

# Inter-event Times Analysis for Planar Linear Event-triggered Controlled Systems

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**Abstract**— We analyse the properties of the inter-event times for planar linear time-invariant systems controlled by an event-triggered state-feedback law. The triggering rule is given by the relative threshold strategy and we assume that the tunable triggering parameter is small. Several cases are distinguished depending on the nature of the eigenvalues of the (continuous-time) closed-loop system matrix in absence of sampling. When these eigenvalues are real, it is shown that the inter-event times lie in a neighborhood of a given constant for all positive times or converge to the neighborhood of a given constant as time grows. When the eigenvalues are complex conjugates, the inter-event times oscillate with a varying period for which we give an estimate. Moreover, the values taken by the inter-event times over this varying period are approximately the same for all initial conditions. As a consequence, one can run a single simulation over a given interval of time to infer properties of the inter-event times for all initial conditions and all positive times. Numerical simulations are provided to support the presented theoretical guarantees. These results help to understand the behaviour of the inter-event times, instead of solely relying on numerical simulations, and can be exploited to evaluate the performance of the considered triggering condition in terms of average inter-transmission times.

## I. INTRODUCTION

Event-triggered control is a sampling paradigm, which consists in generating transmissions between the plant and the controller using a state-dependent criterion, which is continuously monitored [12]. The basic idea is to adapt plant-controller communication based on the current system needs, and not (solely) based on the time elapsed since the last transmission as in traditional time-triggered control. Event-triggered control is relevant in scenarios where the control system is subject to communication or computation constraints, as in networked control systems or embedded systems.

While various event-triggered control techniques are available in the literature, see e.g. [1], [4], [8], [10], [16], [18]–

[23] to mention a few, very little is known about the actual behavior of the inter-event times. In most cases, the analysis of the inter-event times only ensures the existence of a dwell-time also sometimes called a “minimum inter-event time”, that is a (uniform) strictly positive amount of time (away from zero) between any two successive transmissions. This property guarantees the absence of the Zeno phenomenon and is required by practical hardware limitations. Besides the existence of a dwell-time, we generally do not know how the inter-event times behave. Numerical simulations are thus often carried out to figure out the amount of transmissions generated by the particular event-triggered control strategies in a case-by-case manner, and how these depend, for instance, on the system initial conditions. Exceptions exist though. For instance, the works on discrete-time systems in, e.g., [5], [6], which rely on model predictive control techniques, provide analytical guarantees about the average inter-event times. When the plant dynamics evolve in continuous-time, event-triggered control techniques using model-based holding functions [17] can also be employed to derive properties on the inter-event times as advocated in [3], [14] for fixed threshold policies. On the other hand, the results in [15] provide conditions under which the inter-event times grow larger as the solution converges to the origin. More precise information about the inter-event times can be deduced for the dynamic event-triggered control technique proposed in [20, Section V.B]. In this case, the inter-event times converge to a constant value as time grows, which can be computed by analyzing the dynamics of an auxiliary variable around the origin.

Besides the few aforementioned works, our understanding of the inter-event times remains limited, while it is a key characteristic of the event-triggered controlled system. Because the problem is very challenging, we focus in this paper on plant dynamics given by two-dimensional continuous-time linear time-invariant systems. The controller is a static state-feedback law and the triggering rule is given by [21], which is one of the most popular triggering conditions in the field that is at the core of many other techniques, see e.g., [1], [7], [8], [10]. This triggering law relies on the condition  $|x - \hat{x}| \geq \sigma|x|$ , where  $x$  is the current plant state,  $\hat{x}$  is the plant state at the last transmission instant, and  $\sigma > 0$  is a tunable parameter. Our results require  $\sigma$  to be small, which is typically the case, see [21]. Depending on the nature of the eigenvalues of the state matrix of the continuous-time closed-loop system, we prove that the inter-event times either (i) converge to a neighborhood of a given constant as time tends to

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infinity, (ii) lie in a neighborhood of a given constant for all positive times, (iii) or oscillate with a given period and take value, which are almost insensitive to the considered initial conditions. Simulation results are provided, which confirm and show the strengths of the obtained theoretical guarantees. The proofs are omitted for space reasons.

**Notation.** Let  $\mathbb{R}$  be the set of real numbers,  $\mathbb{R}_{\geq 0} := [0, \infty)$ ,  $\mathbb{R}_{> 0} := (0, \infty)$ ,  $\mathbb{R}_{< 0} := (-\infty, 0)$ ,  $\mathbb{Z}$  be the set of integers,  $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$  and  $\mathbb{Z}_{> 0} := \{1, 2, \dots\}$ . Given a set  $E \subseteq \mathbb{R}^n$  with  $n \in \mathbb{Z}_{> 0}$ , we use  $E^*$  to denote  $E \setminus \{0\}$ . The notation  $(x, y)$  stands for  $[x^\top, y^\top]^\top$ , where  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^m$ . For a right-continuous  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$  and  $t \geq 0$ , we write  $f(t^+)$  to denote  $\lim_{t' \searrow t} f(t')$ .

## II. PROBLEM STATEMENT

Consider the planar system

$$\dot{x} = Ax + Bu, \quad (1)$$

where  $x \in \mathbb{R}^2$  is the state,  $u \in \mathbb{R}^m$  is the control input,  $m \in \mathbb{Z}_{> 0}$ , and  $(A, B)$  is stabilizable. The control input  $u$  is given by the feedback law

$$u = Kx, \quad (2)$$

where the matrix  $K \in \mathbb{R}^{m \times 2}$  is such that  $A+BK$  is Hurwitz; such a matrix does exist since  $(A, B)$  is stabilizable.

We study the scenario where controller (2) is implemented on a digital platform and communicates with system (1) at time instants  $t_i$ ,  $i \in \mathcal{I} := \{j \in \mathbb{Z}_{\geq 0} : j \leq N\}$  with  $N \in \mathbb{Z}_{> 0} \cup \{\infty\}$ . Between two successive transmission instants, the control input is held constant, and it is updated at every  $t_i$ ,  $i \in \mathcal{I}$ , which leads to

$$u = K\hat{x} \quad (3)$$

with

$$\begin{aligned} \dot{\hat{x}}(t) &= 0 & \text{for all } t \in (t_i, t_{i+1}) \\ \hat{x}(t_i^+) &= x(t_i). \end{aligned} \quad (4)$$

The overall system is

$$\left. \begin{aligned} \dot{x}(t) &= Ax(t) + BK\hat{x}(t) \\ \dot{\hat{x}}(t) &= 0 \\ x(t_i^+) &= x(t_i) \\ \hat{x}(t_i^+) &= x(t_i) \end{aligned} \right\} \begin{aligned} & \text{for all } t \in (t_i, t_{i+1}) \\ & \text{for all } i \in \mathcal{I}. \end{aligned} \quad (5)$$

To obtain a solution to (5) in the Carathéodory sense, for each  $i \in \mathcal{I}$ , we flow on  $[t_i, t_{i+1})$  and we jump at  $t_{i+1}$ , and so on.

The transmission instants  $t_i$ ,  $i \in \mathcal{I}$ , are defined implicitly by a state-dependent triggering rule. We use the criterion of [21] to define these instants, that is, a transmission occurs whenever

$$|\hat{x}(t) - x(t)| \geq \sigma|x(t)|, \quad (6)$$

where  $\sigma > 0$  is selected to ensure that the origin of system (5) is uniformly globally exponentially stable<sup>1</sup>, see [21]. In particular, the construction of (6) in [21] and the stability proof are based on the existence of a quadratic, positive definite (thus radially unbounded) function  $V : \mathbb{R}^2 \rightarrow \mathbb{R}_{\geq 0}$  such that

$$|\hat{x} - x| \leq \sigma|x| \Rightarrow \langle \nabla V(x), Ax + BK\hat{x} \rangle \leq -a|x|^2, \quad (7)$$

where  $a > 0$  depends on  $\sigma$ .

We assume that a transmission occurs at  $t = 0$ , so that  $t_0 = 0$  and  $x(0) = \hat{x}(0)$ . Since after a transmission  $\hat{x} = x$  in view of (5), the next inter-event time is the time it takes for  $|\hat{x} - x|$  to grow from 0 to  $\sigma|x|$ . We denote this time as  $\tau_\sigma(x(t_i))$  for  $i \in \mathcal{I}$ . It is equal to  $\tau_\sigma(\hat{x}(t))$  for all  $t \in [t_i, t_{i+1})$ ,  $i \in \mathcal{I}$ , since  $\hat{x}$  is constant between two successive events in view of (5). As in [2], the inter-event time is defined, for  $x_0 \in \mathbb{R}^2$ , as

$$\tau_\sigma(x_0) := \inf \{ \eta > 0 : |x_0 - \phi(\eta; x_0)| = \sigma|\phi(\eta; x_0)| \}, \quad (8)$$

where  $\phi(\eta; x_0)$  is the solution<sup>2</sup> to  $\dot{x} = Ax + BKx_0$  at time  $\eta \geq 0$ , initialized at  $x_0$ .

*Remark 1:* Note that we consider the time from  $\eta = 0$  in (8), and not from  $\eta = t_i$  with  $i \in \mathcal{I}$ , which is without loss of generality as system (5) is time-invariant and satisfies the semi-group property.  $\square$

It is shown in [21] that there exists a uniform almost global strictly positive lower bound on the minimum inter-event times for system (5), in the sense that there exists  $\varepsilon > 0$  such that  $\tau_\sigma(x_0) \geq \varepsilon$  for any  $x_0 \in \mathbb{R}^{2,*}$ . The objective of this study is to go further in the analysis of the function  $\tau_\sigma$ : we aim at providing analytical characterizations of the behaviour of  $\tau_\sigma(\hat{x}(t))$  along the solutions to (5). The presented results apply for small  $\sigma$  in (6) and are validated on an example in Section V. First, we establish properties of  $\tau_\sigma$  for this purpose.

## III. PROPERTIES OF $\tau_\sigma$

We first need to make sure that  $\tau_\sigma$  cannot be equal to  $\infty$ . In other words, we want to guarantee that  $\tau_\sigma(\mathbb{R}^2) \subseteq \mathbb{R}_{\geq 0}$ . This is ensured by the next lemma.

*Lemma 1:* For any  $x_0 \in \mathbb{R}^2$ ,  $\tau_\sigma(x_0) \in [0, \infty)$ .  $\square$

Lemma 1 implies that  $\mathcal{I} = \mathbb{Z}_{\geq 0}$ , i.e.  $N = \infty$  in the definition of  $\mathcal{I}$ , for any  $x_0 \in \mathbb{R}^2$ . We also have that  $\tau_\sigma(0) = 0$ , which means that an infinite number of jumps occurs in finite time at the origin<sup>3</sup>. This potential issue is clarified when writing the overall system using the hybrid formalism

<sup>1</sup>Strictly speaking, the uniform global asymptotic stability of  $\{(x, \hat{x}) : x = 0\}$  is proved in [21]. The uniform global asymptotic stability of  $x = \hat{x} = 0$  is established in [20, Section V.C], and the exponential property follows from the linearity of the system under consideration and (6). Also, we consider Carathéodory solutions in this work, which leads to a slight inconsistency because the solution initialized at the origin is trivial, as it can not flow. This issue is overcome when modeling the overall system using the hybrid formalism of [11], see [20]. We nevertheless show in Section III that we can exclude the origin in the forthcoming analysis.

<sup>2</sup>We abandon in the following the notation  $\phi$  to denote a solution, and use instead directly  $x$  (or  $\hat{x}$ ).

<sup>3</sup>See the last part of footnote 1 on page 2.

[11], see [9] and [20, Section IV.B] for more details. Now, any (maximal) solution to system (5) initialized at  $(x_0, x_0)$  with  $x_0 \neq 0$  will never reach the origin and is complete, as formalized in the next lemma. We therefore exclusively consider  $\tau_\sigma$  on  $\mathbb{R}^{2,*}$  in the rest of this study.

**Lemma 2:** For any  $x_0 \in \mathbb{R}^{2,*}$ , any solution  $(x, \hat{x})$  to system (5) initialized at  $(x_0, x_0)$  verifies  $x(t) \neq 0$  and  $\hat{x}(t) \neq 0$  for all  $t \geq 0$ .  $\square$

We also recall a homogeneity property of  $\tau_\sigma$ , which follows from [2, Theorem 4.11 and Remark 4.12].

**Lemma 3:** For any  $x_0 \in \mathbb{R}^{2,*}$  and  $\mu \in \mathbb{R}^*$ ,  $\tau_\sigma(x_0) = \tau_\sigma(\mu x_0)$ .  $\square$

Lemma 3 means that  $\tau_\sigma$  is constant along lines passing through the origin (excluding the origin).

Finally, the next proposition provides an expression of  $\tau_\sigma(x_0)$  for any  $x_0 \in \mathbb{R}^{2,*}$ , when parameter  $\sigma$  in (6) is small.

**Proposition 1:** There exist  $r : \mathbb{R}^2 \times (0, 1) \rightarrow \mathbb{R}$ ,  $c_r > 0$  and  $\sigma_1^* \in (0, 1]$  such that for any  $\sigma \in (0, \sigma_1^*)$  and any  $x_0 \in \mathbb{R}^{2,*}$ ,  $\tau_\sigma(x_0) = \sigma \frac{|x_0|}{|A_c x_0|} + r(x_0, \sigma)$  and  $|r(x_0, \sigma)| \leq c_r \sigma^2$ , where  $A_c := A + BK$ .  $\square$

Proposition 1 means that  $\tau_\sigma(x_0)$  is well approximated by  $\sigma \frac{|x_0|}{|A_c x_0|}$  for small  $\sigma > 0$ , for any  $x_0 \in \mathbb{R}^{2,*}$ . The fact that the constant  $c_r$ , which appears in the upper-bound of the norm  $r$ , is independent of  $x_0$  (and  $\sigma$ ), is crucial in the following. Note that Proposition 1 can be used to derive lower and upper bounds on  $\tau_\sigma(x_0)$ .

It is important to note that the results of this section do not exploit the fact that system (1) is of dimension two. In other words, these results hold when  $x$  is of dimension  $n \in \mathbb{Z}_{>0}$ . This will no longer be the case in the next section, with the exception of Theorem 1.

#### IV. MAIN RESULTS

Lemma 3 reveals an important feature of the inter-event time function: it only depends on which line passing through the origin  $\hat{x}$  lies and not on its actual value. To analyse  $\tau_\sigma(\hat{x}(t))$  along the solutions to (5), we can therefore study the argument (or, angle) of  $\hat{x}(t)$  and then exploit the results of the previous section. We recall that the argument<sup>4</sup> of  $x = (x_1, x_2) \in \mathbb{R}^{2,*}$  can be defined as

$$\arg : \mathbb{R}^{2,*} \rightarrow [-\pi, \pi]$$

$$x \mapsto \arg(x) := \begin{cases} \arctan\left(\frac{x_2}{x_1}\right) & \text{when } x_1 > 0 \\ \arctan\left(\frac{x_2}{x_1}\right) + \pi & \text{when } x_1 < 0 \text{ and } x_2 \geq 0 \\ \arctan\left(\frac{x_2}{x_1}\right) - \pi & \text{when } x_1 < 0 \text{ and } x_2 < 0 \\ \frac{\pi}{2} & \text{when } x_1 = 0 \text{ and } x_2 > 0 \\ -\frac{\pi}{2} & \text{when } x_1 = 0 \text{ and } x_2 < 0. \end{cases} \quad (9)$$

We distinguish in the following different cases according to the type of eigenvalues of  $A_c = A + BK$ , which are denoted  $\lambda_1$  and  $\lambda_2$ .

<sup>4</sup>Often, the argument is defined as  $\arg(x) = \arctan\left(\frac{x_2}{x_1}\right)$  but this is only true when  $x_1 > 0$ .

*A. When  $\lambda_1$  and  $\lambda_2$  are real, equal and of geometric multiplicity two*

The next theorem follows from Proposition 1 and the properties of  $\lambda_1$  and  $\lambda_2$ .

**Theorem 1:** When  $\lambda_1 = \lambda_2 < 0$  and their geometric multiplicity is two, there exist  $c_r > 0$  and  $\sigma_1^* \in (0, 1]$  such that for any initial condition  $(x_0, x_0)$  with  $x_0 \in \mathbb{R}^{2,*}$  and any  $\sigma \in (0, \sigma_1^*)$ , the corresponding solution  $(x, \hat{x})$  to (5) verifies  $\tau_\sigma(\hat{x}(t)) = \frac{\sigma}{|\lambda_1|} + r(\hat{x}(t), \sigma)$  with  $|r(\hat{x}(t), \sigma)| \leq c_r \sigma^2$ .  $\square$

Theorem 1 ensures that, for any initial condition  $(x_0, x_0)$  with  $x_0 \in \mathbb{R}^{2,*}$ , the inter-event times are close to  $\frac{\sigma}{|\lambda_1|}$  for all positive times when  $\lambda_1 = \lambda_2$  and their geometric multiplicity is two. Hence, the considered event-triggering rule essentially leads to periodic sampling, when  $\sigma$  is small. The proof of Theorem 1 does not exploit the fact that the state  $x$  is of dimension two: the results apply to any dimension. Also, function  $r$  and constants  $c_r, \sigma_1^*$  are the same as in Proposition 1, which explains why the same notation is used.

When the geometric multiplicity of  $\lambda_1 = \lambda_2$  is one, different proof techniques are needed, which are not provided in this paper.

*B. When  $\lambda_1$  and  $\lambda_2$  are real and distinct*

We assume without loss of generality that  $0 > \lambda_1 > \lambda_2$ . The next lemma characterizes the (asymptotic) behaviour of the argument of  $\hat{x}$  along the solutions to (5).

**Lemma 4:** When  $\lambda_1 > \lambda_2$ , there exist  $c_{\text{distinct}} > 0$  and  $\sigma_{\text{distinct}}^* \in (0, 1]$  such that for any initial condition  $(x_0, x_0)$  with  $x_0 \in \mathbb{R}^{2,*}$  and any  $\sigma \in (0, \sigma_{\text{distinct}}^*)$ , the corresponding solution  $(x, \hat{x})$  to (5) verifies one of the following properties:

- (i) There exists  $v_1$ , a non-zero eigenvector of  $A_c$  associated with  $\lambda_1$ , such that  $\limsup_{t \rightarrow \infty} |\arg(\hat{x}(t)) - \arg(v_1)| \leq c_{\text{distinct}} \sigma$ .
- (ii) There exists  $v_2$ , a non-zero eigenvector of  $A_c$  associated with  $\lambda_2$ , such that  $|\arg(\hat{x}(t)) - \arg(v_2)| \leq c_{\text{distinct}} \sigma$  for all  $t \geq 0$ .  $\square$

Lemma 4 recovers the properties of the argument of the solutions for the continuous-time closed-loop system in the absence of sampling  $\dot{x}_c = A_c x_c$  when  $\sigma \rightarrow 0$ , see [13, Chapter 2.1]. Indeed, when  $\lambda_1$  and  $\lambda_2$  are real and distinct, either the argument of  $x_c$  converges to  $\arg(v_1)$  for  $v_1$  some non-zero eigenvector of  $A_c$  associated with  $\lambda_1$  if the solution is not initialized on the eigenspace associated to  $\lambda_2$ , otherwise it is constant and equal to  $\arg(v_2)$  at all times, with  $v_2$  some non-zero eigenvector of  $A_c$  associated with  $\lambda_2$ . Similar results are recovered in Lemma 4 up to a perturbation of the order of  $\sigma$  due to sampling.

Properties of  $\tau_\sigma(\hat{x})$  along solutions to (5) are established next by exploiting Proposition 1 and Lemma 4.

**Theorem 2:** When  $\lambda_1 > \lambda_2$ , there exist  $c_1, c_2 > 0$  and  $\sigma_{\text{distinct}}^* \in (0, 1]$  such that for any initial condition  $(x_0, x_0)$  with  $x_0 \in \mathbb{R}^{2,*}$  and any  $\sigma \in (0, \sigma_{\text{distinct}}^*)$ , the corresponding solution  $(x, \hat{x})$  to (5) verifies one of the following properties:

- (i)  $\limsup_{t \rightarrow \infty} \left| \tau_\sigma(\hat{x}(t)) - \frac{\sigma}{|\lambda_1|} \right| \leq c_1 \sigma^2$ .

$$(ii) \left| \tau_\sigma(\hat{x}(t)) - \frac{\sigma}{|\lambda_2|} \right| \leq c_2 \sigma^2 \text{ for all } t \geq 0. \quad \square$$

Theorem 2 states that, when the eigenvalues of  $A_c$  are real and distinct, the inter-event time of system (5) either tends to  $\frac{\sigma}{|\lambda_1|} = \frac{\sigma|v_1|}{|A_c v_1|}$  or it takes values close to  $\frac{\sigma}{|\lambda_2|} = \frac{\sigma|v_2|}{|A_c v_2|}$  for all positive times, up to a perturbation of the order of  $\sigma^2$  in both cases.

### C. When $\lambda_1$ and $\lambda_2$ are complex conjugates and non-real

We write  $\lambda_1 = \lambda + i\beta$  and  $\lambda_2 = \lambda - i\beta$  where  $\lambda < 0$  and  $\beta > 0$ . We first derive properties of  $\tau_\sigma(x)$  along solutions to (5).

**Proposition 2:** When  $\lambda_1$  and  $\lambda_2$  are non-real complex conjugates, there exist  $c_{\text{complex}} > 0$  and  $\sigma_{\text{complex}}^* \in (0, 1]$  such that for any initial condition  $(x_0, x_0)$  with  $x_0 \in \mathbb{R}^{2,*}$  and any  $\sigma \in (0, \sigma_{\text{complex}}^*)$ , the corresponding solution  $(x, \hat{x})$  to (5) verifies the next property: For any  $t \geq 0$ , there exists  $\theta(t) \in \left[ \frac{\pi}{\beta} - c_{\text{complex}}\sigma, \frac{\pi}{\beta} + c_{\text{complex}}\sigma \right]$  such that  $\tau_\sigma(x(t)) = \tau_\sigma(x(t + \theta(t)))$ .  $\square$

Proposition 2 states that for small  $\sigma$ , any solution  $(x, \hat{x})$  to (5) not initialized at the origin, is such that  $\tau_\sigma(x(t))$  oscillates with a varying period close to  $\frac{\pi}{\beta}$ . Note that  $\tau_\sigma(x(t))$  takes value of the order of  $\sigma$ , thus (much) smaller than  $\frac{\pi}{\beta}$ , according to Proposition 1.

The next proposition establishes that the set of values taken by  $t \mapsto \tau_\sigma(x(t))$  over any time interval of length larger than  $\frac{\pi}{\beta} + c_{\text{complex}}\sigma$  is independent of the initial condition  $(x_0, x_0)$ , when  $x_0 \in \mathbb{R}^{2,*}$ . Note that it suffices to show this result for any time interval of length  $\frac{\pi}{\beta} + c_{\text{complex}}\sigma$  in view of Proposition 2. We define for this purpose<sup>5</sup>, for any  $x_0 \in \mathbb{R}^{2,*}$  and  $\sigma \in (0, 1)$ ,

$$\mathcal{T}_\sigma(x_0) := \left\{ \tau_\sigma(x(t)) : t \in \left[ 0, \frac{\pi}{\beta} + \sigma c_{\text{complex}} \right] \right\} \subseteq \mathbb{R}_{>0}, \quad (10)$$

where  $x$  is the corresponding component of the solution to (5) initialized at  $(x_0, x_0)$ , and  $c_{\text{complex}} > 0$  is as in Proposition 2.

**Proposition 3:** When  $\lambda_1$  and  $\lambda_2$  are non-real complex conjugates, for any  $x_0, x'_0 \in \mathbb{R}^{2,*}$ ,  $\sigma \in (0, \sigma_{\text{complex}}^*)$  with  $\sigma_{\text{complex}}^*$  as in Proposition 2,  $\mathcal{T}_\sigma(x_0) = \mathcal{T}_\sigma(x'_0)$ .  $\square$

We now show that the properties in Propositions 2 and 3 are approximately preserved for  $\tau_\sigma(\hat{x})$  along solutions to (5).

**Theorem 3:** When  $\lambda_1$  and  $\lambda_2$  are non-real complex conjugates, there exist  $\hat{c}_r > 0$ ,  $\hat{c}_{\text{complex}} \geq c_{\text{complex}}$  such that for any initial condition  $(x_0, x_0)$  with  $x_0 \in \mathbb{R}^{2,*}$  and any  $\sigma \in (0, \sigma_{\text{complex}}^*)$ , the corresponding solution  $(x, \hat{x})$  to (5) verifies the next property: For any  $t \geq 0$ , there exist  $\hat{\theta}(t) \in \left[ \frac{\pi}{\beta} - \hat{c}_{\text{complex}}\sigma, \frac{\pi}{\beta} + \hat{c}_{\text{complex}}\sigma \right]$  and  $r_{\text{complex}}(t, x_0, \sigma)$  such that  $\tau_\sigma(\hat{x}(t)) = \tau_\sigma(\hat{x}(t + \hat{\theta}(t))) + r_{\text{complex}}(t, x_0, \sigma)$  and  $|r_{\text{complex}}(t, x_0, \sigma)| \leq \hat{c}_r \sigma^2$ .  $\square$

<sup>5</sup>The fact that  $\mathcal{T}_\sigma(x_0) \subseteq \mathbb{R}_{>0}$  follows from Lemma 1 and [21, Theorem III.1].

The constants  $c_{\text{complex}}$  and  $\sigma_{\text{complex}}^*$  in Theorem 3 are the same as in Proposition 2. Theorem 3 implies that the inter-event time function  $t \mapsto \tau_\sigma(\hat{x}(t))$  describes an ‘‘almost’’ periodic pattern of period  $\frac{\pi}{\beta}$  for any initial condition  $(x_0, x_0)$  with  $x_0 \in \mathbb{R}^{2,*}$ . Note that  $\hat{c}_{\text{complex}}\sigma$ , which is the order of  $\sigma$ , is negligible with respect to  $\frac{\pi}{\beta}$ . Also,  $r_{\text{complex}}(t, x_0, \sigma)$  is of the order of  $\sigma^2$  and is therefore negligible with respect to  $\tau_\sigma(\hat{x}(t) + \hat{\theta}(t))$ , which is of the order of  $\sigma$  according to Proposition 1.

Contrary to  $t \mapsto \tau_\sigma(x(t))$ , we have no reason to think that  $t \mapsto \tau_\sigma(\hat{x}(t))$  takes exactly the same values over any interval of length  $\frac{\pi}{\beta} + \hat{c}_{\text{complex}}\sigma$  for any initial condition, as illustrated later in Figure 4 in Section V. Nevertheless, the initial conditions have a weak impact on the values taken by  $t \mapsto \tau_\sigma(x(t))$  over any interval of length<sup>6</sup>  $\frac{\pi}{\beta} + \hat{c}_{\text{complex}}\sigma$ , as formalized in the next theorem. Similarly to (10), we introduce for this purpose<sup>7</sup>, for any  $x_0 \in \mathbb{R}^{2,*}$  and  $\sigma \in (0, 1)$ ,  $\widehat{\mathcal{T}}_\sigma(x_0) := \left\{ \tau_\sigma(\hat{x}(t)) : t \in \left[ 0, \frac{\pi}{\beta} + \hat{c}_{\text{complex}}\sigma \right] \right\} \subseteq \mathbb{R}_{>0}$  where  $\hat{x}$  is the corresponding component of the solution to (5) initialized at  $(x_0, x_0)$ , and  $\hat{c}_{\text{complex}} > 0$  is as in Theorem 3.

**Theorem 4:** When  $\lambda_1$  and  $\lambda_2$  are non-real complex conjugates, for any  $x_0, x'_0 \in \mathbb{R}^{2,*}$  and  $\sigma \in (0, \sigma_{\text{complex}}^*)$ ,  $\widehat{\mathcal{T}}_\sigma(x_0) \subseteq \widehat{\mathcal{T}}_\sigma(x'_0) + \sigma^2[-\hat{c}_r, \hat{c}_r]$ , where  $\sigma_{\text{complex}}^*$  and  $\hat{c}_r$  are as in Theorem 3.  $\square$

Theorem 4 implies that the values taken by  $t \mapsto \tau_\sigma(\hat{x}(t))$  over any time interval of length larger than  $\frac{\pi}{\beta} + \hat{c}_{\text{complex}}\sigma$  are the same for any initial condition, up to a negligible error of the order of  $\sigma^2$ .

## V. NUMERICAL EXAMPLE

To illustrate the results of Section IV, we consider the same linear system as in [21, Section V], namely

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u. \quad (11)$$

The matrix  $K$  is designed such that the corresponding matrix  $A + BK$  is Hurwitz, and three cases are considered depending on the eigenvalues  $\lambda_1, \lambda_2$  of  $A + BK$  being (i) real and distinct, (ii) complex conjugates, (iii) real and equal. To design the triggering rule as in [21], we have taken the real, symmetric, positive matrix  $P$  such that  $(A + BK)^\top P + P(A + BK) = -\mathbb{I}$ . This leads to (6) with  $\sigma \in \left( 0, \frac{1}{2|PBK|} \right)$ .

For each of these cases, we have studied numerically the impact of  $\sigma$  and the initial conditions on the inter-event times.

**Case (i):**  $K = [0 \ -6]$ ,  $\lambda_1 = -1$  and  $\lambda_2 = -2$ . Then  $\sigma \in (0, 0.1179)$ . Figure 1 shows the inter-event times for

<sup>6</sup>Again, it is enough to study the values of the inter-event times on a time interval of length  $\frac{\pi}{\beta} + \hat{c}_{\text{complex}}\sigma$  in view of Theorem 3.

<sup>7</sup>The same comment as in footnote 5 on page 4 applies.

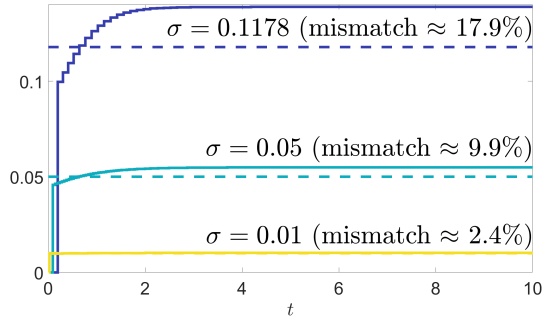


Fig. 1. Inter-event times (solid lines) and value of  $\frac{\sigma}{|\lambda_1|}$  (dashed line) when  $(\lambda_1, \lambda_2) = (-1, -2)$  for different values of  $\sigma$ : 0.1178 (blue), 0.05 (green), 0.01 (yellow). The mismatch is the error percentage between the limit value of the inter-event times and  $\frac{\sigma}{|\lambda_1|}$ .

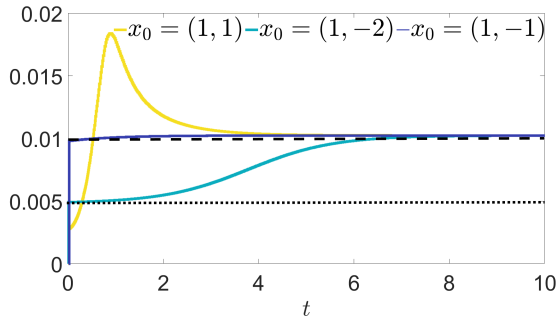


Fig. 2. Inter-event times for different values of  $x_0$  when  $(\lambda_1, \lambda_2) = (-1, -2)$ : (1, 1) (yellow), (1, -2) (green), (1, -1) (blue). The black dashed line corresponds to  $\frac{\sigma}{|\lambda_1|}$ , and the black dotted line to  $\frac{\sigma}{|\lambda_2|}$ .

$\sigma \in \{0.1178, 0.05, 0.01\}$ , and the initial condition  $(x_0, x_0)$  with  $x_0 = (1, 1)$ . According to Theorem 2, the inter-event times converge to a value close to  $\frac{\sigma}{|\lambda_1|}$  as the time tends to infinity or is close to  $\frac{\sigma}{|\lambda_2|}$  for all positive times. We see that the inter-event times indeed converge to a constant close to  $\frac{\sigma}{|\lambda_1|}$  in all the cases considered in Figure 1, and that the mismatch between the limit value and  $\frac{\sigma}{|\lambda_1|}$  is getting smaller as we decrease  $\sigma$ , which is in agreement with the statement of Theorem 2.

We might wonder whether there are solutions for which the inter-event times are close to  $\frac{\sigma}{|\lambda_2|}$  for all positive times, which is allowed by Theorem 2. We have not been able to find such solutions for this example. Figure 2 suggests that the inter-event times converge to  $\frac{\sigma}{|\lambda_1|}$  for a given  $\sigma$ , independently of the initial condition. In particular, we have taken for this set of simulations  $\sigma = 0.01$ ,  $x_0 = (1, 1)$  as above,  $x_0 = (1, -2)$ , which lies in the eigenspace associated to  $\lambda_2$ , and  $x_0 = (1, -1)$ , which lies in the eigenspace associated to  $\lambda_1$ . We have selected a small value for  $\sigma$  to be sure we can distinguish whether the inter-event times converge to  $\frac{\sigma}{|\lambda_1|}$  or  $\frac{\sigma}{|\lambda_2|}$ .

*Case (ii):*  $K = [-3 \ -7]$ ,  $\lambda_1 = -2 + j$  and  $\lambda_2 = -2 - j$ . Then  $\sigma \in (0, 0.0728)$ . We have selected different values of  $\sigma$ , namely  $\sigma \in \{0.0725, 0.03, 0.01\}$ , with initial condition  $(x_0, x_0)$  and  $x_0 = (1, 1)$ . The obtained inter-event times are depicted in Figure 3. We observe a periodic-like behaviour in each case and that the “pseudo” period is getting closer

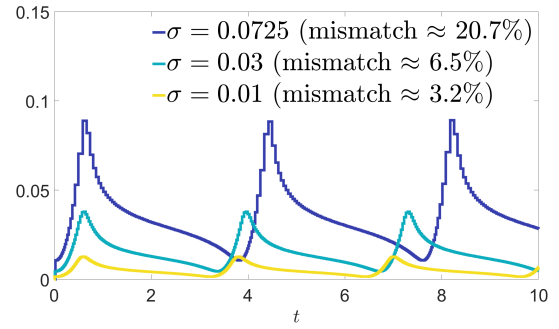


Fig. 3. Inter-event times for different values of  $\sigma$  when  $(\lambda_1, \lambda_2) = (-2 + j, -2 - j)$ : 0.0725 (blue), 0.03 (green), 0.01 (yellow). The mismatch is the error percentage between  $\pi$  and the observed period.

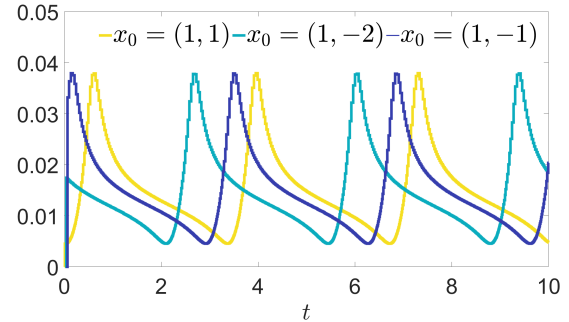


Fig. 4. Inter-event times for different values of  $x_0$  when  $(\lambda_1, \lambda_2) = (-2 + j, -2 - j)$ : (1, 1) (yellow), (1, -2) (green), (1, -1) (blue).

to  $\frac{\pi}{|\beta|} = \pi$  as  $\sigma$  decreases, in agreement with Theorem 3.

We have then selected  $\sigma = 0.03$  and studied the inter-event times for different initial conditions  $(x_0, x_0)$  with  $x_0 \in \{(1, 1), (1, -2), (1, -1)\}$ , see Figure 4. The inter-event times describe similar though slightly different patterns of very similar periods, in agreement with Theorem 4.

*Case (iii):*  $K = [-2 \ -7]$ ,  $\lambda_1 = \lambda_2 = -2$ . Then  $\sigma \in (0, 0.0858)$ . Note that this case is not covered by our analysis, see Section IV-A. We have considered the initial condition  $(x_0, x_0)$  and  $x_0 = (1, 1)$  and different values of  $\sigma$ , namely  $\sigma \in \{0.085, 0.04, 0.01\}$ , see Figure 5. We observe that the inter-event times converge in all cases to a constant, which is in a neighborhood of  $\frac{\sigma}{|\lambda_1|}$ , and that the mismatch reduces with  $\sigma$  like in case (i). We have also varied the initial conditions for  $\sigma = 0.01$ . In particular, we have taken  $x_0 = (1, -2)$ , which is in the eigenspace associated with  $\lambda_1$ , and  $x_0 = (1, -1.9)$  and  $x_0 = (1, -2.1)$ , which are, loosely speaking, on both sides of the eigenspace of  $\lambda_1$ . Again, in all cases the inter-event times converge to a constant close to  $\frac{\sigma}{|\lambda_1|}$ , see Figure 6. We may thus conjecture that, when  $\lambda_1 = \lambda_2 < 0$  and their geometric multiplicity is one, the inter-event times (approximately) converge to  $\frac{\sigma}{|\lambda_1|}$  as time grows.

## VI. CONCLUSION

We have analyzed the inter-event times for planar linear event-triggered control based on the relative threshold technique of [21]. We have shown that these inter-event times (approximately): (i) converge to or lie for all positive times in

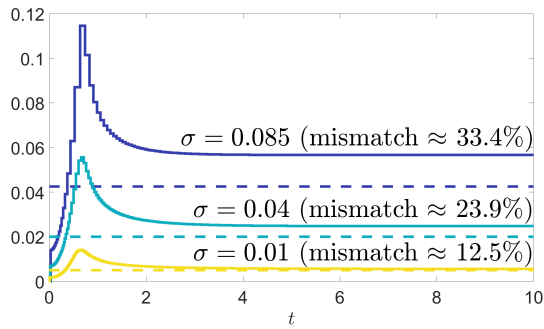


Fig. 5. Inter-event times for different values of  $\sigma$  when  $\lambda_1 = \lambda_2 = -2$ : 0.085 (blue), 0.04 (green), 0.01 (yellow). The mismatch corresponds to the error percentage between  $\sigma \frac{\pi}{2}$  and the limit value of the inter-event times.

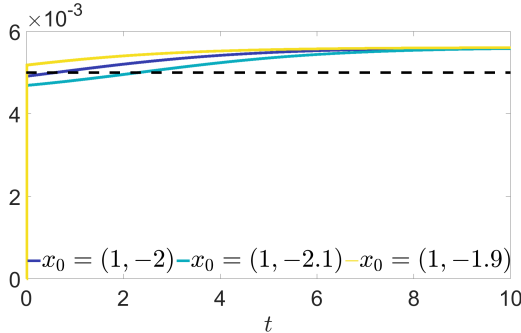


Fig. 6. Inter-event times for different values of  $x_0$  when  $\lambda_1 = \lambda_2 = -2$ :  $(1, -2)$  (blue),  $(1, -2.1)$  (green),  $(1, -1.9)$  (yellow). The dashed line corresponds to the value  $\sigma \frac{\pi}{2}$ .

a neighborhood of given constants when the eigenvalues of the state matrix of the continuous-time closed-loop system in absence of sampling are real and distinct, or real, equal and of geometric multiplicity two; (ii) describe an almost periodic pattern, when these eigenvalues are complex conjugates. In the latter case, an estimation of the period is provided. Importantly, these results apply mutatis mutandis to nonlinear event-triggered control systems, whose linearization around the origin is given by the considered linear model and triggering rules.

We are currently working on the extension of these results to output feedback control, as well as to other triggering rules.

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