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# On robustness of constrained discrete-time systems to state measurement errors☆

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**Abstract**

In this note we show that robustness with respect to *additive disturbances* implies robustness with respect to state *measurement errors* and *additive disturbances* for a class of discrete-time closed-loop nonlinear systems. The main result is formulated in terms of input-to-state stability and includes the possible presence of input and state constraints. Moreover, the state feedback controllers are allowed to be discontinuous and set-valued and thus the result also applies to model predictive control laws.


*Keywords:* Measurement errors; Robustness; Input-to-state stability; Constrained systems; Discrete-time systems; Nonlinear control systems; Difference inclusions; Discontinuous feedback; Model predictive control

## 1. Introduction

In practice state variables of a control system are always corrupted by *state measurement errors*. State measurement errors can be caused by measurement noise present in sensor readouts or by state estimation (observation) errors caused by the usage of observers. It is therefore important that state feedback controllers are designed such that they are robust to state measurement errors. In this note the notion of input-to-state stability (ISS) (Jiang & Wang, 2001) is used to study robustness of *discrete-time* nonlinear systems subject to state measurement errors. Only a few results on ISS with respect to state measurement errors are available in the literature, especially if state and input constraints have to be taken into account. In Freeman and Kokotović (1993) an ISS result is given for *smooth* state feedback control laws perturbed by state measurement errors in closed-loop with a *continuous-time* nonlinear control system. For *discrete-time* nonlinear systems, robustness results to state measurement errors were obtained in Magni, Nicolao, and Scattolini (1998) and Messina, Tuna, and Teel (2005). The result in Magni et al. (1998) holds under the assumption that the state feedback control law is Lipschitz continuous. Although in Messina et al. (2005) no Lipschitz continuity of the state feedback control law is required, state (and control) constraints on the system dynamics are not incorporated and the unconstrained system is assumed to be continuous in both input and state variables.

This note aims at extending the abovementioned work to the case of nonlinear control systems with *input and state constraints* and possibly discontinuous and/or set-valued state feedbacks including model predictive control (MPC) laws. Indeed, to exploit one of the assets of MPC in being one of the few control strategies to deal in a systematic way with constraints, an extension of Magni et al. (1998) and Messina et al. (2005) towards *constrained* systems is valuable. Moreover, it is important to include set-valuedness as it can occur in MPC due to non-uniqueness of the (sub)optimal control sequence of the MPC optimization problem (cf. Mayne, Rawlings, Rao, & Scokaert, 2000). Also discontinuity has to be accounted for, as it is known that MPC can generate discontinuous feedbacks. Another important reason for allowing for discontinuous

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feedbacks is the existence of nonlinear systems that can be stabilized by discontinuous feedbacks, but not by continuous ones.

In this note it is shown how one can infer ISS with respect to *state measurement errors* and *additive disturbances* for a state feedback in closed-loop with a constrained discrete-time nonlinear system from ISS with respect to *additive disturbances*. The value of this transformation result is emphasized by the fact that design methods that result in closed-loop systems that are input-to-state stable with respect to measurement errors, especially in the field of MPC for constrained nonlinear systems, are rare, while there are various ISS results in the MPC literature on *additive disturbances*, see e.g. Lazar, Heemels, Roset, Nijmeijer, and van den Bosch (2007), Limon, Alamo, and Camacho (2002), Limon, Alamo, Salas, and Camacho (2006), and Magni, Raimondo, and Scattolini (2006). Also in Grimm, Messina, Tuna, and Teel (2003) the authors study robustness of constrained MPC to additive disturbances (in a weaker sense than ISS) and, moreover, they mention the problem of measurement errors. Our main result provides a direct method to transform the mentioned MPC laws into control laws that render the closed-loop system input-to-state stable with respect to *state measurement errors* and *additive disturbances*.

## 2. Notation and system theory notions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integers and the set of non-negative integers, respectively. The composition of two functions $f : \mathbb{X}_1 \to \mathbb{X}_2$ and $g : \mathbb{X}_2 \to \mathbb{X}_3$ is denoted by $f \circ g : \mathbb{X}_1 \to \mathbb{X}_3$ meaning that $(f \circ g)(x) = f(g(x))$ for $x \in \mathbb{X}_1$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\gamma(0) = 0$. If in addition $\gamma(s) \to \infty$ when $s \to \infty$, then we say that $\gamma$ is a $\mathcal{K}_\infty$-function. Note that any $\mathcal{K}_\infty$-function $\gamma$ is invertible and we denote its inverse by $\gamma^{-1}$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{KL}$-function if, for each fixed $k \in \mathbb{R}_+$, the function $\beta(\cdot, k)$ is a $\mathcal{K}$-function, and for each fixed $s \in \mathbb{R}_+$, the function $\beta(s, \cdot)$ is non-increasing and $\beta(s, k) \to 0$ as $k \to \infty$. For any $x \in \mathbb{R}^n$, $|x|$ stands for an arbitrary norm on $\mathbb{R}^n$. For any function $\phi : \mathbb{Z}_+ \to \mathbb{R}^n$, we denote $\|\phi\| = \sup\{|\phi_k| \mid k \in \mathbb{Z}_+\}$, where we use the convention that $\phi_k \triangleq \phi(k)$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\mathrm{int}(\mathcal{S})$ its interior. For two arbitrary sets $\mathcal{S} \subseteq \mathbb{R}^n$ and $\mathcal{P} \subseteq \mathbb{R}^n$, let $\mathcal{S} \sim \mathcal{P} \triangleq \{x \in \mathbb{R}^n \mid x + \mathcal{P} \subseteq \mathcal{S}\}$ denote the *Pontryagin* difference. By the notation $\mathcal{F} : \mathbb{X}_1 \hookrightarrow \mathbb{X}_2$, we mean that $\mathcal{F}$ is a set-valued function from $\mathbb{X}_1$ to $\mathbb{X}_2$, i.e. $\mathcal{F}(x) \subseteq \mathbb{X}_2$ for each $x \in \mathbb{X}_1$.

Consider a non-autonomous system described by the discrete-time nonlinear difference inclusion

$$x_{k+1} \in \mathcal{F}(x_k, v_k), \tag{1}$$

where $x_k \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state and $v_k \in \mathbb{V} \subseteq \mathbb{R}^p$ is the disturbance at discrete-time $k \in \mathbb{Z}_+$. The set $\mathbb{V}$ is assumed to be a known set with $0 \in \mathbb{V}$. Furthermore, $\mathcal{F} : \mathcal{X} \times \mathbb{V} \hookrightarrow \mathcal{X}$, is a set-valued mapping with $\mathcal{F}(0, 0) = \{0\}$ and $\mathcal{F}(\xi, \mu) \neq \emptyset$ for all $\xi \in \mathcal{X}$ and all $\mu \in \mathbb{V}$. Hence, for all $\xi \in \mathcal{X}$ and all $\mu \in \mathbb{V}$ we have that $\emptyset \neq \mathcal{F}(\xi, \mu) \subseteq \mathcal{X}$, which guarantees that for each initial state $x_0 \in \mathcal{X}$ at time 0 and disturbance function $v : \mathbb{Z}_+ \to \mathbb{V}$ there exists a global solution, not necessarily unique, to system (1). The set of corresponding solutions of the difference inclusion (1) is denoted by $\mathcal{S}_{\mathcal{F}}(x_0, v)$. The condition $\mathcal{F}(\xi, \mu) \subseteq \mathcal{X}$ for all $\xi \in \mathcal{X}$ and all $\mu \in \mathbb{V}$ is related to robust positive invariance.

**Definition 2.1.** Given a disturbance set $\mathbb{V}$, we call a set $\mathcal{P} \subseteq \mathbb{R}^n$ *robustly positively invariant* (*RPI*) for system (1) if for all $\xi \in \mathcal{P}$ and all $\mu \in \mathbb{V}$ it holds that $\mathcal{F}(\xi, \mu) \subseteq \mathcal{P}$.

**Definition 2.2.** For given sets $\widehat{\mathcal{X}} \subseteq \mathcal{X}$ and $\mathbb{V} \subseteq \mathbb{R}^p$, with $0 \in \mathrm{int}(\widehat{\mathcal{X}})$, we call system (1) *input-to-state stable* in $\widehat{\mathcal{X}}$ with respect to disturbances in $\mathbb{V}$, if there exist a $\mathcal{KL}$*-function* $\beta$ and a $\mathcal{K}$*-function* $\gamma$ such that for each function $v : \mathbb{Z}_+ \to \mathbb{V}$ and each $x_0 \in \widehat{\mathcal{X}}$ all solutions $x \in \mathcal{S}_{\mathcal{F}}(x_0, v)$ satisfy

$$|x_k| \leqslant \beta(|x_0|, k) + \gamma(\|v\|), \quad \forall k \in \mathbb{Z}_+. \tag{2}$$

Before presenting the main result, two remarks are in order. First, as also mentioned in Jiang and Wang (2001), by causality of (1), the same definition of ISS would result if one would replace (2) by

$$|x_k| \leqslant \beta(|x_0|, k) + \gamma(\|v_{[k-1]}\|), \quad \forall k \in \mathbb{Z}_+, \tag{3}$$

where $v_{[k-1]} : \mathbb{Z}_+ \to \mathbb{V}$ denotes the truncation of $v$ at $k-1$, i.e. $v_{[k-1],j} = v_j$ when $j \leqslant k-1$ and $v_{[k-1],j} = 0$ when $j \geqslant k$. Secondly, if for all $\xi \in \mathcal{X}$ and all $\mu \in \mathbb{V}$ the set $\mathcal{F}(\xi, \mu)$ consists of exactly one element, the inclusion (1) becomes a standard difference *equation*.

## 3. Main result

Consider the constrained closed-loop system

$$x_{k+1} = f(x_k, u_k) \quad \text{with } u_k \in \kappa(x_k), \tag{4}$$

where $x_k \in \mathbb{X} \subseteq \mathbb{R}^n$ and $u_k \in \mathbb{U} \subseteq \mathbb{R}^m$ are the state and the control input, respectively, at discrete-time $k \in \mathbb{Z}_+$. The sets $\mathbb{X}$ and $\mathbb{U}$ are known sets with 0 in their interior and represent the state and input constraints, respectively. The function $\kappa : \widetilde{\mathcal{X}} \hookrightarrow \mathbb{U}$ is a set-valued state feedback law defined on $\widetilde{\mathcal{X}} \subseteq \mathbb{X}$ that is allowed to be discontinuous. Finally, the function $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ satisfies $f(0, 0) = 0$ and the following assumption.

**Assumption 3.1.** The function $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is uniformly continuous in $x$ in the sense that there exists a $\mathcal{K}$-function $\widetilde{\eta}_f$ such that $|f(x, u) - f(\overline{x}, u)| \leqslant \widetilde{\eta}_f(|x - \overline{x}|)$ for all $x, \overline{x} \in \mathbb{X}$ and all $u \in \mathbb{U}$.

Note that all functions $f$ that are Lipschitz continuous in $x$ with Lipschitz constant $L_f$, satisfy Assumption 3.1 with $\widetilde{\eta}_f(s) = L_f s$. We consider the following perturbed versions of the closed-loop system (4):

$$\widetilde{x}_{k+1} \in f(\widetilde{x}_k, \kappa(\widetilde{x}_k)) + w_k \triangleq \mathcal{F}_w(\widetilde{x}_k, w_k), \tag{5}$$

$$x_{k+1} \in f(x_k, \kappa(x_k + e_k)) + d_k \triangleq \mathcal{F}_{e,d}(x_k, e_k, d_k), \tag{6}$$

where $\tilde{x}_k, x_k$ are the state variables, $w_k \in \mathbb{W} \subseteq \mathbb{R}^n$, $d_k \in \mathbb{D} \subseteq \mathbb{R}^n$ the *additive disturbances* and $e_k \in \mathbb{E} \subseteq \mathbb{R}^n$ the *state measurement error* at discrete-time $k \in \mathbb{Z}_+$, respectively.

**Assumption 3.2.** Let $\mathbb{W} \triangleq \{\omega \in \mathbb{R}^n \,|\, |\omega| \leqslant \lambda\}$ for some $\lambda > 0$. Suppose that system (5) is input-to-state stable in $\tilde{\mathcal{X}} \subseteq \mathbb{X}$ with respect to additive disturbances in $\mathbb{W}$ with $0 \in \operatorname{int}(\tilde{\mathcal{X}})$, i.e. there exist a $\mathcal{KL}$-*function* $\beta_{\tilde{x}}$ and a $\mathcal{K}$-*function* $\gamma_{\tilde{x}}^w$ such that for all $\tilde{x}_0 \in \tilde{\mathcal{X}}$ and $w : \mathbb{Z}_+ \to \mathbb{W}$ all solutions $\tilde{x} \in \mathcal{S}_{\mathcal{F}_w}(\tilde{x}_0, w)$ satisfy

$$|\tilde{x}_k| \leqslant \beta_{\tilde{x}}(|\tilde{x}_0|, k) + \gamma_{\tilde{x}}^w(\|w\|), \quad \forall k \in \mathbb{Z}_+. \tag{7}$$

Furthermore, assume that $\tilde{\mathcal{X}}$ is RPI for system (5) with *additive disturbances* in $\mathbb{W}$.

To differentiate between various $\mathcal{KL}$- and $\mathcal{K}$-functions, we will adopt the convention to use sub- and superscripts to indicate between which variables the functions apply, e.g. $\gamma_{\tilde{x}}^w$ indicates that this is an ISS gain function from $w$ to $\tilde{x}$.

**Theorem 3.3.** *Suppose that Assumptions* 3.1 *and* 3.2 *hold and define the* $\mathcal{K}_\infty$*-function* $\eta_f(s) = \tilde{\eta}_f(s) + s$ *for* $s \geqslant 0$. *Let* $\lambda_e \geqslant 0$ *and* $\lambda_d \geqslant 0$ *be such that* $\lambda_e + \lambda_d \leqslant \lambda$ *and define*

$$\mathbb{E} \triangleq \{\varepsilon \in \mathbb{R}^n \,|\, |\varepsilon| \leqslant \eta_f^{-1}(\lambda_e)\}, \quad \mathbb{D} \triangleq \{\delta \in \mathbb{R}^n \,|\, |\delta| \leqslant \lambda_d\},$$

*and* $\mathcal{X} \triangleq \tilde{\mathcal{X}} \sim \mathbb{E}$. *Suppose that* $0 \in \operatorname{int}(\mathcal{X})$. *Then, the following statements hold*:

(i) *The set* $\mathcal{X} \subseteq \mathbb{X}$ *is an RPI set for closed-loop system* (6) *with state measurement errors in* $\mathbb{E}$ *and additive disturbances in* $\mathbb{D}$.

(ii) *The state and input constraints are satisfied for all trajectories of* (6) *with initial states* $x_0$ *in* $\mathcal{X}$, *measurement errors in* $\mathbb{E}$ *and additive disturbances in* $\mathbb{D}$, *i.e. for all* $x \in \mathcal{S}_{\mathcal{F}_{e,d}}(x_0, e, d)$ *with* $x_0 \in \mathcal{X}$, $e : \mathbb{Z}_+ \to \mathbb{E}$ *and* $d : \mathbb{Z}_+ \to \mathbb{D}$ *it holds that* $x_k \in \mathbb{X}$ *and* $\kappa(x_k + e_k) \subseteq \mathbb{U}$ *for all* $k \in \mathbb{Z}_+$.

(iii) *The closed-loop system* (6) *is input-to-state stable in* $\mathcal{X}$ *with respect to state measurement errors in* $\mathbb{E}$ *and additive disturbances in* $\mathbb{D}$. *In particular, we have that for all* $x_0 \in \mathcal{X}$, $e : \mathbb{Z}_+ \to \mathbb{E}$ *and* $d : \mathbb{Z}_+ \to \mathbb{D}$ *all solutions* $x \in \mathcal{S}_{\mathcal{F}_{e,d}}(x_0, e, d)$ *satisfy*

$$|x_k| \leqslant \beta_x(|x_0|, k) + \gamma_x^e(\|e\|) + \gamma_x^d(\|d\|), \quad \forall k \in \mathbb{Z}_+, \tag{8}$$

*with* $\beta_x(|x_0|, k) \triangleq \beta_{\tilde{x}}(2|x_0|, k)$, $\gamma_x^d(\|d\|) \triangleq \gamma_{\tilde{x}}^w(2\|d\|)$ *and*

$$\gamma_x^e(\|e\|) \triangleq \beta_{\tilde{x}}(2\|e\|, 0) + \gamma_{\tilde{x}}^w(2\eta_f(\|e\|)) + \|e\|.$$

**Proof.** (i) Let $\xi \in \mathcal{X}$, $\varepsilon \in \mathbb{E}$ and $\delta \in \mathbb{D}$. We will show that for all $\bar{\varepsilon} \in \mathbb{E}$,

$$[f(\xi, \kappa(\xi + \varepsilon)) + \delta] + \bar{\varepsilon} \subseteq \tilde{\mathcal{X}} \tag{9}$$

as this would prove that $\mathcal{X}$ is RPI for (6) according to Definition 2.1. We proceed by observing that

$$f(\xi, \mu) + \delta + \bar{\varepsilon} = f(\tilde{\xi}, \mu) + \omega, \quad \forall \mu \in \kappa(\tilde{\xi}) \subseteq \mathbb{U}, \tag{10}$$

with $\tilde{\xi} \triangleq \xi + \varepsilon$ and $\omega \triangleq f(\xi, \mu) - f(\tilde{\xi}, \mu) + \delta + \bar{\varepsilon}$. Using Assumption 3.1 yields $|f(\tilde{\xi} - \varepsilon, \mu) - f(\tilde{\xi}, \mu)| \leqslant \tilde{\eta}_f(|\varepsilon|)$. Therefore, it holds that for all $\varepsilon, \bar{\varepsilon} \in \mathbb{E}$, $\delta \in \mathbb{D}$ and $\tilde{\xi} \in \tilde{\mathcal{X}}$

$$\begin{aligned} |\omega| &= |f(\tilde{\xi} - \varepsilon, \mu) - f(\tilde{\xi}, \mu) + \delta + \bar{\varepsilon}| \leqslant \tilde{\eta}_f(|\varepsilon|) + |\delta| + |\bar{\varepsilon}| \\ &\leqslant \tilde{\eta}_f \circ \eta_f^{-1}(\lambda_e) + \lambda_d + \eta_f^{-1}(\lambda_e) = \lambda_e + \lambda_d \leqslant \lambda, \\ &\quad \forall \mu \in \kappa(\tilde{\xi}) \subseteq \mathbb{U}, \end{aligned} \tag{11}$$

which shows that $\omega \in \mathbb{W}$. Using now Assumption 3.2, i.e. $\tilde{\mathcal{X}}$ is RPI for system (5) with additive disturbances in $\mathbb{W}$, (10) yields that for all $\xi \in \mathcal{X}$, $\varepsilon, \bar{\varepsilon} \in \mathbb{E}$ and $\delta \in \mathbb{D}$,

$$f(\xi, \mu) + \delta + \bar{\varepsilon} \subseteq \tilde{\mathcal{X}}, \quad \forall \mu \in \kappa(\xi + \varepsilon) \subseteq \mathbb{U},$$

which is equivalent to (9).

(ii) Due to (i), it holds that for any $x_0 \in \mathcal{X}$ and any $e : \mathbb{Z}_+ \to \mathbb{E}$, $d : \mathbb{Z}_+ \to \mathbb{D}$ all trajectories $x \in \mathcal{S}_{\mathcal{F}_{e,d}}(x_0, e, d)$ satisfy $x_k \in \mathcal{X} \subseteq \mathbb{X}$, $x_k + e_k \in \tilde{\mathcal{X}} \subseteq \mathbb{X}$ for all $k \in \mathbb{Z}_+$ and thus $u_k \in \kappa(x_k + e_k) \subseteq \mathbb{U}$ for all $k \in \mathbb{Z}_+$.

(iii) Let $x_0$ in $\mathcal{X}$, $e : \mathbb{Z}_+ \to \mathbb{E}$, $d : \mathbb{Z}_+ \to \mathbb{D}$ and $x \in \mathcal{S}_{\mathcal{F}_{e,d}}(x_0, e, d)$. We perform the following coordinate change on (6):

$$x_k = \tilde{x}_k - e_k, \quad \forall k \in \mathbb{Z}_+, \tag{12}$$

which gives

$$\tilde{x}_{k+1} \in f(\tilde{x}_k - e_k, \kappa(\tilde{x}_k)) + d_k + e_{k+1}, \quad k \in \mathbb{Z}_+ \tag{13}$$

or

$$\tilde{x}_{k+1} \in f(\tilde{x}_k, \kappa(\tilde{x}_k)) + w_k, \quad k \in \mathbb{Z}_+, \tag{14}$$

where

$$\begin{aligned} &w_k \triangleq f(\tilde{x}_k - e_k, u_k) - f(\tilde{x}_k, u_k) + d_k + e_{k+1} \quad \text{for some} \\ &\quad u_k \in \kappa(\tilde{x}_k) \subseteq \mathbb{U}, e_k, e_{k+1} \in \mathbb{E}, d_k \in \mathbb{D}, \tilde{x}_k \in \tilde{\mathcal{X}}. \end{aligned} \tag{15}$$

Hence,

$$\begin{aligned} w_k \in \overline{\mathbb{W}} \triangleq \{ &f(\tilde{\xi} - \varepsilon, \mu) - f(\tilde{\xi}, \mu) + \delta + \bar{\varepsilon} \,| \\ &\mu \in \kappa(\tilde{\xi}) \subseteq \mathbb{U}, \varepsilon, \bar{\varepsilon} \in \mathbb{E}, \delta \in \mathbb{D}, \tilde{\xi} \in \tilde{\mathcal{X}}\}. \end{aligned}$$

We claim that $\overline{\mathbb{W}} \subseteq \mathbb{W}$. Indeed, if $\omega \in \overline{\mathbb{W}}$, then we can use Assumption 3.1 to obtain that for all $\varepsilon, \bar{\varepsilon} \in \mathbb{E}$, $\delta \in \mathbb{D}$ and $\tilde{\xi} \in \tilde{\mathcal{X}}$ (11) holds, which implies that $\overline{\mathbb{W}} \subseteq \mathbb{W}$ and therefore $w_k \in \mathbb{W}$ for all $k \in \mathbb{Z}_+$. Due to the fact that $w_k \in \mathbb{W}$ for all $k \in \mathbb{Z}_+$ and $x_k + e_k \in \tilde{\mathcal{X}}$ for all $k \in \mathbb{Z}_+$ (as shown in item (ii) of the proof) we obtain that $\tilde{x}_k \in \tilde{\mathcal{X}}$ for all $k \in \mathbb{Z}_+$. As a consequence, we can apply (7) of Assumption 3.2 to (14). Via (15) and using Assumption 3.1 in a similar manner as in (11), we obtain that for all $u_k \in \kappa(\tilde{x}_k) \subseteq \mathbb{U}$, $e_k, e_{k+1} \in \mathbb{E}$, $d_k \in \mathbb{D}$, $\tilde{x}_k \in \tilde{\mathcal{X}}$ and $k \in \mathbb{Z}_+$

$$\begin{aligned} |w_k| &\leqslant |f(\tilde{x}_k - e_k, u_k) - f(\tilde{x}_k, u_k) + d_k + e_{k+1}| \\ &\leqslant |f(\tilde{x}_k - e_k, u_k) - f(\tilde{x}_k, u_k)| + \|d\| + \|e\| \\ &\leqslant \tilde{\eta}_f(\|e\|) + \|e\| + \|d\| = \eta_f(\|e\|) + \|d\|. \end{aligned} \tag{16}$$

Substituting the last inequality of (16) into (7) gives

$$|\tilde{x}_k| \leqslant \beta_{\tilde{x}}(|\tilde{x}_0|, k) + \gamma_{\tilde{x}}^w(\eta_f(\|e\|) + \|d\|). \tag{17}$$

Applying (12) and property (17) yields

$$\begin{aligned}|x_k| &= |\widetilde{x}_k - e_k| \leqslant |\widetilde{x}_k| + |e_k| \\ &\leqslant \beta_{\tilde{x}}(|x_0 + e_0|, k) + \gamma_{\tilde{x}}^w(\eta_f(\|e\|) + \|d\|) + |e_k| \\ &\leqslant \beta_{\tilde{x}}(|x_0| + |e_0|, k) + \gamma_{\tilde{x}}^w(2\eta_f(\|e\|)) \\ &\quad + \gamma_{\tilde{x}}^w(2\|d\|) + \|e\| \\ &\leqslant \beta_{\tilde{x}}(2|x_0|, k) + \beta_{\tilde{x}}(2|e_0|, k) + \gamma_{\tilde{x}}^w(2\eta_f(\|e\|)) \\ &\quad + \gamma_{\tilde{x}}^w(2\|d\|) + \|e\| \\ &\leqslant \beta_{\tilde{x}}(2|x_0|, k) + \beta_{\tilde{x}}(2\|e\|, 0) + \gamma_{\tilde{x}}^w(2\eta_f(\|e\|)) \\ &\quad + \gamma_{\tilde{x}}^w(2\|d\|) + \|e\| \\ &= \beta_x(|x_0|, k) + \gamma_x^e(\|e\|) + \gamma_x^d(\|d\|),\end{aligned}$$

which concludes the proof. □

As a corollary, we can obtain a similar result for

$$x_{k+1} \in f(x_k, \kappa(x_k + e_k)) \triangleq \mathscr{F}_e(x_k, e_k), \tag{18}$$

which is a special case of (6), where we only consider measurement errors $e_k \in \mathbb{E}$ and no additive disturbances $d_k$. For illustration purposes, we consider in the corollary below the case where $f$ is Lipschitz continuous in $x$.

**Corollary 3.4.** *Suppose that Assumption 3.2 holds and that $f$ is Lipschitz continuous in $x$, i.e. Assumption 3.1 holds with $\widetilde{\eta}_f(s) = L_f s$, $s \geqslant 0$. Let*

$$\mathbb{E} \triangleq \left\{ \varepsilon \in \mathbb{R}^n \,\middle|\, |\varepsilon| \leqslant \frac{\lambda}{(L_f + 1)} \right\}, \tag{19}$$

*$\mathscr{X} \triangleq \widetilde{\mathscr{X}} \sim \mathbb{E}$ and suppose $0 \in \operatorname{int}(\mathscr{X})$. Then, the following statements hold.*

(i) *The set $\mathscr{X} \subseteq \mathbb{X}$ is an RPI set for closed-loop system (18) perturbed by state measurement errors in $\mathbb{E}$.*
(ii) *The state and input constraints are satisfied for all trajectories of (18) with initial states $x_0$ in $\mathscr{X}$ and measurement errors in $\mathbb{E}$, i.e. for all $x \in \mathscr{S}_{\mathscr{F}_e}(x_0, e)$ with $x_0 \in \mathscr{X}$ and $e : \mathbb{Z}_+ \to \mathbb{E}$ it holds that $x_k \in \mathbb{X}$ and $\kappa(x_k + e_k) \subseteq \mathbb{U}$ for all $k \in \mathbb{Z}_+$.*
(iii) *The closed-loop system (18) is input-to-state stable in $\mathscr{X}$ with respect to state measurement errors in $\mathbb{E}$. In particular, we have that for all $x_0 \in \mathscr{X}$ and $e : \mathbb{Z}_+ \to \mathbb{E}$ all solutions $x \in \mathscr{S}_{\mathscr{F}_e}(x_0, e)$ satisfy*

$$|x_k| \leqslant \beta_x(|x_0|, k) + \gamma_x^e(\|e\|), \quad \forall k \in \mathbb{Z}_+, \tag{20}$$

*with $\beta_x(|x_0|, k) \triangleq \beta_{\tilde{x}}(2|x_0|, k)$ and*

$$\gamma_x^e(\|e\|) \triangleq \beta_{\tilde{x}}(2\|e\|, 0) + \gamma_{\tilde{x}}^w((L_f + 1)\|e\|) + \|e\|.$$

**Remark 3.5.** Corollary 3.4 also applies in the unconstrained case, i.e. when $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{U} = \mathbb{R}^m$, with the unbounded disturbance sets $\mathbb{W} = \mathbb{E} = \mathbb{R}^n$ ($\lambda = \infty$). In this case, the above result applies for $\widetilde{\mathscr{X}} = \mathscr{X} = \mathbb{R}^n$ and yields a global ISS result with respect to measurement noise. A similar remark can be made for Theorem 3.3.

The derived results can be applied in the domain of MPC. In Lazar et al. (2007), Limon et al. (2002, 2006), Magni et al. (2006) MPC laws are proposed that result in closed-loop systems that are input-to-state stable with respect to additive disturbances. In the setting of this paper, $\mathbb{X}$ are the state and $\mathbb{U}$ are the control variable constraints and $\widetilde{\mathscr{X}}$ is the feasible set for the MPC optimization problem. Applying Corollary 3.4 would yield directly an MPC state feedback law that is input-to-state stable in $\mathscr{X}$ with respect to measurement errors in $\mathbb{E}$, where the relation between $\mathbb{W}$ and $\mathbb{E}$ is given in (19).

The result of Corollary 3.4 is also relevant for "certainty equivalence control," where one designs *output* feedback controllers that generate the control input via a state feedback law using an estimate of the state, which is obtained, for instance, from an observer. For linear systems, the separation principle gives a formal justification of this approach in the absence of constraints. Such a principle does not hold generally, when nonlinear systems and/or constraints are considered. In Mayne, Raković, Findeisen, and Allgöwer (2006) one considers for instance the constrained *linear* case using a particular MPC controller, while for *unconstrained* nonlinear discrete-time systems interesting results are available in e.g. Kazakos and Tsinias (1994) and Messina et al. (2005). In the constrained linear and nonlinear case, Corollary 3.4 might be useful as it yields state feedbacks that are input-to-state stable with respect to measurement errors. If observers are available that yield globally asymptotically stable (GAS) estimation error dynamics (or satisfy other ISS properties), one might apply the well-known small gain results (see e.g. Jiang & Wang, 2001) to prove that the closed-loop system is GAS. For the constrained case, it might be necessary to run the observer a sufficiently large period of time to ensure that the estimation error is contained in $\mathbb{E}$, before switching on the state feedback controller using the estimated state. In the unconstrained case with $\mathbb{E} = \mathbb{R}^n$ as discussed in Remark 3.5, this is not necessary. One application of these ideas in the domain of output feedback control of DC–DC converters is presented in Roset, Lazar, Heemels, and Nijmeijer (2007).

## 4. Conclusions

The result in this note shows that feedback laws that can render a closed-loop system input-to-state stable with respect to *additive disturbances* can also render the same closed-loop system input-to-state stable with respect to *state measurement errors* and *additive disturbances*. The result holds for control and state constrained nonlinear systems that are possibly discontinuous with respect to the control variable. Furthermore, the result allows for possible discontinuity and set-valuedness of the state feedback law and as such applies to model predictive controllers. The fact that many results are available that render MPC closed-loop systems input-to-state stable with respect to additive disturbances and only a few for measurement errors, indicates the value of this note. Moreover, we also foresee applications of the results in the context of certainty equivalence output feedback control.

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