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Definitions of incremental stability for hybrid systems

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Abstract— **The analysis of incremental stability properties typically involves measuring the distance between any pair of solutions of a given dynamical system, corresponding to different initial conditions, at the same time instant. This approach is not directly applicable for hybrid systems in general. Indeed, hybrid systems generate solutions that are defined with respect to hybrid times, which consist of both the continuous time elapsed and the discrete time, that is the number of jumps the solution has experienced. Two solutions of a hybrid system do not a priori have the same time domain, and we may therefore not be able to compare them at the same hybrid time instant. To overcome this issue, we invoke graphical closeness concepts. We present definitions for incremental stability depending on whether incremental asymptotic stability properties hold with respect to the hybrid time, the continuous time, or the discrete time, respectively. Examples are provided throughout the paper to illustrate these definitions, and the relations between these three incremental stability notions are investigated. The definitions are shown to be consistent with those available in the literature for continuous-time and discrete-time systems.**

I. Introduction

In the literature, a dynamical system is said to be incrementally asymptotically stable when all its solutions are asymptotically stable, see e.g., [1], [7], [18], [24], [25]. Loosely speaking, this means that: (i) the states of any two solutions, whose initial conditions are 'close' to each other, remain 'close' to each other for all positive time; (ii) that the states of any two solutions converge towards each other as time proceeds. Related stability notions are those of convergent systems (e.g., [4], [15]) and contraction (e.g., [5], [12]). In the current paper, the focus is on novel definitions of incremental asymptotic stability for hybrid systems in the formalism of [8], although we project that this work will also support further developments on establishing sufficient conditions for contraction and novel definitions of convergence for hybrid systems.

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The majority of the literature on incremental stability (and related stability notions) focuses on smooth dynamical systems either in continuous time or in discrete time. Some works addressing such stability properties for classes of non-smooth systems can be found in [16], [17], [21], [22], [23], [25]. Results on incremental stability for hybrid dynamical systems are rare. An exception is the recent work in [11], where incremental stability is studied for a class of hybrid systems in the formalism of [8]. Results on convergence for a class of measure differential inclusions can be found in [9].

The study of incremental stability involves comparing (the distance between) two solutions associated with different initial conditions at a certain time instant. The analysis of incremental stability for hybrid systems in the formalism of [8] is challenging for two reasons (both associated with the hybrid nature of the dynamics). Firstly, in [8], solutions to hybrid systems are defined for hybrid time instants, which are the pairs consisting of the ordinary continuous time and of the discrete time, which is the number of jumps experienced by the solution so far. As any two solutions to the same hybrid system do not necessarily have identical hybrid time domains, we cannot directly use available definitions of incremental stability for continuous- and discrete-time systems for hybrid systems, as it is not a priori clear at which hybrid time instants solutions should be compared. Secondly, earlier works in [2], [6] (although in the scope of tracking control and not in the scope of incremental stability) have shown that the fact that close solutions may exhibit jumps at (close but) distinct time instants implies that a conventional Euclidean distance function is not suitable for generic hybrid systems with state-triggered jumps. Both issues need to be addressed carefully when proposing definitions for incremental stability of hybrid systems.

The authors of [11] have presented a definition of incremental stability in which solutions are compared at the same continuous time instant, but possibly corresponding to different discrete times. This can be justified in many applications, for example in mechanical systems with impacts, where the number of jumps the solution has encountered is typically irrelevant. It is assumed for this purpose that the maximal solutions to the hybrid system (i.e. the solutions that cannot be extended) have an unbounded domain in the continuous time direction and that they cannot jump several times instantaneously. Like in [1], an extended system is then proposed and Lyapunov-based conditions are given.

In this paper, we present different definitions of incremental stability for hybrid systems compared to [11]. We start by recalling the concept of ε-closeness of hybrid arcs (see [8]), which provides a notion of closeness for two solutions

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with possibly different time domains. We then propose a definition of (pre-)incremental stability, which essentially says that for any two (maximal) solutions, with 'close' initial conditions, we have that their state evolutions and their time domains remain 'close' for all time and converge to each other when time progresses. Contrary to the definition of ε-closeness of hybrid arcs, we use a generic mapping to evaluate the distance between the states of the solutions, and not necessarily the Euclidean distance. This is justified by the fact that the latter may be restrictive in the context of incremental stability as demonstrated in [24], [25] for continuous-time systems and in [2] for hybrid systems. Moreover, it has been shown in [24] that incremental stability is not a coordinate-invariant property for continuous-time systems when exclusively considering the Euclidean distance.

We also introduce the weaker notion of *flow uniform (pre-)incremental asymptotic stability* to denote systems which verify uniform incremental asymptotic stability properties with respect to the continuous time. The idea is similar to Definition 3.1 in [19] and [11]. It consists of evaluating the distance between two solutions at 'close' continuous times, while tolerating an offset between the discrete times at which the two solutions are compared. This definition generalizes the one in [11], as it relaxes the assumption on the maximal solutions mentioned above. Furthermore, we use more generic mappings to evaluate the distance between two solutions and we tolerate a mismatch between the continuous times at which the solutions are compared, which provides more flexibility. Additionally, we define the symmetric notion of *jump uniform (pre-)incremental asymptotic stability* for hybrid systems which exhibit incremental stability properties with respect to the discrete time. This definition is relevant for systems for which the discrete time is dominant.

Examples are provided throughout the paper to illustrate the definitions. Moreover, the relations between the three definitions are investigated. We finally show that, if we embed a uniformly incrementally asymptotically stable continuous-time (respectively, discrete-time) system as a hybrid system, it is uniformly incrementally asymptotically stable according to our definitions, thereby showing the consistency of our definitions with existing 'classical' ones. The proofs are omitted for the sake of brevity.

II. Preliminaries

Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \ldots\}$, $\mathbb{Z}_{>0} := \{1, 2, \ldots\}$. For $(x, y) \in \mathbb{R}^{n+m}$, (x, y) stands for $[x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}}$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. A continuous function $\gamma : \mathbb{R}^2_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each $t \in \mathbb{R}_{\geq 0}$, $\gamma(\cdot, t)$ is of class $\mathcal{K}$, and, for each $s \in \mathbb{R}_{\geq 0}$, $\gamma(s, \cdot)$ is decreasing to zero. Let $x \in \mathbb{R}$, $\lceil x \rceil := \min\{a \in \mathbb{Z} : x \leq a\}$. For a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, the domain of F is the set $\operatorname{dom} F := \{x \in \mathbb{R}^n : F(x) \neq \emptyset\}$.

We study hybrid systems of the form [8]

$$\dot{x} \in F(x) \quad x \in C, \qquad x^+ \in G(x) \quad x \in D, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, F is the flow map, G is the jump map, C is the flow set and D is the jump set. We assume that system (1) satisfies the *hybrid basic conditions* (see Assumption 6.5 in [8]), i.e. the following holds: (i) C and D are closed subsets of $\mathbb{R}^n$; (ii) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous[1] and locally bounded[2] relative to C, $C \subset \operatorname{dom} F$, and $F(x)$ is convex for each $x \in C$; (iii) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to D, and $D \subset \operatorname{dom} G$. These conditions ensure that system (1) is *well-posed*, see Chapter 6 in [8] for more details.

We recall some definitions related to [8]. A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, \ldots, J\}) = \bigcup_{j \in \{0,1,\ldots,J-1\}} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \ldots \leq t_J$. A function $\phi : E \to \mathbb{R}^n$ is a *hybrid arc* if E is a hybrid time domain and if for each $j \in \mathbb{Z}_{\geq 0}$, $t \mapsto \phi(t, j)$ is locally absolutely continuous on $I^j := \{t : (t, j) \in E\}$. The hybrid arc $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ is a *solution* to (1) if: (i) $\phi(0, 0) \in C \cup D$; (ii) for any $j \in \mathbb{Z}_{\geq 0}$, $\phi(t, j) \in C$ and $\frac{d}{dt}\phi(t, j) \in F(\phi(t, j))$ for almost all $t \in I^j$; (iii) for every $(t, j) \in \operatorname{dom} \phi$ such that $(t, j+1) \in \operatorname{dom} \phi$, $\phi(t, j) \in D$ and $\phi(t, j+1) \in G(\phi(t, j))$. A solution ϕ to (1) is *maximal* if it cannot be extended, and it is *complete* if $\operatorname{dom} \phi$ is unbounded. Note that a solution may be maximal but not complete. For a solution ϕ to (1), $\sup_t \operatorname{dom} \phi := \sup\{t \in \mathbb{R}_{\geq 0} : \exists j \in \mathbb{Z}_{\geq 0}\ (t, j) \in \operatorname{dom} \phi\}$ and $\sup_j \operatorname{dom} \phi := \sup\{j \in \mathbb{Z}_{\geq 0} : \exists t \in \mathbb{R}_{\geq 0}\ (t, j) \in \operatorname{dom} \phi\}$.

III. From Graphical Closeness to Incremental Asymptotic Stability

To define incremental stability for hybrid systems, we need to evaluate the distance between any two solutions of system (1). A feature of this system is that two solutions do not have the same hybrid time domain in general (examples are provided below). As a consequence, we may not be able to compare them at the same (hybrid) time instant. To avoid that issue, we resort to graphical closeness concepts. In particular, the definitions we propose below are inspired by the notion of ε-closeness of hybrid arcs, see Definition 4.11 in [8], which is related to the Hausdorff distance between the graphs of the hybrid arcs and which we recall below.

Definition 1: Given $\varepsilon > 0$, two hybrid arcs ϕ_1 and ϕ_2 are ε-close if:

(i) for all $(t, j) \in \operatorname{dom} \phi_1$ there exists s such that $(s, j) \in \operatorname{dom} \phi_2$, $|t - s| < \varepsilon$ and $|\phi_1(t, j) - \phi_2(s, j)| < \varepsilon$.

(ii) for all $(t, j) \in \operatorname{dom} \phi_2$ there exists s such that $(s, j) \in \operatorname{dom} \phi_1$, $|t - s| < \varepsilon$ and $|\phi_2(t, j) - \phi_1(s, j)| < \varepsilon$. $\square$

In Definition 1, the hybrid arcs ϕ_1 and ϕ_2 are not compared at the same hybrid time instant (t, j) but at (t, j) for one and (s, j) for the other, with $|t - s| < \varepsilon$. In that way, $\operatorname{dom} \phi_1$ and $\operatorname{dom} \phi_2$ do not need to be equal, they only need to be 'close' enough so that for any $(t, j) \in \operatorname{dom} \phi_1$ there exists an appropriate pair $(s, j) \in \operatorname{dom} \phi_2$ and vice-versa.

[1]The set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous if its graph $\{(y, z) : y \in \mathbb{R}^n,\ z \in F(y)\}$ is closed, see Lemma 5.10 in [8].

[2]See Definition 5.14 in [8].

Definition 1 may therefore be used to compare two solutions to (1) at any time instant, even though these do not have same hybrid time domain.

The distance between two hybrid arcs is evaluated using the Euclidean distance in Definition 1, which may be restrictive in the context of incremental stability as discussed in the introduction. Inspired by [20], we use a generic positive function, which we denote δ, instead of the Euclidean distance, to compare the states of two hybrid solutions and we will talk of incremental stability properties with respect to a certain δ, which also allows for 'output' incremental stability (as opposed to incremental stability for the full state). We concentrate on mappings $\delta : \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$ which belong to the set $\mathcal{D}$ of continuous mappings that verify for any $(x_1, x_2) \in \mathbb{R}^{2n}$: (i) $\delta(x_1, x_2) = \delta(x_2, x_1)$; (ii) $x_1 = x_2 \Rightarrow \delta(x_1, x_2) = 0$. The first condition means that δ is symmetric and the second one states that δ vanishes when $x_1 = x_2$. In that way, the functions in $\mathcal{D}$ are general enough to encompass the metrics considered in [2], [24], [25] as particular cases and to accommodate the features of hybrid systems for which it may be restrictive to ask that $\delta(x_1, x_2) = 0$ implies $x_1 = x_2$. In this manner, we can consider distance functions where the set $\{(x_1, x_2) : \delta(x_1, x_2) = 0\}$ is larger than the diagonal $\{(x_1, x_2) : x_1 = x_2\}$, but still corresponds to a behaviour that is desired in applications, see [2] for instance.

In view of Definition 1 and the discussion above, we propose the following definition of incremental asymptotic stability.

Definition 2: Given $\delta \in \mathcal{D}$, system (1) is *uniformly pre-incrementally asymptotically stable with respect to δ in graphical sense (δ-UpIS)* if the following conditions hold:

(i) for any $\varepsilon > 0$, there exists $s > 0$ such that for any pair of maximal solutions (ϕ_1, ϕ_2) with $\delta(\phi_1(0,0), \phi_2(0,0)) < s$ it holds that, for all $(t,j) \in \operatorname{dom}\phi_1$, there exists $(t', j) \in \operatorname{dom}\phi_2$ with $|t - t'| < \varepsilon$ such that $\delta(\phi_1(t,j), \phi_2(t',j)) < \varepsilon$;

(ii) for any $\varepsilon > 0$ and $r > 0$, there exists $\Theta \geq 0$ such that for any pair of maximal solutions (ϕ_1, ϕ_2) with $\delta(\phi_1(0,0), \phi_2(0,0)) < r$ it holds that, for all $(t,j) \in \operatorname{dom}\phi_1$ with $t + j \geq \Theta$, there exists $(t', j) \in \operatorname{dom}\phi_2$ with $|t - t'| < \varepsilon$ such that $\delta(\phi_1(t,j), \phi_2(t',j)) < \varepsilon$.

System (1) is *uniformly incrementally asymptotically stable with respect to δ in graphical sense (δ-UIS)* when it is δ-UpIS and any maximal solution to (1) is complete. $\square$

Item (i) of Definition 2 is a uniform global stability property of all maximal solutions. It means that for any $\varepsilon > 0$, there exists $s > 0$ such that any two maximal solutions ϕ_1 and ϕ_2 are ε-close (in the distance function δ) as long as $\delta(\phi_1(0,0), \phi_2(0,0)) < s$. Item (ii) of Definition 2 is a uniform global attractivity property of all the maximal solutions. It requires that for any $\varepsilon, r > 0$ there exists $\Theta > 0$ such that any two maximal solutions ϕ_1 and ϕ_2 such that $\delta(\phi_1(0,0), \phi_2(0,0)) < s$ are ε-close (in the distance function δ) after a uniform amount of 'time' Θ. Notice that we do not explicitly state symmetric statements as in Definition 1, as items (i) and (ii) of Definition 2 hold for *any pair* of maximal solutions.

Definition 2 not only requires the states of any two solutions to remain close and to converge to each other, it also requires their hybrid time domains to be close and to converge to each other, which is a strong requirement as confirmed by the proposition below.

Proposition 1: Consider system (1) and suppose it is δ-UIS for a given $\delta \in \mathcal{D}$. Then, one of the following properties holds: (i) $\operatorname{dom}\phi = \mathbb{R}_{\geq 0} \times \{0\}$ for any maximal solution ϕ; (ii) $\operatorname{dom}\phi = \{0\} \times \mathbb{Z}_{\geq 0}$ for any maximal solution ϕ. $\square$

Proposition 1 implies that, if system (1) is δ-UIS (whatever $\delta \in \mathcal{D}$), it is either a purely continuous-time system or a purely discrete-time system, which is clearly restrictive. In the next sections, we present alternative definitions to characterize hybrid systems which exhibit incremental stability properties with respect to the continuous time, or the discrete time, respectively.

IV. Flow incremental asymptotic stability

The definition below is relevant for systems for which the continuous time is considered dominant over the discrete (jump) time.

Definition 3: Given $\delta \in \mathcal{D}$, system (1) is *flow uniformly pre-incrementally asymptotically stable with respect to δ (δ-FUpIS)* if the following conditions hold:

(i) for any $\varepsilon > 0$, there exists $s > 0$ such that for any pair of maximal solutions (ϕ_1, ϕ_2) with $\delta(\phi_1(0,0), \phi_2(0,0)) < s$ it holds that, for all $(t,j) \in \operatorname{dom}\phi_1$, there exists $(t', j') \in \operatorname{dom}\phi_2$ with $|t - t'| < \varepsilon$ such that $\delta(\phi_1(t,j), \phi_2(t',j')) < \varepsilon$;

(ii) for any $\varepsilon > 0$ and $r > 0$, there exists $T \geq 0$ such that for any pair of maximal solutions (ϕ_1, ϕ_2) with $\delta(\phi_1(0,0), \phi_2(0,0)) < r$ it holds that, for all $(t,j) \in \operatorname{dom}\phi_1$ with $t \geq T$, there exists $(t', j') \in \operatorname{dom}\phi_2$ with $|t - t'| < \varepsilon$ such that $\delta(\phi_1(t,j), \phi_2(t',j')) < \varepsilon$.

System (1) is *flow uniformly incrementally asymptotically stable with respect to δ (δ-FUIS)* when, in addition, any maximal solution ϕ to (1) is such that $\sup_t \operatorname{dom}\phi = \infty$. $\square$

Item (i) of Definition 3 is a uniform global stability property. It implies that any two solutions ϕ_1 and ϕ_2, which are initialized close to each other (where δ is used to evaluate the distance between the initial conditions) remain close to each other at some close continuous times, while discarding the numbers of jumps the solutions have experienced. It also implies that $\sup_t \operatorname{dom}\phi_1$ and $\sup_t \operatorname{dom}\phi_2$ are 'close' (otherwise there may not exist $(t', j') \in \operatorname{dom}\phi_2$ such that $|t - t'| < \varepsilon$ in item (i) of Definition 3). Item (ii) is a uniform global attractivity property of every solution, as the constant T is the same for all maximal solutions ϕ_1 and ϕ_2 with $\delta(\phi_1(0,0), \phi_2(0,0)) < r$, given $\varepsilon, r > 0$. It can be noted that the time mismatch $t - t'$ of the solutions in Definition 3 reminds of Zhukovsky stability for continuous-time systems, see e.g., Chapter 8.4 in [10]. If δ is the Euclidean distance, this small time mismatch $t - t'$ does not allow for the 'peaking phenomenon' of the error $\delta(\phi_1(t,j), \phi_2(t,j'))$ to occur as described in e.g., [2], [6], [9], [13]. Still, systems exhibiting the peaking phenomenon can be FU(p)IS in another δ.

Loosely speaking, Definition 3 consists of projecting the state trajectories on the hyperplane (t, x) and evaluating the distance between two solutions on this hyperplane.

Remark 1: Definition 3 differs from the definitions in [11] on several points. First, a solution may experience two consecutive jumps (see Example 1 for instance) and the maximal solutions to system (1) are not required to be complete in the t-direction in the definition of δ-FUpIS, which relaxes Assumption 3.1 in [11]. Second, the solutions ϕ_1 and ϕ_2 in Definition 3 are not compared at the same continuous time t but at two (potentially) distinct times t and t' with $|t - t'| < \varepsilon$, which provides more flexibility. Third, the function δ is not constrained to be the Euclidean distance. □

We derive from Definition 3 that, when there exists a pair of maximal solutions ϕ_1 and ϕ_2 with $\sup_t \operatorname{dom} \phi_1 = \infty$ and $\sup_t \operatorname{dom} \phi_2 < \infty$, the system can never be δ-FUpIS for any $\delta \in \mathcal{D}$, as item (i) of Definition 3 can never be satisfied. Hence, either all maximal solutions should have an unbounded domain in the t-direction or all should have a bounded one for the system to be δ-FUpIS. In the first case, δ-FUpIS would immediately become δ-FUIS. We also remark that when $\sup_t \operatorname{dom} \phi < T' < \infty$ for all maximal solutions ϕ to (1), with $T' > 0$, then item (ii) of Definition 3 trivially holds by taking $T = T'$.

A simple example of a δ-FUIS hybrid system is provided below.

Example 1: [A δ-FUIS system] Consider the system

$$\begin{array}{rclr} \begin{pmatrix} \dot{x} \\ \dot{\sigma} \end{pmatrix} & \in & \begin{pmatrix} -x \\ [0, \rho] \end{pmatrix} & (x, \sigma) \in C \\ \begin{pmatrix} x^+ \\ \sigma^+ \end{pmatrix} & = & \begin{pmatrix} \min\{x, 1\} \\ \sigma - 1 \end{pmatrix} & (x, \sigma) \in D, \end{array} \tag{2}$$

where $C = \{(x, \sigma) : x \in [0, 1] \text{ and } \sigma \in [0, N]\}$, $D = \{(x, \sigma) : x \in [1, \infty) \text{ and } \sigma \in [1, N]\}$, $\rho > 0$ and $N \in \mathbb{Z}_{>0}$. The σ-subsystem is an average dwell-time automaton (see [3], [6]) which is used to rule out solutions which jump infinitely many times when $x = 1$. The parameters ρ and N are arbitrarily selected. Hence, any maximal solution has an unbounded domain in the continuous time direction. It can be noted that the (maximal) solutions to (2) do not a priori have the same hybrid time domain. Take for instance $\phi_1 : (t, 0) \mapsto (e^{-t} x_1(0, 0), 0)$ with $x_1(0, 0) \in [0, 1)$ and $\phi_2(0, 0) = (2, 1)$ and $\phi_2(t, 1) = (e^{-t}, 0)$ for $t \geq 0$. We have $\operatorname{dom} \phi_1 = \mathbb{R}_{\geq 0} \times \{0\} \neq (\{0\} \times \{0\}) \cup (\mathbb{R}_{\geq 0} \times \{1\}) = \operatorname{dom} \phi_2$.

Let $\phi = (x, \sigma)$ be a maximal solution to (2). For any $(t, j) \in \operatorname{dom} \phi$ with $t > 0$,

$$x(t, j) = e^{-t} \min\{x(0, 0), 1\}. \tag{3}$$

We are going to use this expression to prove that system (2) is δ-FUIS where $\delta : \mathbb{R}^4 \to \mathbb{R}$ and $\delta : (x_1, \sigma_1, x_2, \sigma_2) \mapsto |x_1 - x_2|$. We first verify that item (i) of Definition 3 holds. Let $\varepsilon > 0$, $s = \varepsilon$ and $\phi_1 = (x_1, \sigma_1)$, $\phi_2 = (x_2, \sigma_2)$ be two maximal solutions to (2) such that $\delta(\phi_1(0, 0), \phi_2(0, 0)) = |x_1(0, 0) - x_2(0, 0)| < s$. Let $(t, j) \in \operatorname{dom} \phi_1$. When $t = 0$,

- if $x_1(0, j) = x_1(0, 0)$ so $j = 0$ or $x_1(0, 0) \in [0, 1]$, and in both cases $|x_1(0, j) - x_2(0, 0)| = |x_1(0, 0) - x_2(0, 0)| < s = \varepsilon$;
- if $x_1(0, j) \neq x_1(0, 0)$, necessarily $x_1(0, j) = 1$, $j \geq 1$ and $x_1(0, 0) > 1$. If $x_2(0, 0) > 1$, $(0, 1) \in \operatorname{dom} \phi_2$, $x_2(0, 1) = 1$ and $|x_1(0, j) - x_2(0, 1)| = 0$. If $x_2(0, 0) \leq 1$, $|x_1(0, j) - x_2(0, 0)| = x_1(0, j) - x_2(0, 0) < |x_1(0, 0) - x_2(0, 0)| < s = \varepsilon$.

When $t > 0$, in view of (3) and the observation that maximal solutions to (2) have hybrid time domains that are unbounded in t-direction, for any $(t, j) \in \operatorname{dom} \phi_1$ with $t > 0$, there exists $j' \in \mathbb{Z}_{\geq 0}$ such that $(t, j') \in \operatorname{dom} \phi_2$ and $|x_1(t, j) - x_2(t, j')| = e^{-t} |\min\{x_1(0, 0), 1\} - \min\{x_2(0, 0), 1\}|$. Noting that $|\min\{x_1(0, 0), 1\} - \min\{x_2(0, 0), 1\}| \leq |x_1(0, 0) - x_2(0, 0)|$, we have $|x_1(t, j) - x_2(t, j')| \leq e^{-t} |x_1(0, 0) - x_2(0, 0)| < s = \varepsilon$. Hence item (i) of Definition 3 is verified. Note that $t' = t$ here.

Let $\varepsilon, r > 0$ and ϕ_1, ϕ_2 be two maximal solutions to (2) such that $\delta(\phi_1(0, 0), \phi_2(0, 0)) = |x_1(0, 0) - x_2(0, 0)| < r$. For all $(t, j) \in \operatorname{dom} \phi_1$ with $t \geq T$ and $T > \max\{0, \ln(\frac{r}{\varepsilon})\}$, there exists $j' \in \mathbb{Z}_{\geq 0}$ with $(t, j') \in \operatorname{dom} \phi_2$ and $|x_1(t, j) - x_2(t, j')| = e^{-t} |\min\{x_1(0, 0), 1\} - \min\{x_2(0, 0), 1\}| \leq e^{-T} r < \varepsilon$. Item (ii) of Definition 3 is guaranteed with $t' = t$. We have proved that system (2) is δ-FUIS. ■

V. Jump Incremental Asymptotic Stability

Similar to flow incremental asymptotic stability, we define below the symmetric notion of jump incremental asymptotic stability.

Definition 4: Given $\delta \in \mathcal{D}$, system (1) is *jump uniformly pre-incrementally asymptotically stable with respect to δ (δ-JUpIS)* if the following conditions hold:

(i) for any $\varepsilon > 0$, there exists $s > 0$ such that for any pair of maximal solutions (ϕ_1, ϕ_2) with $\delta(\phi_1(0, 0), \phi_2(0, 0)) < s$ it holds that, for all $(t, j) \in \operatorname{dom} \phi_1$, there exists $(t', j) \in \operatorname{dom} \phi_2$ such that $\delta(\phi_1(t, j), \phi_2(t', j)) < \varepsilon$;

(ii) for any $\varepsilon > 0$ and $r > 0$, there exists $J \geq 0$ such that for any pair of maximal solutions (ϕ_1, ϕ_2) with $\delta(\phi_1(0, 0), \phi_2(0, 0)) < r$ it holds that, for all $(t, j) \in \operatorname{dom} \phi_1$ with $j \geq J$, there exists $(t', j) \in \operatorname{dom} \phi_2$ such that $\delta(\phi_1(t, j), \phi_2(t', j)) < \varepsilon$.

System (1) is *jump uniformly incrementally asymptotically stable with respect to δ (δ-JUIS)* when, in addition, any maximal solution ϕ to (1) is such that $\sup_j \operatorname{dom} \phi = \infty$. □

In item (i) of Definition 4, the distance between two solutions is evaluated at the discrete time j, without imposing any conditions on the continuous times by opposition to Definition 3. It has to be noted that the solutions ϕ_1 and ϕ_2 in items (i) and (ii) of Definition 4 are evaluated at the same discrete time j, and not at j and j', respectively, with $|j - j'| < \varepsilon$ as we might expect. That is justified by the fact that when $\varepsilon < 1$, $|j - j'| < \varepsilon$ implies that $j = j'$ since $j, j' \in \mathbb{Z}_{\geq 0}$. Since the satisfaction of items (i) and (ii) of Definition 4 for any $\varepsilon \in (0, 1)$ implies its satisfaction for any $\varepsilon \geq 1$, there is no loss of generality in evaluating ϕ_1 and ϕ_2 at the same discrete time j. We emphasize again that item (ii) of Definition 4 is a uniform attractivity property, as the constant J is the same for all maximal solutions ϕ_1 and

ϕ_2 with $\delta(\phi_1(0,0),\phi_2(0,0)) < r$, given $\varepsilon, r > 0$. Compared to Definition 3, Definition 4 somehow consists of projecting the state trajectories on the hyperplane (j,x) (and not (t,x)) and evaluating the distance between two solutions on this hyperplane.

Similar observations as for Definition 3 can be made. For instance, when there exists a pair of maximal solutions (ϕ_1,ϕ_2) with $\sup_j \operatorname{dom}\phi_1 = \infty$ and $\sup_j \operatorname{dom}\phi_2 < \infty$, the system can never be δ-JUpIS for any $\delta \in \mathcal{D}$, which implies that all maximal solutions either have an unbounded domain in the j-direction or a bounded one for the system to be δ-JUpIS. An example of a hybrid system which is JUIS with respect to the Euclidean distance is provided below.

Example 2 (A δ-JUIS system): Consider the system

$$\dot{x} = -1 \quad x \in [1,\infty), \qquad x^+ = \tfrac{1}{2}x \quad x \in [0,1]. \tag{4}$$

We note that any maximal solution has an unbounded domain in the discrete time direction, as they all reach in finite continuous time the set $D = [0,1]$ and then they jump infinitely many times. We take δ to be the Euclidean distance. Let $\varepsilon > 0$, $s = \varepsilon$ and ϕ_1, ϕ_2 be two maximal solutions to (4) such that $\delta(\phi_1(0,0),\phi_2(0,0)) < s$. We distinguish four cases to study item (i) in Definition 4.

Case 1: $(\phi_1(0,0),\phi_2(0,0)) \in D^2$.
We have that $\operatorname{dom}\phi_1 = \operatorname{dom}\phi_2 = \{0\}\times\mathbb{Z}_{\geq 0}$. Let $(0,j) \in \{0\}\times\mathbb{Z}_{\geq 0}$, then $\delta(\phi_1(0,j),\phi_2(0,j)) = \frac{1}{2^j}|\phi_1(0,0) - \phi_2(0,0)| < s = \varepsilon$.

Case 2: $(\phi_1(0,0),\phi_2(0,0)) \in C^2$.
We have $\operatorname{dom}\phi_i = ([0,\phi_i(0,0)-1]\times\{0\}) \cup (\{\phi_i(0,0)-1\}\times\mathbb{Z}_{>0})$ for $i \in \{1,2\}$. Consider $(t,0) \in \operatorname{dom}\phi_1$. If $(t,0) \in \operatorname{dom}\phi_2$, then $\delta(\phi_1(t,0),\phi_2(t,0)) = |\phi_1(0,0)-t-(\phi_2(0,0)-t)| < s = \varepsilon$. If $(t,0) \notin \operatorname{dom}\phi_2$, that means that $t > \phi_2(0,0)-1$, but $t \leq \phi_1(0,0)-1$, therefore $\phi_2(0,0) < \phi_1(0,0)$. If $\phi_1(t,0) > \phi_2(0,0)$, then $|\phi_1(t,0)-\phi_2(0,0)| = \phi_1(t,0)-\phi_2(0,0) \leq \phi_1(0,0)-\phi_2(0,0) < s = \varepsilon$. Finally, if $\phi_1(t,0) \leq \phi_2(0,0)$, by continuity of $\phi_2(\cdot,0)$ on $[0,\phi_2(0,0)-1)$, there exists $(t',0) \in \operatorname{dom}\phi_2$ such that $\phi_1(t,0) = \phi_2(t',0)$ and $\delta(\phi_1(t,0),\phi_2(t',0)) = 0$. Let $j > 0$ and $(t,j) \in \operatorname{dom}\phi_1$, then we observe $\phi_1(t,j) = \frac{1}{2^j}$. There exists $(t',j) \in \operatorname{dom}\phi_2$ such that $\phi_2(t',j) = \frac{1}{2^j}$. Hence, $\delta(\phi_1(t,j),\phi_2(t',j)) = 0$.

Case 3: $(\phi_1(0,0),\phi_2(0,0)) \in D\times C$.
Let $(t,j) \in \operatorname{dom}\phi_1$. If $j = 0$, necessarily $(t,j) = (0,0)$ and the desired result holds. If $j > 0$, $\phi_1(0,j) = \frac{1}{2^j}\phi_1(0,0)$ and there exists $(t',j) \in \operatorname{dom}\phi_2$ such that $\phi_2(t',j) = \frac{1}{2^j}$. Hence, $\delta(\phi_1(0,j),\phi_2(t',j)) = \frac{1}{2^j}(1-\phi_1(0,0)) \leq \frac{1}{2^j}(\phi_2(0,0) - \phi_1(0,0)) < s = \varepsilon$.

Case 4: $(\phi_1(0,0),\phi_2(0,0)) \in C\times D$.
Let $(t,j) \in \operatorname{dom}\phi_1$. If $j = 0$, $\phi_1(t,0) = \phi_1(0,0)-t$ for $(t,0) \in \operatorname{dom}\phi_1$. Hence $|\phi_1(t,0)-\phi_2(0,0)| = \phi_1(0,0)-t-\phi_2(0,0) \leq \phi_1(0,0)-\phi_2(0,0) < s = \varepsilon$. If $j > 0$, $\phi_1(t,j) = \frac{1}{2^j}$ and $\phi_2(0,j) = \frac{1}{2^j}\phi_2(0,0)$. Hence, $\delta(\phi_1(t,j),\phi_2(0,j)) < s = \varepsilon$ in view of the arguments at the end of Case 3. We have proved that item (i) of Definition 4 holds.

We now study item (ii) of Definition 4. Let $\varepsilon, r > 0$ and ϕ_1, ϕ_2 be two maximal solutions to (4) with $\delta(\phi_1(0,0),\phi_2(0,0)) < r$. For any $j > 0$, it holds that $\phi_i(t,j) = \frac{1}{2^j}\min\{\phi_i(0,0),1\}$ for $(t,j) \in \operatorname{dom}\phi_i$, $i \in \{1,2\}$. Hence, for any $(t,j) \in \operatorname{dom}\phi_1$ with $j > 0$, there exists $(t',j) \in \operatorname{dom}\phi_2$ such that $\delta(\phi_1(t,j),\phi_2(t',j)) = \frac{1}{2^j}|\min\{\phi_1(0,0),1\} - \min\{\phi_2(0,0),1\}| \leq \frac{1}{2^j}|\phi_1(0,0) - \phi_2(0,0)| < \frac{1}{2^j}r$. We obtain the desired result by selecting $J = \lceil \frac{\ln(\frac{r}{\varepsilon})}{\ln(2)} \rceil$. As a conclusion, we have shown that system (4) is JUIS with respect to the Euclidean distance. ■

VI. Relations between the definitions

A system which is δ-FU(p)IS is not necessarily δ-JU(p)IS and vice versa, as demonstrated by the following examples.

Example 3 (δ-FUIS but not δ-JUpIS): System (2) has been shown to be δ-FUIS with $\delta : (x_1,\sigma_1,x_2,\sigma_2) \mapsto |x_1 - x_2|$. Nonetheless, it cannot be δ-JUpIS as some maximal solutions have an unbounded domain in the j-direction (consider those for which $\dot{\sigma} = \frac{\rho}{2}$ for instance) and some have a bounded domain in this direction (when σ remains constant on flows). As a consequence, item (i) of Definition 4 does not hold. ■

Example 4 (δ-JUIS but not δ-FUpIS): Consider system (4) and suppose, in order to attain a contradiction, that it is FUpIS with respect to the Euclidean distance. As a consequence, for $r > 1$ and $\varepsilon \in (0,\frac{r}{2})$, there exists $T \geq 0$ such that the statement in item (ii) of Definition 3 holds. Let ϕ_1 and ϕ_2 be two maximal solutions with $\phi_1(0,0) = (\alpha + \frac{1}{2})r$ and $\phi_2(0,0) = \alpha r$ where $\alpha > 1$ is a parameter we are free to select. We see that $|\phi_1(0,0) - \phi_2(0,0)| = \frac{r}{2} < r$. Moreover, since $\alpha r > 1$, $\operatorname{dom}\phi_i = ([0,\phi_i(0,0)-1]\times\{0\}) \cup (\{\phi_i(0,0)-1\}\times\mathbb{Z}_{>0})$ for $i \in \{1,2\}$. We select α sufficiently large such that $\phi_1(0,0) - 1 = (\alpha + \frac{1}{2})r - 1 \geq T$. Let $t = \phi_1(0,0)-1$ and $j \in \mathbb{Z}_{\geq 0}$ be such that $(t,j) \in \operatorname{dom}\phi_1$. According to item (ii) of Definition 3, there exists $(t',j') \in \operatorname{dom}\phi_2$ such that $|t - t'| < \varepsilon$. Note that $t' \leq \phi_2(0,0)-1$ by definition of $\operatorname{dom}\phi_2$. Consequently, $|t-t'| = \phi_1(0,0)-1-t' \geq \phi_1(0,0)-\phi_2(0,0)$. We deduce that $\frac{r}{2} = \phi_1(0,0)-\phi_2(0,0) \leq |t-t'| < \varepsilon$. This contradicts the fact that $\varepsilon \in (0,\frac{r}{2})$. As a consequence, system (8) is not FUpIS with respect to the Euclidean distance, although it is JUIS with respect to this distance. ■

On the other hand, a system can be both δ-FU(p)IS and δ-JU(p)IS; examples are mentioned in Section VII.

The proposition below shows the connections between Definition 2 and Definitions 3-4.

Proposition 2: Let $\delta \in \mathcal{D}$. The following statements hold.

(i) If system (1) is δ-UpIS, then it is both δ-FUpIS and δ-JUpIS.

(ii) If system (1) is δ-UIS, then it is either δ-FUIS or δ-JUIS.

(iii) If system (1) is both δ-FUpIS and δ-JUpIS, it is not necessarily δ-UpIS. □

Item (iii) of Proposition 2 is due to the fact that the hybrid time domains of the solutions play a very important role for δ-UIS. Indeed, a system may very well be both δ-FUpIS and δ-JUpIS, and not δ-UpIS (for some $\delta \in \mathcal{D}$), because two (maximal) solutions, which have close initial conditions

δ-FUIS	$\Rightarrow$	δ-FUpIS	$\not\Rightarrow$	δ-JUpIS
δ-JUIS	$\Rightarrow$	δ-JUpIS	$\not\Rightarrow$	δ-FUpIS
δ-UIS	$\Rightarrow$	δ-UpIS	$\underset{\not\Leftarrow}{\Rightarrow}$	δ-FUpIS and δ-JUpIS
δ-UIS	$\Rightarrow$	δ-FUIS or δ-JUIS		

Fig. 1. Relations between Definitions 2, 3, and 4.

according to the distance δ do not have 'close' hybrid time domains. A summary of the relations between Definitions 2, 3, and 4 is provided in Figure 1.

VII. Consistency with definitions for continuous-time and discrete-time systems

The proposition below shows that the proposed definitions are consistent with the definitions of incremental stability available in the literature for continuous-time systems.

Proposition 3: Consider the continuous-time system $\dot{x} \in f(x)$, where $x \in \mathbb{R}^n$, and $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded on $\mathbb{R}^n$, and $f(x)$ is convex for each $x \in \mathbb{R}^n$. Suppose that any maximal solution is complete and that there exist $\delta \in \mathcal{D}$ and $\beta \in \mathcal{KL}$ such that any pair of maximal solutions (x_1, x_2) verifies for all $t \geq 0$, $\delta(x_1(t), x_2(t)) \leq \beta(\delta(x_1(0), x_2(0)), t)$. Then, hybrid system (1) with $F(x) = f(x)$, $C = \mathbb{R}^n$, $G(x) = \{x\}$ and $D = \emptyset$, for $x \in \mathbb{R}^n$, is δ-FUIS and δ-UIS. $\square$

Proposition 3 states that if a continuous-time system is uniformly incrementally asymptotically stable, then this property is preserved when this system is embedded as a hybrid system of the form (1). Note that the choice of G in Proposition 3 has no impact on the result. The following proposition states an equivalent result for discrete-time systems. Incremental stability of discrete-time systems is investigated in e.g., [12], [14].

Proposition 4: Consider the discrete-time system $x^+ \in g(x)$, where $x \in \mathbb{R}^n$, $g : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded on $\mathbb{R}^n$, and nonempty for all $x \in \mathbb{R}^n$. Suppose that this system is incrementally asymptotically stable with respect to $\delta \in \mathcal{D}$, in the sense that there exists $\beta \in \mathcal{KL}$ such that for any pair of maximal solutions (x_1, x_2) and $k \in \mathbb{Z}_{\geq 0}$, $\delta(x_1(k), x_2(k)) \leq \beta(\delta(x_1(0), x_2(0)), k)$. Then, hybrid system (1) with $F(x) = \{x\}$, $C = \emptyset$, $G(x) = g(x)$ and $D = \mathbb{R}^n$, for $x \in \mathbb{R}^n$, is δ-JUIS and δ-UIS. $\square$

VIII. Conclusion

We have proposed a definition of incremental stability for hybrid systems based on the notion of ε-closeness of hybrid arcs. This definition was proved to be rather restrictive, as it only covers the case of purely continuous-time or purely discrete-time systems (when all the maximal solutions to the system are complete). This motivated us to pursue two alternative definitions, which are relevant in situations where either the continuous time is dominant and the number of jumps the solutions have encountered is irrelevant, or the opposite, where the number of jumps of the solutions is dominant and the amount of time the solutions have flowed is not important. The relations between these definitions have been investigated and we have shown that the proposed definitions are consistent with those existing in the literature for purely continuous-time and discrete-time systems.

References

[1] D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*, 47(3):410–421, 2002.

[2] J.J.B. Biemond, N. van de Wouw, W.P.M.H. Heemels, and H. Nijmeijer. Tracking control for hybrid systems with state-triggered jumps. *IEEE Transactions on Automatic Control*, 58(4):876–890, 2013.

[3] C. Cai, A.R. Teel, and R. Goebel. Smooth Lyapunov functions for hybrid systems - Part II: (pre)asymptotically stable compact sets. *IEEE Transactions on Automatic Control*, 53(3):734–748, 2008.

[4] B.P. Demidovich. *Lectures on Stability Theory*. Nauka (in Russian), 1967.

[5] F. Forni and R. Sepulchre. A differential Lyapunov framework for contraction analysis. *IEEE Trans. on Aut. Control*, 59:614–628, 2014.

[6] F. Forni, A.R. Teel, and L. Zaccarian. Follow the bouncing ball: global results on tracking and state estimation with impacts. *IEEE Transactions on Automatic Control*, 58:1470–1485, 2013.

[7] V. Fromion, S. Monaco, and D. Normand-Cyrot. A link between input-output stability and Lyapunov stability. *Systems & Control Letters*, 27:243–248, 1996.

[8] R. Goebel, R.G. Sanfelice, and A.R. Teel. *Hybrid dynamical systems*. Princeton University Press, Princeton, U.S.A., 2012.

[9] R. Leine and N. van de Wouw. *Stability and convergence of mechanical systems with unilateral constraints*. Lecture Notes in Applied and Computational Mechanics, vol. 36. Springer-Verlag, Berlin, 2008.

[10] G.A. Leonov, I.M. Burkin, and A.I. Shepeljavyi. *Frequency methods in oscillation theory*. Kluwer Academic Publishers, Dordrecht, 1996.

[11] Y. Li, S. Phillips, and R.G. Sanfelice. Results on incremental stability for a class of hybrid systems. In *IEEE Conference on Decision and Control, Los Angeles: U.S.A.*, pages 3089–3094, 2014.

[12] W. Lohmiller and J.-J. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696, 1998.

[13] L. Menini and A. Tornambè. Asymptotic tracking of periodic trajectories for a simple mechanical system subject to nonsmooth impacts. *IEEE Transactions on Automatic Control*, 46(7):1122–1126, 2001.

[14] A. Pavlov and N. van de Wouw. Steady-state analysis and regulation of discrete-time nonlinear systems. *IEEE Transactions on Automatic Control*, 57(7):1793–1798, 2012.

[15] A. Pavlov, N. van de Wouw, and H. Nijmeijer. *Uniform output regulation of nonlinear systems: a convergent dynamics approach*. Birkhäuser, Boston, U.S.A., 2006.

[16] A. Pavlov, N. van de Wouw, and H. Nijmeijer. Frequency response functions for nonlinear convergent systems. *IEEE Transactions on Automatic Control*, 52(6):1159 – 1165, 2007.

[17] G. Pola, A. Girard, and P. Tabuada. Approximately bisimilar symbolic models for incrementally stable switched systems. *IEEE Transactions on Automatic Control*, 55(1):116–126, 2010.

[18] B. Rüffer, N. van de Wouw, and M. Mueller. Convergent systems vs. incremental stability. *Syst. & Control Letters*, 62(3):277–285, 2013.

[19] R.G. Sanfelice, A.R. Teel, and R. Sepulchre. A hybrid systems approach to trajectory tracking control for juggling systems. In *IEEE Conference on Decision and Control, New Orleans, U.S.A.*, pages 5282–5287, 2007.

[20] A.R. Teel and L. Praly. A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions. *ESAIM: Control, Optimisation and Calculus of Variations*, 5(1):313–367, 2000.

[21] N. van de Wouw and A. Pavlov. Tracking and synchronisation for a class of pwa systems. *Automatica*, 44:2909–2915, 2008.

[22] R.A. van den Berg, A.Y. Pogromsky, G.A. Leonov, and J.E. Rooda. Design of convergent switched systems. In *Group coordination and cooperative control*, Lecture Notes in Control and Information Sciences, vol. 336, K.Y. Pettersen, J.T. Gravdahl & H. Nijmeijer (Eds.), pages 291–311. Berlin, Germany, Springer, 2006.

[23] V.A. Yakubovich. Matrix inequalities method in stability theory for nonlinear control systems: I. absolute stability of forced vibrations. *Automat. Remote Control*, 7:905–917, 1998.

[24] M. Zamani and P. Tabuada. Backstepping design for incremental stability. *IEEE Trans. on Automatic Control*, 56(9):2184–2189, 2011.

[25] M. Zamani, N. van de Wouw, and R. Majumdar. Backstepping controller synthesis and characterizations of incremental stability. *Systems & Control Letters*, 62(10):949–962, 2013.