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State estimation for systems with varying sampling rate[1]

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Abstract

We investigate problems that occur in state estimation when the outputs of a continuous time system are sampled at non-equidistant time instants. The time interval between the samples is assumed to be known, and to belong to a prescribed set. An observer synthesis procedure is proposed which guarantees the stability of the estimation error and in addition minimizes the covariance of the observation error.

1 Introduction

In this paper we study the problem of state estimation for a continuous time linear system, corrupted with process and measurement noise, in the case when the time interval between the samples is not constant, but belongs to some known set of time intervals. By an example we will show that simply choosing static Kalman filters for each of the discretized systems corresponding to each sample time might yield unstable estimation error dynamics. Therefore, we develop a procedure for choosing static observer gains in a Luenberger type of observer, so that the stability of the estimation error in the noise-free case is guaranteed under any sequence of the allowed inter-sample times. Moreover, our procedure guarantees that the covariance of the estimation error will be asymptotically bounded by an upper bound, which can be optimized.

2 Preliminaries

Consider the following continuous time system:

$$\dot{x} = Ax + Bu + Gw \quad (1a)$$

$$y = Cx + v \quad (1b)$$

The process and measurement noise, denoted by w and v, respectively, are assumed to be mutually uncorrelated, Gaussian and have covariances $E[w(t)w(t-s)] = Q\delta(t-s)$ and $E[v(t)v(t-s)] = R\delta(t-s)$. We assume that the system (1) is sampled only at discrete sampling instants t_k, where $t_k - t_{k-1} \in \{h_1, h_2, \dots, h_N\}$. The state evolution of the system at the sampling instants can be described by:

$$x_{k+1} = \Phi^{i_k} x_k + \Gamma^{i_k} u_k + w_k \quad (2a)$$

$$y_k = Cx_k + v_k \quad (2b)$$

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where $\Phi^i = e^{Ah_i}$, $\Gamma^i = \int_0^{h_i} e^{As}Bds$, where the index i_k takes value in $\{1, \dots, N\}$, depending on the time interval between the last two samples. The variables w_k and v_k denote discrete time equivalents of process and measurement noise, which can be taken to be white, zero mean, uncorrelated, with covariance matrices [4,5]:

$$Q_d^i = \int_0^{h_i} e^{A\tau} G Q G^\top e^{A^\top \tau} \mathrm{d}\tau, \quad R_d^i \simeq \frac{R}{h_i}.$$

Note that w_k can be taken as $w_k := \int\limits_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} G w(\tau) d\tau$ and therefore the expression for Q_d^i above is exact. A similar expression for v_k is hard to find, and therefore the chosen R_d^i is an approximation, but is generally accepted [4] and used [5]. The equations of the observer are given by:

$$\hat{x}_{k+1} = \Phi^i \hat{x}_k + \Gamma^i u_k + L_k \left[y_k - \hat{y}_k \right] \quad (3a)$$

$$\hat{y}_k = C\hat{x}_k \quad (3b)$$

where L_k is a time varying observer gain. One way to determine L_k is to use the time varying Kalman filter: $L_k = (\Phi^i P_k C^\top)(R_d^i + C P_k C^\top)^{-1}$, where the covariance of the state estimation error (which is defined as $e_k = x_k - \hat{x}_k$) $P_k = E[e_k e_k^\top]$ at each time step k is given by the well known Riccati recursion formula:

$$P_k = (\Phi^i - L_{k-1}C)P_{k-1}(\Phi^i - L_{k-1}C)^\top + Q_d^i + L_{k-1} R_d^i L_{k-1}^\top \quad (4)$$

For linear time invariant systems, the state estimation error covariance P_k asymptotically converges towards the steady state value P_∞, and in order to reduce computational burden, the asymptotic gain L_∞ is often used.

In our case it is possible to compute the asymptotic observer gain $L^i = L_\infty^i$, for each of the inter-sampling times h_i, and to use these gains in the observer (3), whenever the data is sampled with corresponding sampling rate h_i. Such an approach may not yield a stable observer.

Example 2.1 Consider the stable continuous time system (1) with:

$$A = \begin{bmatrix} 0 & 1 \\ -1000 & -0.1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 1]$$

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \quad R = 0.5$$

Measurements are taken with two different sampling rates $h_1 = 0.004$s and $h_2 = 0.08$s. We obtain the following two asymptotic gains for the system discretized at sampling rates $h_{1,2}$:

$$L^1 = [-0.0010 \; 0.1156]^\top; \quad L^2 = [0.0075 \; -0.8877]^\top \quad (5)$$

Observers for each individual sampling rate $h_{1,2}$ are stable with the corresponding gain $L^{1,2}$, but if the sampling sequence $(h_1, h_1, h_1, h_1, h_2, h_1, h_1, h_1, h_1, h_2, \dots)$ occurs the (noise-free) estimation error dynamics $e_{k+1} = (\Phi^{i_k} - L^{i_k}C)e_k$ is not stable as the spectral radius $\rho((\Phi^1 - L^1C)^4(\Phi^2 - L^2C)) > 1$. A simulation is shown on figure 1, where w and v are set to zero. It is interesting to note that the estimation error of the time varying Kalman filter, for the same sampling sequence readily converges to 0. The price to be paid for such performance is the increased computational effort, which is not always desirable.

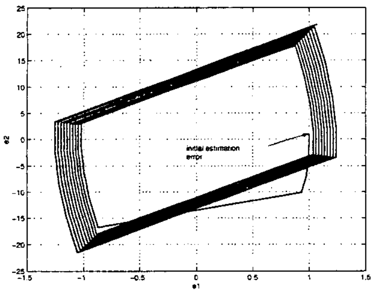

Figure 1: Phase portrait of the of the estimation error: divergent sequence

3 Main results

We will give now conditions under which the observer is stable for any variation among the prescribed sampling rates by using *static* observer gains that only depend on the last sample time.

Theorem 3.1 *Assume that*[1] $Q_d^i + L^i R_d^i {L^i}^\top$ *is positive definite for all* $i = 1, 2, \dots, N$. *If there exist* $P = P^\top > 0$ *and* L^i, $i \in \{1, 2, \dots, N\}$ *such that:*

$$(\Phi^i - L^i C)P(\Phi^i - L^i C)^\top - P + (Q_d^i + L^i R_d^i {L^i}^\top) \leq 0 \textit{ then:} \quad (6)$$

1. *In the noise free case* ($w = v = 0$) *the estimation error dynamics* $e_{k+1} = (\Phi^{i_k} - L^{i_k} C)e_k$ *is asymptotically stable under all sequences of admissible inter-sample times taken from* $\{h_1, h_2, \dots, h_N\}$.
2. *The covariance of the estimation error* $P_k = E[e_k e_k^\top]$ *is asymptotically bounded by* P, *i.e.* $\forall \varepsilon > 0 \ \exists k_0 \forall k \geq k_0 \quad P_k < P + \varepsilon I$. *Moreover, if* $P_{k_0} \leq P$ *for some* k_0 *then* $P_k \leq P$ *for all* $k > k_0$.

Proof. As the term $(Q_d^i + L^i R_d^i {L^i}^\top)$ is positive definite, we have $(\Phi^i - L^i C)P(\Phi^i - L^i C)^\top - P < 0$. By using Schur complements twice we obtain $(\Phi^i - L^i C)^\top P^{-1}(\Phi^i - L^i C) - P^{-1} < 0$, which implies that for $w = v = 0$ the function $V(e) = e^\top P^{-1} e$ can play the role of a common Lyapunov function for all possible linear error dynamics as $V(e_{k+1}) - V(e_k) < 0$. Hence, the estimation error dynamics is guaranteed to be asymptotically stable, under any sampling sequence [1, 2].
To prove the second part, combining (4) with (6) we derive:

$$P_k - P \leq (\Phi^i - L^i C)(P_{k-1} - P)(\Phi^i - L^i C)^\top \quad (7)$$

Note that if $P_{k-1} \leq P$, then the expression on the right hand side is negative definite, and it follows that $P_k \leq P$ as well. This proves the second statement in point 2 of the theorem. If we iterate (7) we get:

$$P_k - P \leq \left(\prod_{j=1}^{k} (\Phi^{i_j} - L^{i_j} C)\right)(P_0 - P)\left(\prod_{j=1}^{k} (\Phi^{i_j} - L^{i_j} C)^\top\right), \quad (8)$$

where P_0 is the error covariance at time $k = 0$. Since there exists a common quadratic Lyapunov function for all the dynamics $(\Phi^i - L^i C)$, namely V, we have that $\lim_{k \to \infty} \prod_{j=1}^{k} (\Phi^{i_j} - L^{i_j} C) = 0$, independently of the values i_j. Hence, the claim of the theorem follows.

Equation (6) can be cast into a linear matrix inequality form under the conditions $R_d^i > 0$ and $Q_d^i > 0$ for all i (this is the case when $R > 0$, and (A, GS) controllable where $Q = SS^\top$). Consequently, (6) can be efficiently solved for the unknown matrix variables. Also, a convex optimization problem can be formulated that minimizes P, in a certain sense, for example min $\lambda_{max}(P)$, where $\lambda_{max}(\cdot)$ is the maximal eigenvalue, or min log det P. This transformation is quite standard and can be found in [1, 3].

[1] Note that $Q_d^i + L^i R_d^i {L^i}^\top$ is already positive semi-definite for all i. Sufficient conditions for strictness are, for instance, (A, GS) controllable where $Q = SS^\top$.

Example 3.2 We consider the setup presented in example 2.1, and we will apply the presented design procedure, in order to obtain asymptotically stable observer, for any sequence of sampling times $h_{1,2}$. Solving the system of linear matrix inequalities we obtain (we minimize min log det P) $L^1 = [0.0018 \ 0.7199]^\top$, $L^2 = [0.0193 \ -0.9310]^\top$ and, as an asymptotic upper bound of the covariance of the estimation error:

$$P = \begin{bmatrix} 0.1275 & -0.4959 \\ -0.4959 & 289.1227 \end{bmatrix}$$

The behavior of the estimation error is depicted in figure 2.

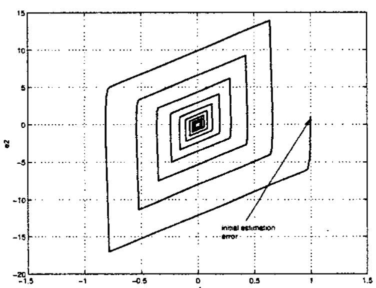

Figure 2: Phase portrait of the of the estimation error: convergent sequence

4 Conclusions

The paper discussed the problem of observer design for a continuous time system, when the output is sampled at non-equidistant time instances, but the time interval between the samples belongs to a prescribed set. An example demonstrated that simply applying asymptotic Kalman filters might yield unstable estimation error dynamics. Therefore, we presented an observer design procedure that is robust against the variations in sampling rate, and, in addition bounds the covariance of the estimation error. These bounds can be minimized via a convex optimization problem.

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