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On the Equivalence of Extended and Oblique Projected Dynamics with Applications to Hybrid Integrator-Gain Systems

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***Abstract*— The class of projected dynamical systems (PDS) provides a powerful framework for modeling dynamical systems of which the trajectories are constrained to a set by means of projection. This work is concerned with establishing equivalence results among two recent variations of PDS. These are (i) extended PDS (ePDS), which enable partial projection of dynamics, and (ii) oblique PDS, (oPDS) where projections can be done with respect to non-Euclidean norms. We present two sets of sufficient conditions for equivalence among these two system classes. These results enable the transfer of system theoretical properties and tools from one class to the other, which we illustrate in this paper. As an application, we study hybrid integrator-gain systems (HIGS), which are recently introduced hybrid control elements aiming at overcoming fundamental limitations of linear time-invariant control, and are formally described in the ePDS framework. We use our results to also describe these control elements as oPDS, thereby enabling the study of HIGS-controlled systems in this framework.**

***Index Terms*— Extended projected dynamical systems, oblique projected dynamical systems, hybrid integrator gain systems.**

I. INTRODUCTION

Projected dynamical systems (PDS) [9], [16] form an important class of discontinuous dynamical systems, which have proven to be useful in the study of many different applications. Examples include control and optimisation [4], [10], [14], [18] as well as the study of oligopolistic markets, urban transportation networks, traffic networks, international trade, agricultural and energy markets [16]. PDS are particularly useful in describing constrained dynamical systems, where the trajectories are ensured to be within a given constraint set at all times by means of projection.

In this paper, we are interested in two recent variations of PDS. One variation is formed by *extended projected dynamical systems* (ePDS), which were introduced in [18] and extend the class of PDS in two essential ways. Firstly, while PDS have been shown to be well-defined for constraint sets that satisfy certain regularity conditions (convexity in [2], [16] and Clarke regularity and prox-regularity in [6]), ePDS are well-defined for other constraint sets not satisfying these conditions. Secondly, while in the PDS literature the complete vector field is projected onto the constraint set, the ePDS framework considers the possibility of partial projection of dynamics (see Section III for details). This is particularly useful in the context of systems and control, where one can only modify (and thus project) the controller dynamics and not the plant dynamics, which abide the laws of physics and can not be altered by means of projections.

The other class consists of so-called *oblique projected dynamical systems* (oPDS) [10] which are extensions of PDS in the sense that in contrast to the PDS literature, where projections are performed with respect to the standard Euclidean norm, oPDS provide the possibility of having projections with respect to (varying) non-Euclidean norms. The class of oPDS are particularly interesting in the context of feedback-based optimization [11], [12].

The main contributions of this work are establishing conditions under which ePDS and oPDS are equivalent in the sense that they lead to same system trajectories. In particular, we provide two sets of sufficient conditions under which ePDS can be written as oPDS and vice versa. These results are motivated by the fact that while many constrained systems (especially in the context of systems and control, such as HIGS-based controllers [8] discussed later) are naturally modelled in the ePDS framework, the PDS and oPDS frameworks are currently much more extensively studied. In particular, there exists many results on existence and uniqueness of solutions [2], [6], [10], [16], stability analysis [10], [16] and equivalence to other classes of discontinuous dynamical systems [4], [5], [14] for PDS and oPDS. As such, establishing an equivalence would be useful as it facilitates transferring existing theoretical properties and tools between the classes of oPDS and ePDS. In this work, we particularly highlight this transfer of system-theoretic results, for properties such as (incremental) stability.

As an interesting case study, we consider *hybrid integrator-gain systems* (HIGS) [7], [8], which are recently introduced hybrid control elements aiming at overcoming fundamental limitations of linear time-invariant (LTI) control and, as reported in [7], [15], [19], have enjoyed substantial engineering success, particularly in the field of high precision mechatronics. In [8], [18], HIGS-controlled systems are formally described in the ePDS framework, which turns out to be the natural framework for their formalization. As another contribution of this work, we use our equivalence results to obtain a mathematical description of HIGS-controlled systems in the oPDS framework, which in turn, provides the possibility of analyzing HIGS-controlled systems based on existing results available in the PDS and oPDS literature. The remainder of this paper is organized as follows. Section II

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contains preliminary definitions and notation. In Section III, the system classes considered in this paper are described. Sufficient conditions for establishing equivalence between ePDS and oPDS are presented in Section IV. In Section V we present some implications of the equivalence results. Section VI is concerned with HIGS-controlled systems and their oPDS description. The paper is concluded in Section VII, where conclusions and future directions of research are provided.

II. Preliminaries

We say that a matrix $A \in \mathbb{R}^{n\times n}$ is positive definite, denoted by $A \succ 0$ if it is symmetric and $x^\top Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. The Euclidean inner product between two vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$, denoted by $\langle a,b\rangle$, is defined as $\langle a,b\rangle = a^\top b$. We will also use a "weighted" inner product based on a symmetric positive definite matrix G, denoted by $\langle .,.\rangle_G$ and given by $\langle a,b\rangle_G = a^\top Gb$. For the null space (or kernel) of a matrix $H \in \mathbb{R}^{n\times m}$ we write $\ker H = \{x \in \mathbb{R}^m \mid Hx = 0\}$, and for its column space (or image) we write $\mathrm{im} H = \{Hx \mid x \in \mathbb{R}^m\}$. Moreover, for $J \subseteq \{1,\dots,n\}$, we denote by H_J the matrix of size $|J|\times m$, with $|J|$ the cardinality of set J, consisting of the rows of H with indices in J. The tangent cone to a set $K \subset \mathbb{R}^n$ at a point $x \in K$, denoted by $T_K(x)$, is the set of all vectors $w \in \mathbb{R}^n$ for which there exist sequences $\{x_i\}_{i\in\mathbb{N}} \in K$ and $\{\tau_i\}_{i\in\mathbb{N}}$, $\tau_i > 0$ with $x_i \to x$, $\tau_i \downarrow 0$ and $i \to \infty$, such that $w = \lim_{i\to\infty} \frac{x_i - x}{\tau_i}$. A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to class $\mathscr{K}$ if it is strictly increasing and $\alpha(0) = 0$. A continuous function $\beta : [0,a)\times[0,\infty) \to [0,\infty)$ is said to belong to class $\mathscr{KL}$ if, for each fixed s, the mapping $r \mapsto \beta(r,s)$ belongs to class $\mathscr{K}$ with respect to r and, for each fixed r, the mapping $s \mapsto \beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$. Finally, L_∞^{loc} denotes the set of locally essentially bounded functions on $\mathbb{R}_{\geq 0} := [0,\infty)$.

III. System Classes

A. Extended Projected Dynamical Systems

Extended projected dynamical systems (ePDS) are discontinuous dynamical systems introduced in [18] given by

$$\dot{x}(t) = \Pi_{S,E}(x(t), f(x(t), w(t))), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $w(t) \in \mathbb{R}^{n_w}$ denote the state and exogenous (disturbances) inputs at time $t \in \mathbb{R}_{\geq 0}$, $f : \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ is a vector field and $\Pi_{S,E}$ is a projection operator that projects the dynamics onto the tangent cone of the constraint set $S \subseteq \mathbb{R}^n$ in the direction specified by the image of the matrix $E \in \mathbb{R}^{n\times n_E}$. Without loss of generality we can assume E to have full column rank. In particular, $\Pi_{S,E}$ is given by

$$\Pi_{S,E}(x,v) = P_{T_S(x),E}(v) := \mathrm{argmin}_{a\in T_S(x), a-v\in \mathrm{im} E} \|a-v\|. \tag{2}$$

Note that ePDS have PDS as a special case. Indeed, with $\mathrm{im} E = \mathbb{R}^n$, ePDS becomes the classical PDS formulation

$$\Pi_S(x,v) = P_{T_S(x)}(v) := \mathrm{argmin}_{a\in T_S(x)} \|a-v\|. \tag{3}$$

Alternatively, (2) can be written as

$$\Pi_{S,E}(x,v) = v + \eta^\star(x,v) \tag{4}$$

with

$$\eta^\star(x,v) = \mathrm{argmin}_{\eta\in\Lambda(x,v)} \|E\eta(x,v)\| \tag{5}$$

and

$$\Lambda(x,v) = \{\eta \in \mathbb{R}^{n_E} \mid v + E\eta \in T_S(x)\}. \tag{6}$$

In [18], equivalent representations of (2) which are similar to the ones frequently used in PDS literature are presented (see Section IV in [18] for details).

For now, we consider constraint sets that are convex polyhedral cones given by

$$S := \{x \in \mathbb{R}^n \mid Fx \geq 0\}, \tag{7}$$

for some matrix $F \in \mathbb{R}^{n_f\times n}$, although in the HIGS application (see Section VI) and also [10], more general (non-convex) constraint sets are studied. Note that the tangent cone to S as in (7) is given by

$$T_S(x) = \{a \in \mathbb{R}^n \mid F_{I(x)} a \geq 0\}, \tag{8}$$

where $I(x) = \{i \in \{1,\dots,n_f\} \mid F_i x = 0\}$, denotes the set of active constraints at $x \in S$.

Lemma 3.1: $\Pi_{S,E}(x,v)$ is well-defined in the sense that the right-hand side of (2) is a single point, for all $x \in S$ and $v \in \mathbb{R}^n$, if and only if

$$S + \mathrm{im} E = \mathbb{R}^n. \tag{9}$$

The proof of Lemma 3.1 follows from standard arguments and is omitted due to space reasons.

B. Projected Dynamical Systems with Oblique Projections

Projected dynamical systems with oblique projections (oPDS) were introduced in [10] and are given by

$$\dot{x}(t) = \Pi_S^G(x(t), f(x(t), w(t))), \tag{10}$$

where $x(t) \in \mathbb{R}^n$ and $w(t) \in \mathbb{R}^{n_w}$ denote the state and exogenous inputs at time $t \in \mathbb{R}_{\geq 0}$, $f : \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ is a vector field, as before, and Π_S^G is a projection operator given by

$$\Pi_S^G(x,v) = P_{T_S(x)}^G(v) := \mathrm{argmin}_{a\in T_S(x)} \|a-v\|_G, \tag{11}$$

with

$$\|a-v\|_G := \sqrt{(a-v)^\top G(a-v)}, \tag{12}$$

where $G \in \mathbb{R}^{n\times n}$ is symmetric positive definite, to make $\|.\|_G$ a well-defined norm. Note that with $G = I_n$, (11) also reduces to the classical PDS formulation (3).

It should be pointed out that (11) is well-defined for all $x \in S$ and all $v \in \mathbb{R}^n$ when constraint sets of the form (7) are considered, as the vector field is simply obtained through projection (with respect to a particular norm $\|.\|_G$, and without any restrictions as opposed to ePDS (c.f. (9))) onto the closed convex constraint set (7).

Remark 3.1: Although in this work we consider (11) with a constant matrix G, this need not be the case in general for oPDS. In particular G could vary as a function of the state x by using the notion of Riemannian metrics to define a variable inner product on $T_S(x)$ that changes as a function of x (see [10] for more details).

Our study of oPDS with a constant matrix G is motivated by the fact that they can be linked to the classical PDS (and thus ePDS) by means of similarity transformations and thus, the results available for PDS can be used for studying oPDS as in (10). We make this more explicit in Lemma 3.2.

Lemma 3.2: Every oPDS of the form (10), with a constant $G \succ 0$ can be written as a PDS and thus an ePDS.

Proof. The equivalence between (10) with a constant matrix $G \succ 0$ and PDS, follows from introducing a similarity transformation as done in equation (10) of [13]. Moreover, by using this equivalent PDS representation and the fact that PDS can be written as ePDS of the form (1), with $E = I_n$, an equivalent ePDS representation can be obtained. □

The reverse question, i.e., when ePDS can be written as oPDS (and thus through a similarity transformation as PDS) is considered in the next section.

IV. Sufficient Conditions For Equivalence

In this section, we present sufficient conditions for the equivalence of the ePDS (1) and the oPDS (10). As mentioned before, we consider constraint sets that are convex polyhedral cones described by (7), for which the tangent cone is given by (8).

Theorem 4.1: Given a constraint set $S \subseteq \mathbb{R}^n$ of the form (7), a matrix $E \in \mathbb{R}^{n\times n_E}$ such that (9) holds, and a positive definite matrix $G \in \mathbb{R}^{n\times n}$, then

$$\Pi_{S,E}(x,v) = \Pi_S^G(x,v), \tag{13}$$

for all $x \in S$ and for all $v \in \mathbb{R}^n$, if

$$G^{-1}F^\top = E(E^\top E)^{-1}E^\top F^\top. \tag{14}$$

Sketch of proof. The proof follows from comparing the optimality conditions for the projection operators $\Pi_{S,E}(x,v)$ and $\Pi_S^G(x,v)$. In particular, the steps of the proof are as follows:

1) Consider $v^\star = \Pi_{S,E}(x,v)$. Using (4) and (8) one has $v^\star = v + E\eta^\star$, where $\eta^\star$ is the solution to a quadratic programming (QP) problem. By writing the well-known Karush-Kuhn-Tucker (KKT) conditions [3] for this QP, one obtains an expression for $\eta^\star$, the substitution of which in $v^\star = v + E\eta^\star$ results in an expression for $v^\star$ in terms of v, E, and F.
2) Next, we consider $\bar{v}^\star = \Pi_S^G(x,v)$. It follows from (11) and (8) that $\bar{v}^\star$ is the solution to a QP problem. By writing the KKT conditions for this QP, an expression in terms of v, G, and F is found for $\bar{v}^\star$.
3) By comparing the expressions found in 1) and 2) it follows that $v = \bar{v}^\star$ and thus (13) is true, if (14) holds.

□

The following theorem, proposes an alternative, easy-to-check geometric condition for verifying whether an ePDS can be written as an oPDS.

Theorem 4.2: Given matrices $E \in \mathbb{R}^{n\times n_E}$ and $F \in \mathbb{R}^{n_f\times n}$, there exists a positive definite matrix $G \in \mathbb{R}^{n\times n}$ satisfying (14) if and only if

$$\ker F + \operatorname{im} E = \mathbb{R}^n. \tag{15}$$

Sketch of proof. The main steps in the proof are as follows:

1) For proof of necessity we note that existence of a matrix $G \succ 0$ satisfying (14), is equivalent to existence of a matrix $R \succ 0$ satisfying $RF^\top = PF^\top$, with $R := G^{-1}$ and $P := E(E^\top E)^{-1}E^\top$. Using the definition of P and the fact that $R \succ 0$, it can be shown that $\operatorname{im}F^\top \cap \ker E^\top = \{0\}$. By taking the orthogonal complement of this expression, one obtains (15), showing that (15) is indeed necessary for existence of G.
2) For the proof of sufficiency, we start by noting that (15) is equivalent to $x^\top Px > 0$ when $x \neq 0$ and $x \in \operatorname{im}F^\top$, with P as defined in the previous step. Next, one can show that (14) or, alternatively, $RF^\top = RF^\top$ with R and P as defined in the previous step holds, if and only if

$$T^{-1}RT \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} = T^{-1}PT \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}, \tag{16}$$

where the first n_1 columns of the matrix T form an orthonormal basis for $\operatorname{im}F^\top$ and its remaining columns form an orthonormal basis for $\ker F$. It can then be shown, that if $x^\top Px > 0$ when $x \neq 0$ and $x \in \operatorname{im}F^\top$ and thus (15) holds, it is always possible to find a matrix $R \succ 0$ satisfying (16), concluding the sufficiency proof.

□

Remark 4.1: Note that (15) implies (9), as $\ker F \subseteq S$. Hence, an ePDS satisfying Theorem 4.2, also satisfies the condition of Lemma 3.1 for being well-defined. This is natural as the equivalent oPDS is always well-defined.

It should be noted that the equivalence conditions in Theorem 4.1 and Theorem 4.2 are independent of the particular vector fields and thus hold for all vector fields f in the ePDS (1) and the oPDS (10). However, if (14) does not hold, it may still be true that for *particular* vector fields the equivalence between (1) and (10) can be established. Corollary 4.1 proposes extensions of Theorem 4.1 and Theorem 4.2 that aim at exploiting knowledge of the vector field.

Corollary 4.1: Let a vector field $f : \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^n$, a constraint set $S \subseteq \mathbb{R}^n$ of the form (7), a matrix $E \in \mathbb{R}^{n\times n_E}$ such that (9) holds, and a positive definite matrix $G \in \mathbb{R}^{n\times n}$ be given. Moreover, let us denote by $W \subseteq \mathbb{R}^{n_w}$ the set of disturbance values of interest. Define

$$J := \{i \in \{1,\dots,n_f\} \mid \exists x \in S \cap \ker F_i, \exists w \in W \text{ with } f(x,w) \notin T_S(x)\}. \tag{17}$$

Then it holds for all $x \in S$ and all $w \in W$ that $\Pi_{S,E}(x, f(x,w)) = \Pi_S^G(x, f(x,w))$, if

$$G^{-1}F_J^\top = E(E^\top E)^{-1}E^\top F_J^\top. \tag{18}$$

Moreover, given $f : \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^n$, $S \subseteq \mathbb{R}^n$ and $E \in \mathbb{R}^{n\times n_E}$, there exists $G \succ 0$ such that (18) holds, if and only if

$$\ker F_J + \operatorname{im} E = \mathbb{R}^n. \tag{19}$$

The proof of Corollary 4.1 follows from the fact that $f(x,w)$ lies in the tangent cone $T_S(x)$ when $i \notin J$, and has been omitted for space reasons.

It follows from Theorem 4.2 that given an ePDS (1), if (15) holds, we can translate it into an oPDS (10). As such, the following two-step procedure is proposed for writing an ePDS as an oPDS:

1) Given an ePDS (1), check whether $\ker F + \mathrm{im} E = \mathbb{R}^n$.
2) If the first step is satisfied, find a matrix $G \succ 0$ satisfying (14). This problem can be formulated as a linear matrix inequality (LMI)-based feasibility test, subject to the constraint (14), which can be solved using available semi-definite programming solvers. Alternatively, the procedure in the proof of Theorem 4.2 can be followed.

If the first step above fails and particular knowledge of the vector field is available, one can use the results of Corollary 4.1 in the procedure above, as an alternative.

V. Some Implications of Equivalence: Incremental Stability and More

In this section, we showcase the usefulness of the results established in Section IV, by providing examples where utilizing the equivalence results are advantageous.

Definition 5.1: [1], Consider a dynamical system $\dot{x} = g(x,w)$ with state x taking values in $\mathbb{R}^n$. Assume that for each exogenous disturbance $w \in L_\infty^{loc}$ and each initial condition $x(0) = \xi$ a unique locally absolutely continuous solution exists, whose value at time $t \in \mathbb{R}_{>0}$ is denoted by $x(t,\xi,w)$. The system is said to be incrementally asymptotically stable (δGAS), if there exists a $\mathscr{KL}-$ function β such that for all ξ, $\eta \in \mathbb{R}^n$ and all $w \in L_\infty^{loc}$

$$\|x(t,\xi,w) - x(t,\eta,w)\| \leq \beta(\|\xi - \eta\|,t) \quad \text{for all } t \in \mathbb{R}_{\geq 0}.$$

Definition 5.2: [13], A function $f : \mathbb{R}^m \to \mathbb{R}^n$, is called α-strongly G-monotone, for $\alpha > 0$, if

$$\langle f(x) - f(y), x - y\rangle_G \geq \alpha \|x-y\|_G^2, \quad \text{for all } x,y \in \mathbb{R}^m.$$

Theorem 5.3: Given a constraint set $S \subseteq \mathbb{R}^n$ of the form (7), a matrix $E \in \mathbb{R}^{n \times n_E}$ such that (9) holds, and a vector field $f(x,w) = -p(x) - h(w)$, with $p : \mathbb{R}^n \to \mathbb{R}^n$, and $h : \mathbb{R}^{n_w} \to \mathbb{R}^n$, the ePDS (1) is δGAS, if there exists a symmetric positive definite matrix $G \in \mathbb{R}^{n \times n}$, such that

1) G, F and E satisfy (14), and
2) The function $p : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and α-strongly G-monotone for $\alpha > 0$.

Proof. The first condition in the theorem ensures via Theorem 4.1 that (1) can be written as the oPDS (10). Using this oPDS-based representation, one can utilize condition (2) in the theorem in conjunction with Theorem 2 in [13] to prove δGAS of the system. □

Remark 5.1: Other properties such as incremental input-to-state stability (δISS) [1], as well as uniform/exponential and input-to-state convergence [17] and periodicity of steady-state solutions for periodic inputs, can be established for the ePDS (1) under the conditions stated in Theorem 5.3. The proof for each case would also build upon first obtaining an equivalent oPDS representation for (1), which is possible as a result of the first condition in Theorem 5.3, and then using the second condition in the theorem to prove these properties for the equivalent oPDS, as shown in Remark 1, [13].

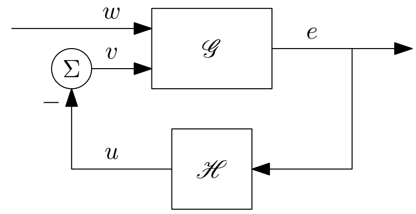

Fig. 1: Closed-loop system in Lur'e form.

In addition to Theorem 5.3 and the results pointed out in Remark 5.1, as explained in Lemma 3.2, one can perform similarity transformations to turn (10) into a classical PDS. As such the equivalence results established in Section IV enable one to also use several existing PDS results for analysis of ePDS.

VI. Equivalent HIGS representations

In this section, we apply the results obtained in the previous sections to HIGS-controlled systems. HIGS are discontinuous dynamical systems introduced in [7] with the aim of realizing performance beyond levels attainable with LTI control, and have enjoyed substantial engineering success (see, for example, [15], [19]). To recall HIGS, we start by describing the closed-loop system under consideration leading to an ePDS-based formulation. Motivated by using the results available for oPDS, the ePDS-based representation together with the results of Section IV are used to derive an alternative description for the closed-loop system in the oPDS framework of Section III-B.

A. System Description

We consider the interconnection in Fig. 1, consisting of an LTI plant $\mathscr{G}$ controlled in feedback with a single-input single-output (SISO) HIGS element $\mathscr{H}$.

The plant $\mathscr{G}$ contains the linear part of the closed-loop system including the plant to be controlled and possibly an LTI controller, given by the state-space representation

$$\begin{cases} \dot{x}_g &= A_g x_g + B_{gv} v + B_{gw} w, \\ e &= C_g x_g, \end{cases} \tag{20}$$

with states x_g taking values in $\mathbb{R}^{n_g}$, performance output e in $\mathbb{R}$, control input v in $\mathbb{R}$ and exogenous disturbances denoted by w taking values in $\mathbb{R}^{n_w}$. As the key area of application for HIGS is formed by motion systems containing floating masses, the following assumption is typically satisfied [8].

Assumption 6.1: The LTI system $\mathscr{G}$ as in Fig. 1 is such that $C_g B_{gw} = 0$ and $C_g B_{gv} = 0$.

The HIGS element $\mathscr{H}$ has as its primary mode of operation the linear integrator dynamics

$$\begin{aligned} \dot{x}_h &= \omega_h e, \\ u &= -v = x_h, \end{aligned} \tag{21}$$

where the state x_h takes values in $\mathbb{R}$, the (HIGS) input e and the (HIGS) output u both take values in $\mathbb{R}$ and $\omega_h \in [0,\infty)$ denotes the integrator frequency. This mode of operation of the HIGS is referred to as the *integrator mode*. The integrator mode (21) can only be used as long as the input-output pair (e,u) of $\mathscr{H}$ remains inside

$$\mathscr{F} := \left\{ (e,u) \in \mathbb{R}^2 \mid eu \geq \frac{1}{k_h} u^2 \right\}, \tag{22}$$

where $k_h \in (0,\infty)$ denotes the gain parameter of $\mathscr{H}$. At moments when the input-output pair (e,u) of $\mathscr{H}$ tends to leave the sector $\mathscr{F}$ the integrator dynamics in (21) are projected such that $(e,u) \in \mathscr{F}$ remains true along the trajectories of the system. Note that we can only change (project) the controller dynamics to satisfy $(e,u) \in \mathscr{F}$, while we cannot change the physical plant dynamics by means of projection. This setup clearly and naturally is described by *partial* projection and thus ePDS, as shortly discussed next.

B. ePDS Based Representation

Consider the closed-loop interconnection in Fig. 1, with state $x = [x_h^\top \;\; x_g^\top]^\top \in \mathbb{R}^n$. By combining (20) and (21), we arrive at the following state space representation for the primary mode of operation for the closed-loop system:

$$\begin{aligned} \dot{x} &= A_{\text{int}} x + Bw, \\ y &= Cx, \end{aligned} \tag{23}$$

where $y = [e \;\; u]^\top$, and

$$\left[\begin{array}{c|c} A_{\text{int}} & B \\ \hline C & \end{array}\right] = \left[\begin{array}{cc|c} 0 & \omega_h C_g & 0_{1\times n_w} \\ -B_{gv} & A_g & B_{gw} \\ \hline 0 & C_g & \\ 1 & 0_{1\times n_g} & \end{array}\right]. \tag{24}$$

Using the ePDS framework of Section III-A, we can formally describe the dynamics of the system as

$$\begin{cases} \dot{x}(t) = \Pi_{S,E}(x(t), A_{\text{int}} x(t) + Bw(t)), \\ y(t) = Cx(t) \end{cases} \tag{25}$$

with

$$S = \mathscr{K} \cup -\mathscr{K}, \tag{26}$$

where $\mathscr{K}$ is a polyhedral cone given by

$$\mathscr{K} := \{x \in \mathbb{R}^n \mid Fx \geq 0\}, \tag{27}$$

where $F = \begin{bmatrix} F_1^\top & F_2^\top \end{bmatrix}^\top$ with $F_1 = [-1 \;\; k_h C_g]$, and $F_2 = \begin{bmatrix} 1 & 0_{n_g} \end{bmatrix}$. In fact, $F_1 x = k_h e - u$ and $F_2 x = u$ such that $(e,u) \in \mathscr{F}$ if and only if $x \in S$. Moreover, $E = [1 \;\; 0_{n_g}^\top]^\top$ such that the x_h-dynamics are projected and x_g (representing physical states that cannot be projected or controller states that should not be projected) is not changed by means of projection (see [8], [18] for more details). Note that $S + \text{im}E = \mathbb{R}^n$, which makes the right-hand side of (25) well-defined (see [18]).

C. oPDS Based Representation

We will now present an alternative representation of HIGS-controlled systems in the oPDS framework discussed in Section III-B. Prior to doing so, let us make note of the fact that in Sections III and VI we considered convex polyhedral cones as constraint sets, while the constraint set (26) is a non-convex set and is given by a union of such cones. As a result, we can not directly use Theorem 4.1 and Theorem 4.2. Instead, we first consider the case where $x \in \mathscr{K}$ (see (27)) and subsequently use the symmetry of (26) to obtain a *switched* oPDS formulation for the HIGS-controlled system.

Let us turn our attention to

$$\begin{aligned} \dot{x}(t) &= \Pi_{\mathscr{K},E}(x(t), A_{\text{int}} x(t) + Bw(t)) \\ &= P_{T_{\mathscr{K}}(x),E}(A_{\text{int}} x(t) + Bw(t)). \end{aligned} \tag{28}$$

Note that even for (28), the results in Theorem 4.1 and Theorem 4.2 can not be directly used since $\ker F + \text{im}E = \{x \in \mathbb{R}^n \mid x = [\alpha \;\; 0_{n_g}^\top]^\top, \; \alpha \in \mathbb{R}\}$. As such, it is not possible to obtain an oPDS representation of HIGS-controlled systems based on Theorem 4.1. However, note that the conditions in Theorem 4.1 and Theorem 4.2 guarantee equivalence of ePDS and oPDS irrespective of the actual vector field. It turns out that for HIGS-controlled systems we can exploit particular knowledge of the vector field in relation to the constraint set, in line with Corollary 4.1. In particular, for the HIGS-controlled system (25), exploiting the form of the vector field (23) in combination with Assumption 6.1 would enable us to indeed make use of the results in Corollary 4.1. To do so, let us make the observation that for $x \in \ker F_2 \cap \mathscr{K}$, it can be verified by using Assumption 6.1, that $A_{\text{int}} x + Bw \in T_S(x)$ (see also the proof of Theorem 11 in [8]). This implies that when $F_2 x = 0$, the projections onto the tangent cone of S result in the integrator mode dynamics (regardless of the choice of E and G). Hence, to rewrite the ePDS (28) as an oPDS we only have to choose $G \succ 0$ to match the projections when $F_1 x = 0$, i.e., one of the facets of $\mathscr{K}$, as $\Pi_{\mathscr{K},E}(x, A_{\text{int}} x + Bw) = \Pi_{\mathscr{K}}^G(x, A_{\text{int}} x + Bw) = A_{\text{int}} x + Bw$, already holds for all points in $\{x \in \mathscr{K} \mid x \notin \ker F_1 \setminus \ker F_2\}$ and for all $w \in \mathbb{R}^{n_w}$, irrespective of the choice of G. As a result, (18) should only be satisfied when $F_1 x = 0$, i.e., if we can find a $G \succ 0$ satisfying

$$G^{-1} F_1^\top = E(E^\top E)^{-1} E^\top F_1^\top, \tag{29}$$

then for $x \in \mathscr{K}$, we can write the ePDS (28) as an oPDS (10). Note that F_1 and E do satisfy the condition (19) in Corollary 4.1, i.e.,

$$\ker F_1 + \text{im}E = \mathbb{R}^n, \tag{30}$$

which implies that it should be possible to find a $G \succ 0$ satisfying (29). Moreover, since $E = [1 \; 0_{n_g}^\top]^\top$ is orthonormal, (29) simplifies to $G^{-1} F_1^\top = EE^\top F_1^\top$, implying that G should satisfy the expression $F_1^\top = GEE^\top F_1^\top$. The latter equality holds for any $G \succ 0$ of the form

$$G = \begin{bmatrix} 1 & -k_h C_g \\ -k_h C_g^\top & G_{22} \end{bmatrix} \tag{31}$$

with $G_{22} \succ 0$, a free variable, sufficiently large such that $G_{22} - k_h^2 C_g^\top C_g \succ 0$ (to have $G \succ 0$ by Schur's lemma). As such, we have shown that for $x \in \mathscr{K}$, the system (28) can be written as an oPDS. Next, we will use this result to obtain an oPDS representation for the HIGS-controlled system (25).

Let us note that the tangent cone to the constraint set (26), is given by

$$T_S(x) = \begin{cases} T_{\mathscr{K}}(x), & \text{if } x \in \mathscr{K} \setminus -\mathscr{K} \\ -T_{\mathscr{K}}(-x), & \text{if } x \in -\mathscr{K} \setminus \mathscr{K} \\ \mathscr{K} \cup -\mathscr{K}, & \text{if } x \in \mathscr{K} \cap -\mathscr{K} \end{cases} \quad (32)$$

with $T_{\mathscr{K}}(x) = \{\nu \in \mathbb{R}^n \mid F_{I(x)}\nu \geq 0\}$, where $I(x) = \{i \in \{1,2\} \mid F_i x = 0\}$, is the set of active constraints at $x \in \mathscr{K}$. Thus, when considering the complete constraint set (26), (25) can be alternatively written as the switched ePDS

$$\begin{aligned} \dot{x}(t) &= \Pi_{S,E}(x, A_{\text{int}}x + Bw) \\ &= \begin{cases} P_{T_{\mathscr{K}}(x),E}(A_{\text{int}}x + Bw), & \text{if } x \in \mathscr{K} \setminus -\mathscr{K} \\ P_{-T_{\mathscr{K}}(-x),E}(A_{\text{int}}x + Bw), & \text{if } x \in -\mathscr{K} \setminus \mathscr{K} \\ A_{\text{int}}x + Bw, & \text{if } x \in -\mathscr{K} \cap \mathscr{K}. \end{cases} \end{aligned} \quad (33)$$

Now, using the discussion above and the symmetry of (26), (25) (or equivalently (33)) can be alternatively written as

$$\begin{aligned} \dot{x}(t) &= \Pi_S^G(x, A_{\text{int}}x + Bw) \\ &= \begin{cases} P^G_{T_{\mathscr{K}}(x)}(A_{\text{int}}x + Bw), & \text{if } x \in \mathscr{K} \setminus -\mathscr{K} \\ P^G_{-T_{\mathscr{K}}(-x)}(A_{\text{int}}x + Bw), & \text{if } x \in -\mathscr{K} \setminus \mathscr{K} \\ A_{\text{int}}x + Bw, & \text{if } x \in -\mathscr{K} \cap \mathscr{K}. \end{cases} \end{aligned} \quad (34)$$

with G as in (31). As such, we have obtained a *switched* oPDS representation of the HIGS-controlled system (25).

This representation of HIGS-controlled systems is of high interest as there are many results established for oPDS (with a constant matrix G) and PDS that come now into reach for analysis of HIGS-controlled systems as well. For instance, the results presented in Section V, indicate that this representation may prove instrumental in studying properties such as incremental stability as well as periodicity of steady-state solutions to HIGS-controlled systems. A hurdle in taking this step however, is the fact that $S = \mathscr{K} \cup -\mathscr{K}$ is not a (polyhedral) cone, which was a condition used in Section V, or stated differently, the switched nature of the switched oPDS (34) obstructs the direct application of the results in Section V. Taking this hurdle forms an important topic of future work. However, we believe that the step from ePDS to oPDS is a valuable one for future analysis of HIGS-controlled systems, as it provides a good basis for studying, among others, the properties mentioned above.

VII. Conclusions

In this paper, we have connected two classes of discontinuous dynamical systems consisiting of extended projected dynamical systems (ePDS) and oblique projected dynamical systems (oPDS). We have presented sufficient conditions for establishing equivalence between these two recently introduced varations of PDS, thereby enabling the transfer of theoretical tools and properties from one class to the other. We have particularly highlighted the transfer of system-theoretic results, for properties such as (incremental) stability, and periodicity of steady-state solutions for periodic inputs. Moreover, the results have been used to obtain a new description for popular hybrid integrator-gain systems (HIGS)-based controllers in terms of (switched) oPDS. This new description can be used to exploit properties and results that are available for oPDS and PDS, for analysis of HIGS-controlled systems. Future research directions include an in-depth study of properties of ePDS in general and HIGS-controlled systems in particular, exploiting the equivalence established in this paper, where handling the switched nature of the dynamics of HIGS requires special consideration.

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