction, we present a generalization to PDS.

#### A. Model representation

Let  $S \subseteq \mathbb{R}^n$  be a given non-empty closed set, on which we impose additional conditions later to obtain a well-posed system. We are interested in dynamical systems of the form

$$\dot{x}(t) = f(x(t)), \quad (10)$$

in which the state of the system has to reside inside the set  $S$ . In classical PDS, as recalled in Subsection II-B, the latter is ensured by “projecting” the *complete* vector field on the tangent cone of the set  $S$ , cf. (7) and (8). This projection is along *all* possible directions of the state in the sense that it just takes the vector  $\Pi_S(x, v) \in T_S(x)$  that is “closest” to  $v = f(x)$ , irrespective of the direction  $\Pi_S(x, v) - v$ . So, it is allowed to alter the complete vector field and thus the velocities of all the states  $x$  in (10). Clearly, if (10) is a closed-loop system in the sense of an interconnection of a physical plant and a controller (and thus the state  $x$  consists of physical plant states  $x_p$  and controller states  $x_c$ ), one cannot project in all directions (see the discussion in the introduction). Indeed, the physical state dynamics cannot be modified by straightforward projection. It is only possible to “project” the controller ( $x_c$ ) dynamics and possibly part of the plant ( $x_p$ ) dynamics in which the control input appears explicitly. Hence, in contrast to PDS, we only have limited directions by which we can “correct” the vector field  $f(x)$  at the boundary, if needed, to keep the state trajectories inside  $S$ . To formalize this, we model the restricted “correction / projection” directions by the image of a matrix  $E \in \mathbb{R}^{n \times n_E}$ , which is assumed, without loss of generality, to have full column rank. Formally,

$$\dot{x}(t) = \Pi_{S,E}(x(t), f(x(t))), \quad (11)$$

with

$$\Pi_{S,E}(x, v) = \operatorname{argmin}_{w \in T_S(x), w - v \in \operatorname{im} E} \|w - v\|, \quad (12)$$

for  $x \in S$  and  $v \in \mathbb{R}^n$ . Hence, the projection  $\Pi_{S,E}$  projects the velocity  $v$  at  $x \in S$  onto the set of admissible velocities (tangent cone) along  $\operatorname{im} E$  in such a way that the correction  $w - v \in \operatorname{im} E$  is minimal in norm. For these systems, we coin the term *extended Projected Dynamical Systems* (ePDS), as they include the classical PDS (7) as a special case by taking  $\operatorname{im} E = \mathbb{R}^n$  (and restricting  $S$  to meet the required regularity conditions).

#### B. Well-posed projection operator $\Pi_{S,E}$

Clearly, we have to show that the introduced projection  $\Pi_{S,E}(x, v)$  is well-defined in the sense that it provides a unique outcome for each  $x \in S$  and each  $v \in \mathbb{R}^n$ . As in the case of classical PDS, this requires conditions on the set  $S$  and, in this case, also on  $E$ . Although we envision that we can work under more general conditions, for the sake of setting the scene in this paper and inspired by the application of the HIGS, we focus on the setting below. Future work will include relaxation of these conditions.

*Assumption 3.1:* The set  $S \subseteq \mathbb{R}^n$  and  $E \in \mathbb{R}^{n \times n_E}$  satisfy

- $S = K \cup -K$  in which  $K$  is a convex polyhedral cone given by  $K = \operatorname{pos} G + \operatorname{im} H$  with the matrix  $[G \ H] \in \mathbb{R}^{n \times n}$  square and of rank  $n$ .
- $K \cap -K = \operatorname{im} H$ .
- $E$  has full column rank.
- $\operatorname{im} E \cap S = \{0\}$  and  $S + \operatorname{im} E = \mathbb{R}^n$ .

Note that  $K$  can also be written as

$$K := \{x \in \mathbb{R}^n \mid Fx \geq 0\}, \quad (13)$$

for some suitable matrix  $F \in \mathbb{R}^{n_f \times n}$  of full rank. We are particularly interested in this setup as it can describe *sector conditions* as are used in reset controllers and the HIGS, cf. (2). Sector conditions also appear in circle and Popov criteria for the analysis of Lur’e type of systems. Moreover, note that such sets do not necessarily meet the previously mentioned regularity requirements (convexity, Clarke regularity and prox-regularity), which are used in the PDS literature. To prove the well-posedness of (12), observe first that

$$T_S(x) = \begin{cases} T_K(x), & \text{if } x \in K \setminus -K \\ K \cup -K, & \text{if } x \in K \cap -K, \\ -T_K(x), & \text{if } x \in -K \setminus K \end{cases} \quad (14)$$

with

$$T_K(x) = \{w \in \mathbb{R}^n \mid F_I(x)w \geq 0\}, \quad (15)$$

where

$$I(x) = \{i \in \{1, 2, \dots, n_f\} \mid F_i x = 0\},$$

is the set of active constraints at  $x$ . We used here the notation  $F_J$  for  $J \subseteq \{1, \dots, n_f\}$  to denote the matrix consisting of the rows of  $F$  with row numbers in  $J$ . Also observe that we can rewrite (12) as

$$\Pi_{S,E}(x, v) = v + E\eta^*(x, v), \quad (16)$$

with

$$\eta^*(x, v) = \operatorname{argmin}_{\eta \in \Lambda(x, v)} \|E\eta\|, \quad (17)$$

and

$$\Lambda(x, v) = \{\eta \in \mathbb{R}^{n_E} \mid v + E\eta \in T_S(x)\}. \quad (18)$$

*Lemma 3.1:* Under Assumption 3.1, it holds for each  $x \in S$  and each  $v \in \mathbb{R}^n$  that  $\Lambda(x, v)$  is a non-empty closed polyhedral set.

The proof follows from standard convex analysis arguments and is omitted due to space reasons.

Due to the fact that the constraint set of (17) is a closed polyhedral set and that the square of the cost function of (16) being  $\eta^T E^T E \eta$  is a quadratic positive definite function (as  $E$  has full column rank), a unique minimizer exists, showing the well-posedness of (16) and thus (12).

### IV. CONNECTING TO ALTERNATIVE PDS REPRESENTATIONS

As already indicated in Subsection II-B, there is an equivalence for classical PDS between the definitions (8) and (9) under certain conditions on the set  $S$ . In this section, the objective is to establish a similar equivalence for ePDS. To do so, let us first introduce

$$P_{S,E}(x) = \operatorname{argmin}_{s \in S, s - x \in \operatorname{im} E} \|s - x\|. \quad (19)$$

Clearly, we can rewrite (19) as

$$P_{S,E}(x) = \operatorname{argmin}_{s \in C_x} \|s - x\| \quad \text{with } C_x := \{s \in S \mid s - x \in \operatorname{im} E\}. \quad (20)$$

Note that, although this formulation has similarities with (8), the set  $C_x = S \cap (x + \operatorname{im} E)$  is dependent on  $x$ , which is not

the case in (8). Observe that  $C_x$  is a non-empty closed and convex set, Hence,  $P_{S,E}(x) = P_{K,E}(x)$  (if  $x \in K + \text{im}E$ ) or  $P_{S,E}(x) = P_{-K,E}(x)$  (if  $x \in -K + \text{im}E$ ) and thus  $P_{S,E}(x)$  gives a unique outcome, see, e.g., the reasoning on page 116 of [15]. In line with (9) for classical PDS, we consider also

$$\tilde{\Pi}_{S,E}(x, v) = \lim_{\delta \downarrow 0} \frac{P_{S,E}(x + \delta v) - x}{\delta}. \quad (21)$$

**Theorem 4.1:** Under Assumption 3.1, it holds that  $\tilde{\Pi}_{S,E}(x, v) = \Pi_{S,E}(x, v)$  for all  $x \in S$  and  $v \in \mathbb{R}^n$ . The proof follows the arguments of the proof of Proposition 5.3.5 in III, [15] and is omitted here for reasons of space.

Hence, also for ePDS we have the equivalence between (12) and (21).

**Remark 4.1:** In [18], the authors treat oblique projections by formulating PDS as

$$\begin{aligned} \dot{x} &= \Pi_S^g(x, f(x)) = \underset{w \in T_S(x)}{\text{argmin}} \|w - f(x)\|_{g(x)}^2 \\ &= \underset{w \in T_S(x)}{\text{argmin}} (w - f(x))^\top \mathcal{G}(x) (w - f(x)), \end{aligned} \quad (22)$$

where  $\mathcal{G}(x)$  is a symmetric, positive definite matrix of appropriate dimensions, for all  $x \in S$ . This is a generalization of the classical PDS that correspond to (22) with  $\mathcal{G}(x)$  equal to the identity matrix. However, since  $\mathcal{G}(x)$  is positive definite and thus full rank, (22) still projects the dynamics  $\dot{x}(t) = f(x)$ , along all possible directions of the state  $x$  and thus differs from our representation (11). The framework (22) is of high interest in other settings though, see [18].

## V. DESCRIPTION OF HIGS-CONTROLLED SYSTEM

In this section we describe a closed-loop system including a HIGS and formalize it through the ePDS framework introduced in Section III. The interconnection under consideration is depicted in Fig. 1. Herein, a HIGS element  $\mathcal{H}$  is in feedback interconnection with an LTI plant  $G$  (including the plant to be controlled and possibly an LTI controller), where  $G$  admits the following state-space representation

$$\dot{x}_p(t) = Ax_p(t) + B(u(t) + d(t)), \quad (23a)$$

$$y(t) = Cx_p(t), \quad (23b)$$

with  $x_p(t) \in \mathbb{R}^{n_x}$ ,  $u(t) \in \mathbb{R}^{n_u}$ ,  $d(t) \in \mathbb{R}^{n_d}$ ,  $y(t) \in \mathbb{R}^{n_y}$  the state, input, disturbance and output of the system, respectively, at time  $t \in \mathbb{R}_{\geq 0}$ . Furthermore it holds that  $A \in \mathbb{R}^{n_x \times n_x}$ ,  $B \in \mathbb{R}^{n_x \times n_u}$ , and  $C \in \mathbb{R}^{n_y \times n_x}$ . Moreover, for ease of exposition, the external disturbances including the reference signal  $r: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_r}$  and the input disturbance signal  $d: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_d}$  are assumed to belong to the class of Bohl functions, generated by exo-systems  $\Sigma_r$  and  $\Sigma_d$  with states  $x_r(t)$  and  $x_d(t)$ , respectively. Consequently, these signals can be described as

$$\begin{aligned} r(t) &= H_r x_r(t) = H_r e^{F_r t} v_r, \\ d(t) &= H_d x_d(t) = H_d e^{F_d t} v_d, \end{aligned} \quad (24)$$

for some matrices  $H_r \in \mathbb{R}^{n_r \times n_{F_r}}$ ,  $H_d \in \mathbb{R}^{n_d \times n_{F_d}}$ ,  $F_r \in \mathbb{R}^{n_{F_r} \times n_{F_r}}$ ,  $F_d \in \mathbb{R}^{n_{F_d} \times n_{F_d}}$ , and vectors  $v_r \in \mathbb{R}^{n_{F_r}}$ ,  $v_d \in \mathbb{R}^{n_{F_d}}$ , see also Definition 2.1. Note that the results below can be easily extended to include signals belonging to the class of piece-wise Bohl functions (see Definition 2.2) by allowing these matrices to switch over time and “concatenate” the resulting solutions. It should be noted that sines, cosines,

exponentials, polynomials and their sums and products are all Bohl functions and thus the class of piece-wise Bohl functions, is sufficiently rich to accurately describe (deterministic) disturbances frequently encountered in practice. Using

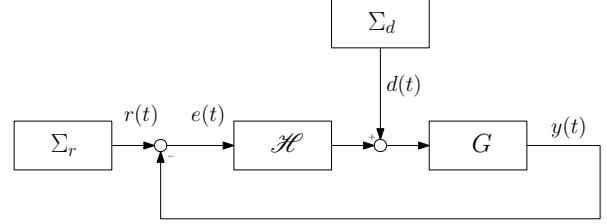


Fig. 1. The interconnection of a HIGS element  $\mathcal{H}$  with an LTI plant  $G$  and exosystems  $\Sigma_r$  and  $\Sigma_d$ .

the definitions above the following state space representation

$$\dot{x} = \begin{bmatrix} \dot{x}_c \\ \dot{x}_p \\ \dot{x}_r \\ \dot{x}_d \end{bmatrix} = \begin{bmatrix} \mathcal{H}(x) \\ Ax_p + B(x_c + H_d x_d) \\ F_r x_r \\ F_d x_d \end{bmatrix}, \quad (25)$$

is obtained for the overall interconnection with  $x = [x_c^\top, x_p^\top, x_r^\top, x_d^\top]^\top \in \mathbb{R}^n$  and  $x_c$  the state of the HIGS element  $\mathcal{H}$ . As explained earlier, the intended operation of  $\mathcal{H}$  is such that it primarily exhibits integrator dynamics, (1). In this mode of operation, (25) can be written as

$$\begin{aligned} \begin{bmatrix} \dot{x}_c \\ \dot{x}_p \\ \dot{x}_r \\ \dot{x}_d \end{bmatrix} &= \begin{bmatrix} \omega_h e \\ Ax_p + B(x_c + H_d x_d) \\ F_r x_r \\ F_d x_d \end{bmatrix} = \\ &= \begin{bmatrix} \omega_h (H_r x_r - C x_p) \\ Ax_2 + B(x_c + H_d x_d) \\ F_r x_r \\ F_d x_d \end{bmatrix} := A_i x. \end{aligned} \quad (26)$$

The philosophy behind the HIGS aims at using the dynamics (26) as long as the input and output of the HIGS stay inside (or on the boundary of) a sector  $\mathcal{F}$ , which translates on the level of the state  $x$  to the state being contained in  $S = \mathcal{H} \cup -\mathcal{H}$  with

$$\mathcal{H} = \{x \in \mathbb{R}^n \mid \underbrace{[F_1^\top \quad F_2^\top]^\top}_F x \geq 0\}, \quad (27)$$

and

$$F_1 = [-1 \quad -C \quad H_r \quad 0], \quad F_2 = [1 \quad 0 \quad 0 \quad 0], \quad (28)$$

and only “correct” the  $x_c$  dynamics when  $x(t)$  tends to leave the sector. Based on the ePDS framework introduced in Section III, we can now formalize this philosophy of the HIGS-controlled system exactly as

$$\dot{x} = \Pi_{S,E}(x, A_i x), \quad (29)$$

with  $E = [1 \quad 0 \quad 0 \quad 0]^\top$ . Note that this choice of  $E$  results in projections only in the direction of the HIGS state  $x_c$ , and not  $x_p$ ,  $x_r$  and  $x_d$ , as this would be physically impossible. To obtain a more explicit expression of (29) recall that

$$\Pi_{S,E}(x, A_i x) = A_i x + E \eta^*(x), \quad (30)$$

where

$$\eta^*(x) = \underset{A_i x + E\eta \in T_S(x)}{\operatorname{argmin}} |\eta|. \quad (31)$$

To explicitly compute (29) via (30), observe that

$$I(x) = \{i \in \{1, 2\} \mid F_i x = 0\}, \quad (32)$$

$$T_{\mathcal{H}}(x) = \{w \in \mathbb{R}^n \mid F_{I(x)} w \geq 0\}, \quad (33)$$

resulting in the following expression for  $T_S(x)$

$$T_S(x) = \begin{cases} T_{\mathcal{H}}(x), & \text{if } x \in \mathcal{H} \setminus -\mathcal{H}, \\ \mathcal{H} \cup -\mathcal{H}, & \text{if } x \in \mathcal{H} \cap -\mathcal{H}, \\ -T_{\mathcal{H}}(x), & \text{if } x \in -\mathcal{H} \setminus \mathcal{H}. \end{cases} \quad (34)$$

It can be verified by some algebraic manipulations that  $A_i x \in T_S(x)$  and  $x \in S$  if and only if

$$x \in S_1 = \{x \in \mathbb{R}^n \mid F_2 x \geq 0 \wedge (F_1 x, F_1 A_i x) \geq_l 0\} \cup \{x \in \mathbb{R}^n \mid F_2 x \leq 0 \wedge (F_1 x, F_1 A_i x) \leq_l 0\}, \quad (35)$$

and in this case we have that  $\eta^*(x) = 0$  and thus

$$\dot{x} = \Pi_{S,E}(x, A_i x) = A_i x.$$

However, when  $A_i x \notin T_S(x)$  and  $x \in S$ , i.e., when

$$x \in S_2 = \{x \in \mathbb{R}^n \mid F_2 x \geq 0 \wedge F_1 x = 0 \wedge F_1 A_i x < 0\} \cup \{x \in \mathbb{R}^n \mid F_2 x \leq 0 \wedge F_1 x = 0 \wedge F_1 A_i x > 0\}, \quad (36)$$

$\eta^*(x)$  is such that  $|\eta|$  is minimal and

$$F_1(A_i x + E\eta^*(x)) \geq 0,$$

when  $F_1 A_i x < 0$  and  $F_2 x \geq 0$  and

$$F_1(A_i x + E\eta^*(x)) \leq 0,$$

when  $F_1 A_i x > 0$  and  $F_2 x \leq 0$ . By solving for the minimal  $|\eta|$  that satisfies the above inequalities, we obtain

$$F_1(A_i x + E\eta^*(x)) = 0,$$

resulting in the following ‘‘gain’’ mode dynamics

$$\dot{x} = \Pi_{S,E}(x, A_i x) = A_i x + E\eta^*(x) = A_g x,$$

with

$$A_g = (I - E(F_1 E)^{-1} F_1) A_i. \quad (37)$$

By reworking (37) we find the expression

$$\dot{x} = \begin{bmatrix} \omega_h e \\ A x_p + B(x_c + H_d x_d) \\ F_r x_r \\ F_d x_d \end{bmatrix} + \begin{bmatrix} \dot{e} - \omega_h e \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (H_r F_r x_r - C(A x_2 + B(x_c + H_d x_d))) \\ A x_p + B(x_c + H_d x_d) \\ F_r x_r \\ F_d x_d \end{bmatrix} = A_g x, \quad (38)$$

for the gain mode dynamics, which is similar to the gain mode described in [5]–[7], see e.g., Eq. (8) in [5]. Based on the discussion above, one can see that (29) is equivalent to

$$\dot{x} = \begin{cases} A_i x, & \text{if } x \in S_1, \\ A_g x, & \text{if } x \in S_2, \end{cases} \quad (39)$$

which is similar to the formulation of the HIGS-controlled system described in [5]–[8].

## VI. LOCAL WELL-POSEDNESS OF THE HIGS-CONTROLLED SYSTEM

In this section we show that the HIGS-controlled system is a locally well-posed system in the sense of existence of solutions, on  $[0, \varepsilon]$  for some  $\varepsilon > 0$  given an initial state and exogenous continuous reference signals  $r(t)$  and (not necessarily continuous) disturbances  $d(t)$  belonging to the class of (piece-wise) Bohl functions. The developments in this section are instrumental in proving global existence of solutions, i.e., on  $[0, \infty)$ , to the HIGS-controlled system, which is the subject of Section VII. Note that here we focus on the description (29) (or (39)) in which the exogenous signals  $r$  and  $d$  are already embedded. However, the existence result below is independent of the used  $r$  and  $d$  and thus applies for any Bohl function. Let us first formally introduce the adopted solution concept.

*Definition 6.1:* An absolutely continuous function  $x : [0, T] \rightarrow \mathbb{R}^n$  is called a solution to the HIGS-controlled system (29) on  $[0, T]$  with initial state  $x_0 \in S$  if  $x(0) = x_0$ ,  $x(t) \in S$ , for all  $t \in [0, T]$  and (29) holds almost everywhere in  $[0, T]$ .

The solution concept defined in Definition 6.1, is called a *Carathéodory* solution to (29).

*Definition 6.2:* We call the HIGS-controlled system (29) (or equivalently (39)) locally well-posed if for all  $x_0 \in S$ ,  $r \in PB$  and  $d \in PB$ , there exists an  $\varepsilon > 0$  such that the system admits a solution on  $[0, \varepsilon]$  with initial state  $x_0$ .

To show local well-posedness of (29) (or equivalently (39)), let us consider the set

$$S_{\text{int}} = \{x_0 \in S \mid \exists \varepsilon > 0, \forall t \in [0, \varepsilon], e^{A_i t} x_0 \in S\}. \quad (40)$$

Simply stated,  $S_{\text{int}}$  denotes the set of all initial states from which we can operate in the integrator mode without leaving the set  $S$  for a non-trivial time span.

*Lemma 6.3:* If for some  $x \in \mathbb{R}^n$  and some  $N \in \mathbb{N}$

$$F_1 A_i^k x = 0, \quad k = 0, 1, 2, \dots, N,$$

then

$$F_1 A_i A_g^k x = 0, \quad k = 0, 1, 2, \dots, N-1, \quad (41)$$

and

$$F_1 A_i A_i^N x_g = F_1 A_i^{N+1} x. \quad (42)$$

*Proof:* By substituting the expression (37) for  $A_g$  in (41) and (42), the result can be verified. ■

We are ready to state the main result of this section.

*Theorem 6.4:* The HIGS-controlled system is locally well-posed.

*Sketch of proof:* The steps of the proof are as follows:

- 1) By definition of  $S_{\text{int}}$  it is clear that for all  $x_0 \in S_{\text{int}}$  a solution of the form  $x(t) = e^{A_i t} x_0 \in S$  exists for all  $t \in [0, \varepsilon]$  and some  $\varepsilon > 0$ . Moreover, since  $S_{\text{int}} \subseteq S_1$ , this solution is also a solution to (39) and thus (29).
- 2) To show existence of solutions for  $x_0 \in S \setminus S_{\text{int}}$ , it is first shown that  $x_0 \in S \setminus S_{\text{int}}$  implies  $F_1 x_0 = 0$  and  $F_2 x_0 \neq 0$ . Subsequently, using the definition of the set  $S_2$  as in (36) together with Lemma 6.3, it can be proven that a solution of the form  $\bar{x}(t) = e^{A_g t} x_0 \in S_2$  exists, for all  $t \in [0, \varepsilon]$  and some  $\varepsilon > 0$ , and thus  $\bar{x}(t)$  is a solution to (39).

Therefore, for every initial  $x_0 \in S$  the HIGS-controlled system admits a solution on  $[0, \varepsilon]$  for some  $\varepsilon > 0$  and thus is locally well-posed. ■

## VII. GLOBAL WELL-POSEDNESS ANALYSIS OF THE HIGS-CONTROLLED SYSTEM

Building on Section VI, in this section we show that the HIGS-controlled system is globally well-posed.

*Definition 7.1:* We call the HIGS-controlled system (29) (or equivalently (39)) globally well-posed if for every initial condition  $x_0 \in S$ ,  $r \in PB$  and  $d \in PB$ , it admits a solution on  $[0, \infty)$ .

In proving global well-posedness, we will use the existence of a constant  $M \in \mathbb{R}$  such that (29) (or (39)) satisfies the linear growth condition

$$\|\Pi_{S,E}(x, A_i x)\| \leq M\|x\|, \text{ for all } x \in \mathbb{R}^n, \quad (43)$$

which simply holds as  $\Pi_{S,E}(x, A_i x) \in \{A_i x, A_g x\}$  for all  $x \in S$ .

*Theorem 7.2:* The HIGS-controlled system (29) (or equivalently (39)) is globally well-posed.

*Sketch of proof:* Let initial state  $x_0 \in S$ ,  $r \in PB$  and  $d \in PB$  (see Definition 2.2) be given. The closed-loop system admits the representation (29) on  $t \in [t_0, t_1]$  with  $t_0 = 0$  and  $t_1 > 0$ , coming from Definition 2.2 of  $PB$  functions. One can then proceed by contradiction to prove existence of solutions on  $[0, t_1]$  with the following steps

- 1) First it is assumed that a solution  $x(t)$  exists only on  $[0, T)$  with  $T < t_1$ . Using (43) together with (29) and Gronwall's Lemma, one concludes that  $\|x(t)\| < L$  for some  $L > 0$ , and for all  $t \in [0, T)$ .
- 2) Using the result of the previous step together with (43), one can show that  $x(t)$ , is Lipschitz continuous on  $[0, T)$  and thus, uniformly continuous and also absolutely continuous. As a result, a solution exist on  $[0, T]$ .
- 3) Next, based on the local existence of solutions shown in Theorem 6.4, one can extend the solution from initial condition  $x(T)$  to get a solution to (29) on  $[0, T + \varepsilon]$  for some  $\varepsilon > 0$ . As such, a contradiction is reached. This proves the existence of solutions on  $[0, t_1]$ .

Similar reasoning can be used to prove the existence of solutions for the next Bohl part, i.e., on  $[t_1, t_2]$ , starting from initial state  $x(t_1)$ . By concatenation, a solution is proven to exist on  $[0, \infty)$ . Note that the sequence  $\{t_k\}_{k \in \mathbb{N}}$  does not have accumulation points since  $t_k \rightarrow \infty$  as  $k \rightarrow \infty$  due to Definition. 2.2 of  $PB$  functions. ■

## VIII. CONCLUSIONS

This paper introduced the class of extended projected dynamical systems (ePDS). This class includes as a special case the well-established PDS and extends their usage to accommodate for partial projections on a larger range of constraint sets. We have established a connection between our representation of ePDS and representations that resemble those frequently utilized in the PDS literature, cf. (21). The extension of PDS to ePDS was instrumental in (and, in fact, motivated by) the mathematical formalization of HIGS-controlled systems, which was lacking in the literature so far. Additionally, we have proven the fundamental property of well-posedness of HIGS-controlled systems [5]–[7], in the sense of global existence of solutions.

Future research directions include the relaxation of the conditions in Assumption 3.1 in order to enable the modeling

of a larger variety of systems in the ePDS framework. Moreover, well-posedness analysis of ePDS in general, and HIGS-controlled systems in particular, beyond references and disturbances being piece-wise Bohl functions as well as proving uniqueness of solutions will be considered in the future. In addition, the study of multiple HIGS elements in a control loop and connections to other classes of dynamical systems including complementarity systems (see e.g., [19], [20]), are of special interest.

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