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Tracking Control for Hybrid Systems via Embedding of Known Reference Trajectories

Ricardo G. Sanfelice, J. J. Benjamin Biemond, Nathan van de Wouw, and W. P. Maurice H. Heemels

Abstract— **We study the problem of designing controllers to track time-varying state trajectories for plants modeled as hybrid dynamical systems, which are systems with both continuous and discrete dynamics. The reference trajectories are given by functions that may exhibit jumps. The class of controllers considered are also modeled as hybrid systems. These are designed to guarantee stability of tracking and that the difference between the plant's state and the reference trajectory converges to zero. Using recently developed tools for the study of asymptotic stability in hybrid systems, we recast the tracking problem as the problem of stabilizing a closed set and derive conditions for the design of tracking controllers for hybrid reference trajectories with the property that the jump times of the plant coincide with those of the given reference trajectories. The approach is illustrated in examples.**

I. Introduction

The literature on stability analysis and stabilization of equilibria for systems with state jumps is relatively well developed. On the other hand, in many control problems, such as tracking, output regulation, synchronization, and observer design, the goal consists of stabilizing time-varying trajectories. To effectively tackle such problems for hybrid systems, results on the stability and stabilization of time-varying trajectories of such systems are imperative. Unfortunately, general results for stabilizing impulsive/discontinuous, or, more generally, hybrid trajectories are not currently available. Notable specific solutions to stabilization of such trajectories are given by the work in [11], [10], [4], in which the state estimation and tracking problems for particular classes of mechanical systems with impacts are addressed, the work in [7], in which the observer design problem for a class of complementarity systems is studied, the work in [9], in which a tracking control for a class of measure differential inclusions is solved, the work in [1], [2], [3], in which a tracking problem for a class of mechanical systems with unilateral constraints is addressed, and the work in [16] considering the juggling problem as a tracking problem.

In this paper, we present sufficient conditions characterizing controllers solving a state tracking control problem. We consider plants given in terms of a constrained flow equation

$$\dot{\xi} = f_p(\xi, u) \qquad (\xi, u) \in C_p \tag{1}$$

R. G. Sanfelice is with the Department of Aerospace and Mechanical Engineering, University of Arizona, 1130 N. Mountain Ave, AZ 85721, USA. Email: sricardo@u.arizona.edu

J. J. B. Biemond, Nathan van de Wouw, and W. P. M. H. Heemels are with the Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. Email: j.j.b.biemond,n.v.d.wouw,m.heemels@tue.nl. J. J. B. Biemond and Nathan van de Wouw are supported by the Netherlands organisation for scientific research (NWO).

and a constrained jump inclusion

$$\xi^+ \in G_p(\xi, u) \qquad (\xi, u) \in D_p, \tag{2}$$

with output $y = \xi$. For this class of hybrid systems, a controller assigning the input u and measuring ξ is to be designed so that the difference between ξ and the reference trajectory r, which may both flow and jump, is well behaved. Without being precise about a notion of tracking, it should be expected that the tracking controller guarantees both stability and attractivity properties relative to the reference trajectory. The former consists of the property that solutions to the plant starting close to the reference stay close to it while the latter consists of the property that the distance between the plant's solution component and the reference decreases asymptotically. A challenge in guaranteeing these properties for hybrid systems is discussed in Section II. The proposed approach in this note consists of recasting a state tracking problem for hybrid systems, which is defined in Section IV, as the stabilization of a closed set that embeds the reference trajectory. Exploiting sufficient conditions for asymptotic stability of closed sets for hybrid systems, in Section V we present sufficient conditions for a class of hybrid tracking controllers enforcing that the jump times of the plant to coincide with those of the given reference trajectory. The approach is illustrated in examples in Section VI.

II. An Obstacle to Tracking Control Design for Hybrid Systems

Consider a scalar, single-valued hybrid plant as in (1)-(2) with $y = \xi$ and the reference trajectory to be tracked given by the sawtooth signal shown in Figure 1, which has discontinuities when reaching 1. Trajectories ξ to the plant can be defined as functions defined on hybrid time domains $\operatorname{dom}\xi$, which are subsets of $\mathbb{R}_{\geq 0} \times \mathbb{N} := [0, +\infty) \times \{0, 1, 2, \ldots\}$ and parameterize the trajectories by flow time t and jump time j [6]; see Section III for more details. A typical approach used in tracking control of continuous-time and discrete-time plants consist of defining the tracking error and then analyzing the resulting time-varying error dynamics. Following this approach, the reference trajectory r on the hybrid time domain $\operatorname{dom} r$ is given by

$$r(t, j) \quad = \quad t - t_j^r \qquad \forall t \in [t_j^r, t_{j+1}^r], \tag{3}$$

where $t_j^r = j$, $j \in \mathbb{N}$. Note that $r(t, j) \in [0, 1]$ for all $(t, j) \in \operatorname{dom} r$, where $\operatorname{dom} r$ is the union of $[t_j^r, t_{j+1}^r] \times \{j\}$ for every $j \in \mathbb{N}$. Let

$$\mathcal{T}_r := \bigcup_{j \in \mathbb{N}_{>0}} (t_j^r, j - 1) \tag{4}$$

denote the fixed values of (t, j) at which r jumps. Then, the dynamics of the tracking error

$$e := y - r(t,j) = \xi - r(t,j)$$

are given by the flow equation

$$\dot{e} = f_p(e + r(t,j), u) - 1 \tag{5}$$

when

$$(e + r(t,j), u) \in C_p \qquad \text{and} \qquad t \in [t_j^r, t_{j+1}^r], \tag{6}$$

and by the jump equation

$$e^+ = G_e(e + r(t,j), u, t, j) \tag{7}$$

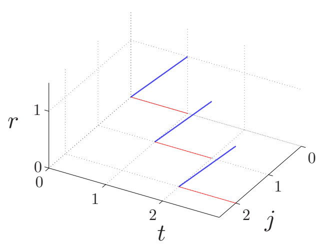
when

$$(e + r(t,j), u) \in D_p \qquad \text{or} \qquad (t,j) \in \mathcal{T}_r, \tag{8}$$

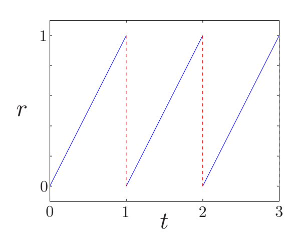
where G_e is defined at every point satisfying (8) as

$$G_e = \begin{cases} G_p(e+r,u) - r & (e+r,u) \in D_p, (t,j) \notin \mathcal{T}_r \\ e + r & (e+r,u) \notin D_p, (t,j) \in \mathcal{T}_r \\ G_p(e+r,u) & (e+r,u) \in D_p, (t,j) \in \mathcal{T}_r. \end{cases}$$

(For notational convenience, we removed the arguments of some of the functions.)



(a) Hybrid arc r.



(b) Projection onto $\mathbb{R}_{\geq 0}$.

Fig. 1. Reference trajectory for the tracking control problem in Section II.

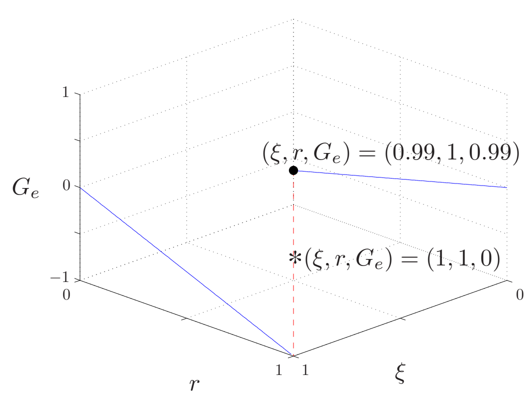


Fig. 2. A resulting jump map G_e for the error system in the tracking control problem of Section II when $G_p(\xi, u) = 0$ and $D_p = \{(\xi, u) \,:\, \xi = 1\}$. The map G_e is defined for each $(\xi, r) \in (\mathbb{R} \times \{1\}) \times (\{1\} \times [0, 1])$.

First, note that, in particular, the constraints (6) and (8) cannot be written in terms of the tracking error solely. Now, suppose that a feedback law $u = \kappa_c(y, r)$ is designed to map the error to zero when both the plant's state ξ and r jump simultaneously, that is, the third case in the definition of G_e is zero. It is possible that, from points ξ in C_p that are nearby D_p and times $(t,j) \in \mathcal{T}_r$, G_e updates e to $|e^+| = e+1$, which is far from zero. In fact, Figure 1 depicts a particular map G_e as a function of ξ and $r(t,j)$ when the jumps of the plant occur when $\xi = 1$, that is, $D_p := \{(\xi, u) \,:\, \xi = 1\}$, and with $G_p(\xi, u) = \xi + u$, $\kappa_c(y, r) = -\xi$. Since, for the given reference, condition $(t,j) \in \mathcal{T}_r$ is equivalent to $r(t,j) = 1$, the jump map G_e is written as a function of (ξ, r) only and is defined at every point satisfying (8), which is the set of points (ξ, r) in $(\mathbb{R} \times \{1\}) \times (\{1\} \times [0,1])$. Note that when $(t,j) \in \mathcal{T}_r$ (equivalently, $r(t,j) = 1$) if $\xi = 1$ then $e^+ = 0$ but if ξ is slightly below 1, then $|e^+|$ will be close to 1 after the jump. This "peaking phenomenon," which is due to the jump instants of plant and reference not coinciding, has also been recognized in [8], [11], [3] and imposes a difficulty in guaranteeing that the norm of e converges to zero.

We consider tracking controllers that avoid the issue of an increasing error signal by ensuring that jumps of the plant occur at the same instant as the jumps of the reference trajectories. For the illustrative example above, a controller designed with the said approach will assign u so that the jumps of the plant and the reference trajectory occur jointly. For this purpose, we recast the tracking control problem as the stabilization of a closed set which embeds the time-varying reference trajectory. For the design of the tracking controllers we exploit sufficient conditions for asymptotic stability of hybrid systems in [5] (see also [14] and [15]). An alternative approach based on generating the reference trajectories from an exosystem was proposed in [13].

III. Preliminaries

Below, given a set S, $\overline{S}$ denotes its closure; given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm; given a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, and $|x|_S := \inf_{y \in S} |x - y|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if is continuous, zero at zero, and strictly increasing and to belong to class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if it belongs to class-$\mathcal{K}$ and is unbounded. $\mathcal{PD}$ denotes the set of real-valued positive definite functions.

A hybrid system $\mathcal{H}$ with state x, input u, and output y is modeled as

$$\mathcal{H} \begin{cases} \dot{x} &= f(x,u) & (x,u) \in C \\ x^+ &\in G(x,u) & (x,u) \in D \\ y &= h(x), \end{cases} \tag{9}$$

where $\mathbb{R}^n$ is the space for the state x, $\mathcal{U} \subset \mathbb{R}^m$ is the space for inputs u, the set $C \subset \mathbb{R}^n \times \mathcal{U}$ is the *flow set*, the function $f : C \to \mathbb{R}^n$ is the *flow map*, the set $D \subset \mathbb{R}^n \times \mathcal{U}$ is the *jump set*, $G : D \rightrightarrows \mathbb{R}^n$ is the *jump map*, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is the *output map*. The data of the hybrid system $\mathcal{H}$ is given by (C, f, D, G, h). Solutions to hybrid systems $\mathcal{H}$ are defined by hybrid arcs on hybrid time domains, which are functions defined on subsets of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ given by the union of intervals of the form $[t_j, t_{j+1}] \times \{j\}$, $t_j \leq t_{j+1}$; see [6] for more details.

We define stability and Lyapunov functions for closed hybrid systems (no inputs and outputs) given by

$$\mathcal{H}\left\{\begin{array}{lcll} \dot{x} & = & f(x) & x\in C \\ x^{+} & \in & G(x) & x\in D. \end{array}\right. \quad (10)$$

The following definition introduces stability for general sets of the state space. Given $\phi(0,0)\in\mathbb{R}^n$, $\mathcal{S}_{\mathcal{H}}(\phi(0,0))$ denotes the set of maximal solutions ϕ to $\mathcal{H}$ with $\phi(0,0)$.

Definition 3.1 (stability): A set $\mathcal{A}\subset\mathbb{R}^n$ is said to be

- *uniformly globally stable* (UGS) if there exists $\alpha\in\mathcal{K}_\infty$ such that each solution $\phi\in\mathcal{S}_{\mathcal{H}}(\phi(0,0))$ satisfies $|\phi(t,j)|_{\mathcal{A}}\le\alpha(|\phi(0,0)|_{\mathcal{A}})$ for all $(t,j)\in\operatorname{dom}\phi$;
- *uniformly globally attractive* (UGA) if for each $\varepsilon>0$ and $\lambda>0$ there exists $T>0$ such that, for any solution $\phi\in\mathcal{S}_{\mathcal{H}}(\phi(0,0))$ with $|\phi(0,0)|_{\mathcal{A}}\le\lambda$, $(t,j)\in\operatorname{dom}\phi$ and $t+j\ge T$ imply $|\phi(t,j)|_{\mathcal{A}}\le\varepsilon$;
- *uniformly globally asymptotically stable* (UGAS) if it is both uniformly globally stable and uniformly globally attractive.

Definition 3.2 (Lyapunov function candidate): A function $V:\operatorname{dom}V\to\mathbb{R}$ is said to be a *Lyapunov function candidate* for the hybrid system $\mathcal{H}=(C,f,D,G)$ with respect to the closed set $\mathcal{A}$ if the following conditions hold:

1) $\overline{C}\cup D\cup G(D)\subset\operatorname{dom}V$,
2) V is continuously differentiable on an open set containing $\overline{C}$.

The following result for asymptotic stability of closed sets will be employed in the design of hybrid controllers for tracking. It is a Lyapunov stability theorem for hybrid systems.

Theorem 3.3: (Lyapunov theorem [5]) Let $\mathcal{H}=(C,f,D,G)$ be a hybrid system and let $\mathcal{A}\subset\mathbb{R}^n$ be closed. If V is a Lyapunov function candidate for $\mathcal{H}$ with respect to $\mathcal{A}$ and there exist $\alpha_1,\alpha_2\in\mathcal{K}_\infty$, and a positive definite and continuous function ρ such that

$$\alpha_1(|x|_{\mathcal{A}})\le V(x)\le\alpha_2(|x|_{\mathcal{A}}) \qquad \forall x\in C\cup D\cup G(D), \quad (11a)$$

$$\langle\nabla V(x),f(x)\rangle\le-\rho\left(|x|_{\mathcal{A}}\right) \qquad \forall x\in C, \quad (11b)$$

$$V(g)-V(x)\le-\rho\left(|x|_{\mathcal{A}}\right) \qquad \forall x\in D,\ g\in G(x), \quad (11c)$$

then $\mathcal{A}$ is uniformly globally asymptotically stable for $\mathcal{H}$.

The following result introduces relaxed Lyapunov conditions.

Corollary 3.4: (relaxed Lyapunov conditions [5]) Let $\mathcal{H}=(C,f,D,G)$ be a hybrid system and let $\mathcal{A}\subset\mathbb{R}^n$ be closed. Suppose that V is a Lyapunov function candidate for $\mathcal{H}$ with respect to $\mathcal{A}$ and there exist $\alpha_1,\alpha_2\in\mathcal{K}_\infty$, and a continuous $\rho\in\mathcal{PD}$ such that (11a) *and either A) or B) below holds:*

A) *Condition* (11c) *holds,*

$$\langle\nabla V(x),f(x)\rangle \quad\le\quad 0 \quad \forall x\in C, \quad (12)$$

and, for each $\lambda>0$, there exist $\gamma_\lambda\in\mathcal{K}_\infty$, $N_\lambda\ge0$ such that for every solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}}\in(0,\lambda]$ we have that $(t,j)\in\operatorname{dom}\phi$, $t+j\ge T$ imply $j\ge\gamma_\lambda(T)-N_\lambda$;

B) *Condition* (11b) *holds,*

$$V(g)-V(x) \quad\le\quad 0 \quad \forall x\in D,\ g\in G(x)\ , \quad (13)$$

and, for each $\lambda>0$, there exist $\gamma_\lambda\in\mathcal{K}_\infty$, $N_\lambda\ge0$ such that for every solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}}\in(0,\lambda]$ we have that $(t,j)\in\operatorname{dom}\phi$, $t+j\ge T$ imply $t\ge\gamma_\lambda(T)-N_\lambda$;

then $\mathcal{A}$ is uniformly globally asymptotically stable.

This corollary states that uniform attractivity can be asserted as long as the Lyapunov function decreases, along solutions, over sufficiently long hybrid time intervals. More precisely, A) is about the Lyapunov function being nonincreasing during flows but strictly decreasing during jumps and the jumps occur frequently enough while B) is about the Lyapunov function being nonincreasing during jumps but strictly decreasing during flows and the flows occur for long enough.

IV. Problem Statement

We consider plants $\mathcal{H}_p$ modeled as hybrid systems $\mathcal{H}$ with state $\xi\in\mathbb{R}^{n_p}$, input $u\in\mathbb{R}^{m_p}$, and output $y=\xi$ given by

$$\mathcal{H}_p\left\{\begin{array}{lcll} \dot{\xi} & = & f_p(\xi,u) & (\xi,u)\in C_p \\ \xi^{+} & \in & G_p(\xi,u) & (\xi,u)\in D_p \end{array}\right. \quad (14)$$

with data (C_p,f_p,D_p,G_p). We consider hybrid arcs $r:\operatorname{dom}r\to\mathbb{R}^{n_p}$ defining reference trajectories to be tracked. The following class of tracking hybrid controllers with state $\eta\in\mathbb{R}^{n_c}$ and data $(C_c,f_c,D_c,G_c,\kappa_c)$ is considered:

$$\mathcal{H}_c\left\{\begin{array}{lcll} \dot{\eta} & = & f_c(\eta,y,r) & (\eta,y,r)\in C_c \\ \eta^{+} & \in & G_c(\eta,y,r) & (\eta,y,r)\in D_c \\ u & = & \kappa_c(\eta,y,r). & \end{array}\right. \quad (15)$$

The input of $\mathcal{H}_c$ has been assigned to (y,r) while its output u to the input of the plant $\mathcal{H}_p$. The closed-loop system (14)-(15) resulting from the interconnection of $\mathcal{H}_p$ and $\mathcal{H}_c$ is denoted $\mathcal{H}_{cl}$, has state

$$x:=(\xi,\eta)\in\mathbb{R}^{n_p}\times\mathbb{R}^{n_c}$$

and is given by[1]

$$\begin{array}{l} \left.\begin{array}{lcl} \dot{\xi} & = & f_p(\xi,\kappa_c(\eta,\xi,r)) \\ \dot{\eta} & = & f_c(\eta,\xi,r) \end{array}\right\} \begin{array}{c}(\xi,\kappa_c(\eta,\xi,r))\in C_p \\ \text{and } (\eta,\xi,r)\in C_c\end{array} \\ \left.\begin{array}{lcl} \xi^{+} & \in & G_p(\xi,\kappa_c(\eta,\xi,r)) \\ \eta^{+} & = & \eta \end{array}\right\} (\xi,\kappa_c(\eta,\xi,r))\in D_p \\ \left.\begin{array}{lcl} \xi^{+} & = & \xi \\ \eta^{+} & \in & G_c(\eta,\xi,r) \end{array}\right\} \quad (\eta,\xi,r)\in D_c, \end{array} \quad (16)$$

where, for notational simplicity, we have omitted the argument (t,j) of the time-varying reference r.

Using the above definitions, we state a tracking control problem for hybrid systems.

[1]When the jump conditions $(\xi,\kappa_c(\eta,\xi,r))\in D_p$ and $(\eta,\xi,r)\in D_c$ are satisfied simultaneously, either jump map can be used. This can be captured with a set-valued jump map; see the model in Section V-A.

Tracking Control Problem ($\star$)**:** Given a plant $\mathcal{H}_p$ and a complete reference trajectory r design the data $(C_c, f_c, D_c, G_c, \kappa_c)$ of the controller $\mathcal{H}_c$ so that the set of points ξ satisfying

$$\xi = r(t,j) \tag{17}$$

is uniformly globally asymptotically stable.[2]

Problem ($\star$) asks for a controller such that the set of points (17) has the UGS and UGA properties (see Definition 3.1) for the closed-loop system. The attractivity property implies that complete solutions to $\mathcal{H}_{cl}$ satisfy

$$\lim_{t+j\to\infty} |\xi(t,j) - r(t,j)| = 0.$$

Moreover, the stability property implies that solutions to the plant with initial conditions $\xi(0,0) = r(0,0)$, if they exist, satisfy

$$\xi(t,j) = r(t,j) \qquad \text{for all } (t,j) \in \operatorname{dom}\xi.$$

Note that unless further conditions are imposed on r, the set in (17) is time varying and not compact. Furthermore, boundedness of the state of the controller is not guaranteed by UGAS of (17) and has to be established separately.

V. A Class of Hybrid Controllers for State Tracking with Known Reference Trajectories

A. Main Approach

In smooth systems, a well-known approach is to introduce the coordinate transformation $e = \xi - r$ and then analyze the resulting system. This approach is used for systems with time-triggered state jumps in [12]. However, in general, the flow and jump sets as well as the flow and jump maps of the error dynamics become time dependent. To avoid this issue, we recast Problem ($\star$), which pertains to the stabilization of a time-varying set, as the stabilization of a closed, not necessarily bounded, time-invariant set. To this end, given a reference $r : \operatorname{dom} r \to \mathbb{R}^{n_p}$, following (4), we define the set $\mathcal{T}_r$ collecting all of the points (t,j) in the domain of r at which r jumps, that is, every point $(t_j^r, j) \in \operatorname{dom} r$ for which $(t_j^r, j+1) \in \operatorname{dom} r$. Auxiliary variables $\tau \in \mathbb{R}_{\geq 0}$ and $k \in \mathbb{N}$ are incorporated as states to parametrize a given reference trajectory r. That is, τ evolves continuously according to the flow time parameter t, while k evolves discretely according to the jump time parameter j at jumps of r. In this setting, the set to be stabilized is given by

$$\mathcal{A} = \{(x,\tau,k) \ : \ \xi = r(\tau,k)\}, \tag{18}$$

which is a subset of $\mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}_{\geq 0} \times \mathbb{N}$. For instance, for the example of Section II, the set to be stabilized with the proposed approach is given by

$$\left\{(x,\tau,k) \ : \ \xi + t_k^r = \tau \in [t_k^r, t_{k+1}^r], (t_k^r, k) \in (0,0) \cup \mathcal{T}_r \ \right\},$$

where $\mathcal{T}_r$ is given in (4). This set is closed and unbounded in the τ and k components.

[2]The definition of UGAS for a time-varying hybrid system follows Definition 3.1; see [14].

The next ingredient of the approach is to guarantee, by design of the controller, that the jumps of the plant and of the reference trajectory occur simultaneously. This will be a constraint in the design of the controller, which, while it restricts the type of systems for which the tracking problem can be solved, it allows for a solution to certain tracking problems as Section VI illustrates.

With a controller satisfying such a property, our approach is to recast the problem under study as the stabilization of the set $\mathcal{A}$ for the resulting closed-loop system

$$\left.\begin{aligned} \dot\xi &= f_p(\xi, \kappa_c(\eta,\xi,r(\tau,k))) \\ \dot\eta &= f_c(\eta,\xi,r(\tau,k)) \\ \dot\tau &= 1 \\ \dot k &= 0 \end{aligned}\right\} \begin{array}{c} (\xi,\kappa_c(\eta,\xi,r(\tau,k))) \in C_p \\ \text{and } (\eta,\xi,r(\tau,k)) \in C_c \\ \text{and } \tau \in [t_k^r, t_{k+1}^r], k \in \mathbb{N} \end{array}$$

$$\left.\begin{aligned} \xi^+ &\in G_p(\xi,\kappa_c(\eta,\xi,r(\tau,k))) \\ \eta^+ &= \eta, \ \tau^+ = \tau \\ k^+ &= k+1 \end{aligned}\right\} \begin{array}{c} (\xi,\kappa_c(\eta,\xi,r(\tau,k))) \in D_p \\ \text{and } (\tau,k) \in \mathcal{T}_r \end{array}$$

$$\left.\begin{aligned} \xi^+ &= \xi \\ \eta^+ &\in G_c(\eta,\xi,r(\tau,k)) \\ \tau^+ &= \tau, \ k^+ = k \end{aligned}\right\} \quad (\eta,\xi,r(\tau,k)) \in D_c. \tag{19}$$

The resulting closed-loop system, denoted $\mathcal{H}_{cl}^\star$, can be modeled by data (C, f, D, G) given by

$$\begin{aligned} C \ &:= \ \{(x,\tau,k) \ : \ (\xi,\kappa_c(\eta,\xi,r(\tau,k))) \in C_p, \\ &\qquad \tau \in [t_k^r, t_{k+1}^r], k \in \mathbb{N}, (\eta,\xi,r(\tau,k)) \in C_c\}, \end{aligned}$$

$$f(x,\tau,k) := \begin{bmatrix} f_p(\xi,\kappa_c(\eta,\xi,r(\tau,k))) \\ f_c(\eta,\xi,r(\tau,k)) \\ 1 \\ 0 \end{bmatrix},$$

$$\begin{aligned} D \ &:= \ D_1 \cup D_2 \\ D_1 \ &:= \ \{(x,\tau,k) \ : \ (\xi,\kappa_c(\eta,\xi,r(\tau,k))) \in D_p, \\ &\qquad (\tau,k) \in \mathcal{T}_r\} \\ D_2 \ &:= \ \{(x,\tau,k) \ : \ (\eta,\xi,r(\tau,k)) \in D_c\}, \end{aligned}$$

$$G(x,\tau,k) := \begin{cases} G_1(x,\tau,k) := \begin{bmatrix} G_p(\xi,\kappa_c(\eta,\xi,r(\tau,k))) \\ \eta \\ \tau \\ k+1 \end{bmatrix} & (x,\tau,k) \in D_1 \setminus D_2, \\ G_2(x,\tau,k) := \begin{bmatrix} \xi \\ G_c(\eta,\xi,r(\tau,k)) \\ \tau \\ k \end{bmatrix} & (x,\tau,k) \in D_2 \setminus D_1, \\ \{G_1(x,\tau,k), G_2(x,\tau,k)\} & (x,\tau,k) \in D_1 \cap D_2. \end{cases}$$

Then, asymptotic stability of $\mathcal{A}$ can be asserted using the sufficient conditions provided by Theorem 3.3 and Corollary 3.4.

B. Characterization of Hybrid Controllers

The data $(C_c, f_c, D_c, G_c, \kappa_c)$ is designed so that:

- There exist a Lyapunov function candidate $V : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \to \mathbb{R}$ for $\mathcal{H}_{cl}^\star$ with respect to $\mathcal{A}$, functions

$\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a continuous $\rho \in \mathcal{PD}$ such that

$$\alpha_1(|(x,\tau,k)|_{\mathcal{A}}) \leq V(x,\tau,k) \leq \alpha_2(|(x,\tau,k)|_{\mathcal{A}}) \quad \forall (x,\tau,k) \in C \cup D \cup G(D), \tag{20}$$

$$\langle \nabla V(x,\tau,k), f(x,\tau,k)\rangle \leq -\rho\left(|(x,\tau,k)|_{\mathcal{A}}\right) \quad \forall (x,\tau,k) \in C, \tag{21}$$

$$V(g) - V(x,\tau,k) \leq -\rho\left(|(x,\tau,k)|_{\mathcal{A}}\right) \quad \forall (x,\tau,k) \in D_1 \setminus D_2,\ g \in G_1(x,\tau,k), \tag{22}$$

$$V(g) - V(x,\tau,k) \leq -\rho\left(|(x,\tau,k)|_{\mathcal{A}}\right) \quad \forall (x,\tau,k) \in D_2 \setminus D_1,\ g \in G_2(x,\tau,k), \tag{23}$$

$$V(g) - V(x,\tau,k) \leq -\rho\left(|(x,\tau,k)|_{\mathcal{A}}\right) \quad \forall (x,\tau,k) \in D_1 \cap D_2,\ g \in \{G_1(x,\tau,k), G_2(x,\tau,k)\}. \tag{24}$$

Remark 5.1: The conditions above imply that complete solutions to the closed-loop system are such that $|(x,\tau,k)(t,j)|_{\mathcal{A}} \to 0$ as $t+j \to \infty$, that is,

$$|\xi(t,j) - r(\tau(t,j), k(t,j))| \to 0 \qquad \text{as } t+j \to \infty.$$

This includes all possible solutions with unconstrained initial conditions of τ and k, in particular, $\tau(0,0) = k(0,0) = 0$, for which $r(\tau(t,j), k(t,j)) = r(t,j)$ and, consequently,

$$|\xi(t,j) - r(t,j)| \to 0 \qquad \text{as } t+j \to \infty.$$

Note that complete solutions to $\mathcal{H}_{cl}$ have the property that $\tau(t,j) + k(t,j)$ is unbounded as $t+j \to \infty$. Furthermore, it implies that $\xi(t,j) = r(t,j)$ on the domain of definition of solutions starting from $\xi(0,0) = r(0,0), \tau(0,0) = k(0,0) = 0$, when solutions from such points exist. While the conditions above could have been expressed in terms of the tracking error e, as illustrated in Section II it is rarely the case that its dynamics can be written as a function of e and η only.

The following result summarizes the discussion above on characterization of tracking controllers.

Theorem 5.2: Given a complete reference trajectory $r : \operatorname{dom} r \to \mathbb{R}^{n_p}$ and associated closed set $\mathcal{A}$, if there exists a hybrid controller $\mathcal{H}_c$ guaranteeing that the jumps of r and $\mathcal{H}_p$ occur simultaneously and there exist a Lyapunov function candidate $V : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \to \mathbb{R}$ for $\mathcal{H}_{cl}^\star$ with respect to $\mathcal{A}$ as in (18)*, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a positive definite and continuous function ρ such that* (20)-(24) *hold, then $\mathcal{H}_c$ provides a solution to Problem ($\star$).*

Remark 5.3: Theorem 5.2 characterizes controllers solving the tracking problem. The conditions in Theorem 5.2 can be relaxed according to items A) and B) of Corollary 3.4. In general, the data of the hybrid controller has to be chosen so that (21)-(24) hold. In particular, condition (21) depends on f_c, C_c and κ_c; (22) depends on κ_c; and (23) depends on G_c and D_c, which are all to be chosen in the design. We foresee that for specific classes of hybrid systems (such as those with linear flow and jump maps), constructive controller design techniques can be developed. The examples in the next section illustrate the feasibility of the design of controllers satisfying the conditions of the theorem.

VI. EXAMPLES

Example 6.1 (Tracking a square wave signal): Consider the scalar hybrid plant $\mathcal{H}_p$

$$\dot{\xi} = -a\xi + u_1 \qquad \xi\, u_1 \geq 0, |\xi| > 0, \tag{25}$$

$$\xi^+ = b + u_2 \qquad \xi\, u_1 \leq 0, |\xi| > 0, \tag{26}$$

where[3] $a, b > 0$, and consider the problem of tracking the square wave signal

$$r(t,j) = (-1)^{j+1} \quad t \in [t^r_j, t^r_{j+1}],\ j \in \mathbb{N},\ t^r_j = j.$$

Then, following the approach proposed in Section V, the goal is to solve Problem ($\star$) with $\mathcal{A}$ given by the points such that $\xi = (-1)^{k+1}, \tau \in [t^r_k, t^r_{k+1}], (t^r_k, k) \in (0,0) \cup \mathcal{T}_r$. For this purpose, we consider the static controller

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \kappa_c(\xi, r(\tau,k)) = \begin{bmatrix} a\, r(\tau,k) \\ -b - r(\tau,k) + \lambda(\xi - r(\tau,k)) \end{bmatrix},$$

with $\lambda \in [0,1)$. It follows that, for every $\xi(0,0) < 0$, every jump of r triggers a jump of the plant. In fact, if $\xi(0,0) < 0$, since $u_1 = a\, r(\tau,k)$, we have that $a\xi(0,0)r(0,0) > 0$ and solutions initially flow. Flows of ξ will not trigger a jump since the sign of ξ remains constant. Jumps of the closed-loop system occur only when r changes sign, which is at (t,j)'s in $\mathcal{T}_r$, $\mathcal{T}_r = \{(1,0),(2,1),(3,2),\ldots\}$. Then, the closed-loop system $\mathcal{H} = (C, f, D, G)$ given by

$$\left.\begin{array}{rcl} \dot{\xi} &=& -a(\xi + r(\tau,k)) \\ \dot{\tau} &=& 1, \quad \dot{k} = 0 \end{array}\right\} \begin{array}{l} a\,\xi\, r(\tau,k) \geq 0, |\xi| > 0, \\ \tau \in [t^r_k, t^r_{k+1}], k \in \mathbb{N}, \end{array}$$

$$\left.\begin{array}{rcl} \xi^+ &=& -r(\tau,k) + \lambda(\xi - r(\tau,k)) \\ \tau^+ &=& \tau, \quad k^+ = k+1 \end{array}\right\} \begin{array}{l} a\,\xi\, r(\tau,k) \leq 0, \\ |\xi| > 0, (\tau,k) \in \mathcal{T}_r \end{array}$$

captures all of the solutions to the original system with initial conditions $\xi(0,0) < 0$, $\tau(0,0) = k(0,0) = 0$. To establish asymptotic stability of $\mathcal{A}$, consider the Lyapunov function

$$V(\xi,\tau,k) = \frac{1}{2}(\xi - r(\tau,k))^2,$$

for which condition (20) holds trivially. For each (ξ,τ,k) satisfying $a\,\xi\, r(\tau,k) \geq 0, |\xi| > 0, \tau \in [t^r_k, t^r_{k+1}]$, $k \in \mathbb{N}$

$$\langle \nabla V(\xi,\tau,k), f(\xi,\tau,k)\rangle = -2aV(\xi,\tau,k);$$

and for each (ξ,τ,k) satisfying $a\,\xi\, r(\tau,k) \leq 0, |\xi| > 0, (\tau,k) \in \mathcal{T}_r$ we have

$$V(G(\xi,\tau,k)) - V(\xi,\tau,k) = -(1-\lambda^2)V(\xi,\tau,k).$$

[3]Condition $|\xi| > 0$ removes solutions that only jump at the origin.

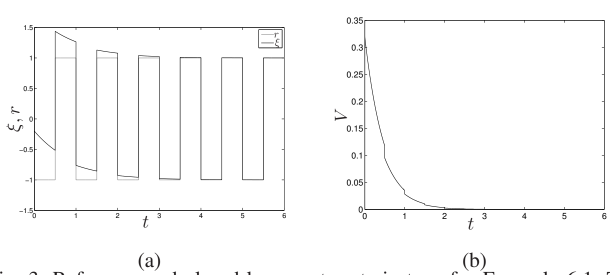


Fig. 3. Reference and closed-loop system trajectory for Example 6.1. The Lyapunov function along the trajectories is also shown. Parameters: $a = b = 1$, $\lambda = 0.9$.

Then, Theorem 5.2 implies uniform global asymptotic stability of $\mathcal{A}$ for the closed-loop system. Figure 3(a) depicts a closed-loop system trajectory converging to the reference asymptotically, both along flows and jumps. Figure 3(b) depicts the Lyapunov function along the trajectory.

Example 6.2 (Tracking for a motion control system): Consider a particle with mass M actuated by a force input u. The position of the particle is denoted by ξ_1 and its velocity by ξ_2. The controller force u contains a Lebesgue integrable part u_1 and an impulsive part u_2 with impulses at instants t_i. The plant is impulsive and modeled as

$$\dot{\xi} = \begin{bmatrix} \xi_2 \\ \frac{u_1}{M} \end{bmatrix} \text{ when } t \neq t_i, \ \xi^+ = \xi + \begin{bmatrix} 0 \\ \frac{u_2}{M} \end{bmatrix} \text{ when } t = t_i,$$

where $M > 0$, the state ξ is completely measured. The input u will be designed, such that the state ξ tracks a reference $r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$, given in Figure 4. The component r_2 jumps at times $(t, j) \in \mathcal{T}_r = \bigcup_{j \in \mathbb{N}} (j + 1, j)$. Such a reference trajectory can be desirable for the position of the end effector of a robot system. A controller that stabilizes the set $\mathcal{A}$ for the given reference trajectory is given by

$$u_1 = -\lambda_1(\xi_1 - r_1) - \lambda_2(\xi_2 - r_2)$$

$$u_2 = \begin{cases} 0, \quad (t,j) \notin \mathcal{T}_r, \\ M, \ (t,j) \in \bigcup_{k \in \mathbb{N}} (4k+3, 4k+2) \cup (4k+4, 4k+3) \\ -M, \ (t,j) \in \bigcup_{k \in \mathbb{N}} (4k+1, 4k) \cup (4k+2, 4k+1), \end{cases}$$

where $\lambda_1, \lambda_2 > 0$. Using the change of coordinates $z = \xi - r(\tau, k)$, the closed-loop system $\mathcal{H} = (C, f, D, G)$ is

$$\left.\dot{z} = \begin{bmatrix} 0 & 1 \\ -\frac{\lambda_1}{M} & -\frac{\lambda_2}{M} \end{bmatrix} z, \ \dot{\tau} = 1, \ \dot{k} = 0 \right\} \begin{array}{c} \tau \in [t_k^r, t_{k+1}^r] \\ k \in \mathbb{N}, \end{array}$$

$$\left. z^+ = z, \ \tau^+ = \tau, \ k^+ = k + 1 \right\} \ (\tau, k) \in \mathcal{T}_r.$$

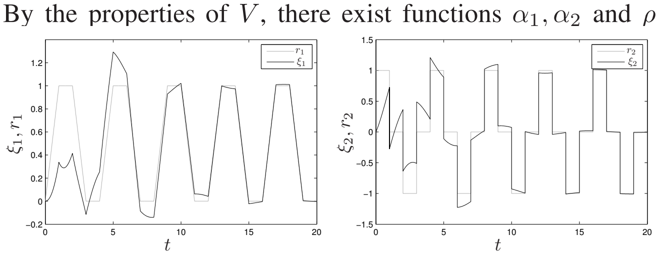
The feedforward signal u_2 assures that z is not affected by the jumps of the reference. Furthermore, if the initial conditions are $\xi(0,0) = r(0,0)$, $\tau(0,0) = k(0,0) = 0$, then the solution satisfies $r(t,j) = \xi(t,j)$ for all $(t,j) \in \operatorname{dom} r$. Take $V(z, \tau, k) = z^\top P z$ with $P = P^\top > 0$ such that

$$\langle \nabla V(z, \tau, k), f(z, \tau, k) \rangle \leq -V(z, \tau, k).$$

Such a matrix P is guaranteed to exist due to the continuous dynamics of z. Since z does not change at jumps, we get

$$V(G(z, \tau, k)) - V(z, \tau, k) = 0 \qquad \forall (z, \tau, k). \tag{27}$$

By the properties of V, there exist functions α_1, α_2 and ρ such that both (20) and (21) are satisfied. Moreover, the hybrid time domain of each solution to the closed-loop system is unbounded in the t direction. Hence, following Remark 5.3, global uniform asymptotic stability of the set $\mathcal{A}$ for the closed-loop system follows using Theorem 5.2 and Corollary 3.4. In Figure 4, a closed-loop trajectory is shown for parameters $M = 1$, $\lambda_1 = 1$ and $\lambda_2 = 0.5$.



(a) (b)

Fig. 4. Reference and closed-loop trajectory for Example 6.2. Parameters: $M = 1$, $\lambda_1 = 1$, and $\lambda_2 = 0.5$.

VII. Conclusion

We state a tracking control problem for tracking of reference signals with jumps. The proposed technique consists of embedding the reference trajectory into a set and then apply Lyapunov stability tools to the closed-loop system. The class of controllers considered have to guarantee the strict property of jump times of the plant coinciding with those of the given reference trajectories. Relaxation of this stringent condition is part of current research.

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