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Automatica 39 (2003) 2139–2147
www.elsevier.com/locate/automatica

Brief Paper

On solution concepts and well-posedness of linear relay systems☆

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Received 24 August 2001; received in revised form 30 June 2003; accepted 7 July 2003

Abstract

In this paper we study the well-posedness (existence and uniqueness of solutions) of linear relay systems with respect to two different solution concepts, Filippov solutions and forward solutions. We derive necessary and sufficient conditions for well-posedness in the sense of Filippov of linear systems of relative degree one and two in closed loop with relay feedback. To be precise, uniqueness of Filippov (and also forward) solutions follows in this case if the first non-zero Markov parameter is positive. By means of an example it is shown that this intuitively clear condition is not true for systems with relative degree larger than two. The influence of the Zeno phenomenon (an infinite number of relay switching times in a finite length time interval) on well-posedness is highlighted and although linear relay systems form a rather limited subclass of hybrid dynamical systems, the consequences of the presence of the Zeno behaviour is typical for many other classes of non-smooth and hybrid systems.

Keywords: Hybrid systems; Relay feedback systems; Well-posedness

1. Introduction

Relay systems are important as they are used in various control problems like sliding mode control (Tsypkin, 1984; Utkin, 1978; Filippov, 1988; Johansson, 1997; Johansson, Rantzer, & Åström, 1999c), and in idealised models of (Coulomb) friction phenomena. Still quite some fundamental issues of such systems (even if the underlying system is linear) are unclear and have received considerable attention in recent years. Analysis of simulation methods of such systems (Mattson, 1996; Heemels, Camlibel, & Schumacher, 2000a), well-posedness, that is the existence and uniqueness of solutions (Filippov, 1988; Lootsma, van der Schaft, & Camlibel, 1999), existence of fast switches (Johansson, 1997; Johansson et al., 1999c) are partially explored problem areas within this context.

The focus of this paper is on so-called *linear relay systems* constituted by linear single input single output (SISO) systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t); \qquad y(t) = Cx(t), \tag{1}$$

in closed loop with the relay feedback

$$u(t) = -\mathrm{sgn}(y(t)). \tag{2}$$

Here $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}^1$ and the matrices A, B, C are of corresponding dimensions and sgn is the relay function.

Although this class of systems has a somewhat small scope, the mathematical difficulties within this class are abundant, and are illustrative for other classes of so-called *hybrid systems*, i.e. systems having mixed discrete (logic/switching) and continuous (analog) dynamics. Such systems are intensively studied recently (Pnueli & Sifakis (guest Eds.), 1995; Antsaklis & Nerode (guest Eds.), 1998; Morse, Pantelides, Sastry, & Schumacher (guest Eds.), 1999). Especially in this area the definition of a solution concept and the basic question of existence and uniqueness of trajectories is certainly non-trivial and of interest (Lygeros, Johansson, Sastry, & Egerstedt, 1999; Johansson, Egerstedt, Lygeros, & Sastry, 1999a). An important related phenomenon that is typical for continuous-time hybrid dynamical systems is the occurrence of an infinite number of (relay) switches in a finite time interval, which is called *Zeno behaviour* and causes many difficulties in analysis and

☆ A preliminary version of this paper was presented at the IFAC NOLCOS Symposium, St. Petersburg, 2001. This work was partially sponsored by the EU project "SICONOS." (IST–2001–37172). This paper was recommended for publication in revised form by Associate Editor Naomi E. Leonard under the direction of Editor Hassan Khalil.

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0005-1098/$ - see front matter © 2003 Elsevier Ltd. All rights reserved.
doi:10.1016/S0005-1098(03)00237-1

simulation. Filippov (Filippov, 1988) already noticed the influence of this phenomenon on uniqueness of solutions to relay systems as he came up with an example containing two relays in which the uniqueness of solutions does not hold due to the presence of Zeno behaviour (see page 116 of Filippov, 1988). Also one of his theorems (Theorem 2.10.4 in Filippov, 1988) excludes Zeno behaviour as one of the conditions to assure uniqueness. However, such a condition is difficult to verify a priori as results guaranteeing the absence of Zeno behaviour are extremely rare (Johansson, Lygeros, Sastry, and Egerstedt, 1999b). Therefore it is challenging and interesting to find conditions excluding Zeno trajectories and to discuss its effects within the context of mathematical properties of models like well-posedness.

As might be conjectured given the form of the relay-characteristic sgn, "positiveness" of the linear system (1) in the sense that its first non-zero Markov parameter is positive is necessary and sufficient for existence and uniqueness of solutions for arbitrary initial states. In Lootsma et al. (1999) this conjecture is proven for so-called forward solutions (see Theorem 1 below). However, for relay systems, the Filippov concept seems more natural and the associated uniqueness is of independent interest. Moreover, uniqueness of Filippov solutions play also an important role for proving consistency of a numerical simulation method based on time-stepping (Heemels et al., 2000a) and the Filippov solution concept has some favorable properties that can be used for analysis purposes (see the Conclusions section). However, the extension of the results obtained in Lootsma et al. (1999) for (1)–(2) are not straightforward as the Filippov solution concept is more general than the forward solution concept; in Lootsma et al. (1999) a particular kind of Zeno behaviour has been excluded in the solution concept by definition (left accumulations of relay switching times are not allowed). The main results of this paper show that the conjecture will hold only partially for Filippov solutions; the "positivity" conditions derived in (Lootsma et al., 1999) will be shown to be necessary and sufficient for Filippov uniqueness for linear relay systems for which the underlying linear system (1) has relative degree one or two. However, by a counterexample of a triple integrator in closed loop with negative relay feedback, it is shown that the conditions are not sufficient in general and the conjecture is therefore wrong. This example shows that there is a clear relation between the well-posedness question and the solution concept that is being used; for two solution concepts the answer to the well-posedness question turns out to be different (even also for a third solution concept used in Imura and van der Schaft (2000), see Remark 2 below). Hence, the solution concept and the well-posedness question cannot be decoupled.

In this paper we consider only the case of single-input-single-output (SISO) linear systems for which the two well-known solution concepts given by Filippov's convex definition and Utkin's equivalent control coincide. However, it is worth mentioning some other solution concepts for systems with discontinuous right-hand side like the ones presented in Gelig, Leonov, and Yakubovich (1978) and Clarke, Ledyaev, Stern, and Wolenski (1998), which are not equivalent to Filippov solutions. Another interesting line of research is related to the question, if the solutions that are generated by an ideal relay are approximations of the system (1)–(2) where sgn is replaced by an non-ideal relay. This question is beyond the scope of this paper, but see e.g. Filippov (1988), Johansson et al. (1999b), Camlibel, Cevik, Heemels, and Schumacher (2000) for a discussion on this topic for various classes of hybrid systems.

The following notations and definitions are used in the paper. A point $\tau \in \mathbb{R}$ is called a right-accumulation point of $\mathscr{E}$, if there exists a sequence $\{\tau_i\}_{i\in\mathbb{N}}$ such that $\tau_i \in \mathscr{E}$ and $\tau_i < \tau$ for all i and furthermore, $\lim_{i\to\infty} \tau_i = \tau$. A left-accumulation point is defined similarly by replacing "$<$" by "$>$." An accumulation point of $\mathscr{E}$ is a left- or a right-accumulation point of $\mathscr{E}$.

The closure of a set $S \subset \mathbb{R}^n$ is denoted by cl$S$.

2. Filippov solutions

Consider the following differential equation:

$$\dot{x} = f(x), \tag{3}$$

where $f : G \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$, is a piece-wise smooth vectorfield undergoing (possible) jumps on a set $M \subset G$ of zero measure. In the simplest "convex" definition of Filippov (Filippov, 1988) (which is the same as Utkin's equivalent control definition (Utkin, 1978) for system (1), (2)) Eq. (3) is transformed into a differential inclusion $\dot{x} \in F(x)$, where for each $x \in G$ the set $F(x)$ is defined to be the smallest convex closed set containing all the limit values of the function f at x. To be precise, $F(x)$ is the smallest convex closed set containing

$$\{z \in \mathbb{R}^n \mid \text{there is } \{x_n\}_{n\in\mathbb{N}} \text{ with } x_n \in G\backslash M,$$
$$x_n \to x \text{ and } z = \lim_{n\to\infty} f(x_n)\}. \tag{4}$$

Definition 1. A Filippov solution of (3) on the interval I is a solution of the differential inclusion

$$\dot{x} \in F(x), \tag{5}$$

that is, an absolutely continuous function $x(t)$ defined on I, for which $\dot{x}(t) \in F(x(t))$ almost everywhere on I.

Note that for system (1)–(2) M is the set $\{x \in \mathbb{R}^n \mid Cx = 0\}$ and $F(x) = \{Ax + B\}$ for $Cx < 0$, $F(x) = \{Ax - B\}$ for $Cx > 0$ and $F(x) = \{Ax + B\bar{u} \mid \bar{u} \in [-1, 1]\}$ when $Cx = 0$.

Suppose that $F(x)$ is bounded on G. The obtained set-valued function $F(x)$ is upper semicontinuous (see Lemma 2.6.3 in Filippov, 1988) and hence, according to Theorem 2.7.1 in Filippov (1988) for arbitrary initial conditions from G the solution of the differential inclusion (5) locally exists.

A solution (in the sense of Filippov) $x(t,x_0)$ with initial condition x_0 is said to be (locally) *right unique* if for each two solutions $x_1(\cdot,x_0)$ and $x_2(\cdot,x_0)$ satisfying $x_1(0,x_0)=x_0$, $x_2(0,x_0)=x_0$ there exists a $t_1>0$ such that $x_1(\cdot,x_0)$ and $x_2(\cdot,x_0)$ coincide on $[0,t_1)$. A similar definition for *left uniqueness* can be given. If a solution $x(t,x_0)$ is both (locally) right and left unique, we will say that it is (locally) unique (Filippov, 1988).

Here we mention some important properties of Filippov solutions:

(1) *Compactness of the set of solutions.* The set of solutions defined on $\alpha \leqslant t \leqslant \beta$ with initial conditions from a given compact set is compact with respect to the $\mathscr{C}[\alpha,\beta]$ topology (see Theorem 2.7.3, Filippov, 1988).
(2) *Continuous dependence on initial conditions.* Uniqueness of solutions implies continuous dependence on the initial data (see Theorem 2.8.2, Filippov, 1988).

3. Forward solutions

The solution concept used in (Lootsma et al., 1999) stems from hybrid dynamical systems (Pnueli & Sifakis (guest Eds.), 1995; Antsaklis & Nerode (guest Eds.), 1998; Morse et al., 1999) and more precisely from (linear) complementarity systems (LCS), (van der Schaft & Schumacher, 1996; van der Schaft & Schumacher, 1998; Heemels, Schumacher, & Weiland, 2000b; Heemels, Schumacher, & Weiland, 1999). The discrete part of the behaviour is related to the idea that an ideal relay element is given by three modes of operations ("discrete states") corresponding to the three branches:

A1. $y(t) \geqslant 0, \quad u(t) = -1,$
A2. $y(t) \leqslant 0, \quad u(t) = 1,$
A3. $y(t) = 0, \quad -1 \leqslant u(t) \leqslant 1.$

During the evolution of the system, the relay switches between these three modes, which have their own characteristic laws of motion. In such hybrid systems the solutions are usually considered in the following "forward sense" (Lootsma et al., 1999; Johansson et al., 1999a).

Definition 2. Suppose that there is an $\varepsilon>0$ and a triple $(u,x,y):[0,\varepsilon)\mapsto \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$ satisfying for $x_0\in\mathbb{R}^n$

(1) (u,x,y) is analytic on $[0,\varepsilon)$;
(2) $x(0)=x_0$;
(3) (1) is satisfied on $[0,\varepsilon)$; and
(4) there exists an $i\in\{1,2,3\}$ such that for all $t\in[0,\varepsilon)$ Ai holds.

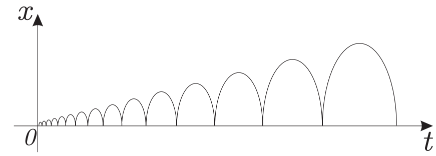

Fig. 1. An example of a left-accumulation point. A solution starts at the point for which the switching instances (points where the trajectory hits the t-axis) form a sequence with a left accumulation point. During any arbitrarily small time period $[0,\varepsilon)$ the solution undergoes an infinite number of switchings.

Then x is called a local forward solution on $[0,\varepsilon)$ to (1)–(2) with initial condition x_0.

This means that a local forward solution satisfies the dynamics of one mode only on an interval of the form $[0,\varepsilon)$ and does not account for the possibility of left accumulation of the relay switching times (see Fig. 1). As a consequence, solutions starting with an infinite number of time instances of leaving and reaching the switching surface $y=0$ in (1), (2) (as e.g. in the example in (Filippov, 1988, p. 116) and the triple integrator example of Section 4) are excluded from the definition of forward solutions. In Lootsma et al. (1999) one proves existence and uniqueness of forward solutions under suitable "positivity" conditions for the *multiple* relay case. This result can be formulated for the *single* relay case in terms of the *Markov parameters* of the system (1), which are defined by $H^i := CA^{i-1}B$ for $i=1,2,\ldots$. The leading Markov parameter is defined as the first Markov parameter that is non-vanishing, i.e. it is given by H^ρ with

$$\rho := \min\{i=1,2,\ldots \quad |H^i \neq 0\} \tag{6}$$

provided that not all Markov parameters are zero. We call ρ the *relative degree* of the system (1).

The following result gives a necessary and sufficient condition for uniqueness of forward solutions. The sufficiency part follows from Lootsma et al. (1999), while the necessity part is almost obvious for SISO systems.

It is worth mentioning, that according to the definition the forward solution is defined only for positive ("forward") time and the definition is asymmetric in time (see also the Conclusions section for time reversing of solutions).

Theorem 1 (Lootsma et al., 1999). *Let the relay system (1)–(2) be given. From any initial condition x_0 there exists a unique local forward solution if and only if the leading Markov parameter H^ρ is positive.*

Proof. To prove the necessity part of this theorem one can notice that for zero initial conditions there are three forward solutions corresponding to the modes A1, A2 and A3 in case H^ρ is negative. Indeed, it is easily verified that the solution to $\dot{x}=Ax+B$ with $x(0)=0$ is real-analytic and satisfies $y^{(r)}(0)=0$, $r=1,\ldots,\rho-1$ and $y^{(\rho)}(0)=H^\rho<0$, which implies that $y(t)\leqslant 0$ for some interval of the form $[0,\varepsilon)$ with $\varepsilon>0$. This means that there exists a non-zero forward (and Filippov) solution in mode $A2$. Similarly, it can be shown that mode $A1$ produces a different non-zero forward solution. □

Hence, uniqueness of forward solutions can be checked simply by means of the calculation of the sign of the leading Markov parameter. In van der Schaft and Schumacher (1999) it is observed in p. 110, that the conditions of Filippov (see Section 2.10 in Filippov, 1988) have to be checked on a point-by-point basis, while here the determination of the leading Markov parameter suffices. However, note that the results of Filippov are applicable to general nonlinear systems coupled to relays, while Theorem 1 is valid for linear relay systems only.

In principle we have restricted ourselves here to *local* forward solutions, but they can be extended to obtain forward solutions that evolve through several modes by concatenation of the local solutions as defined here and thereby possibly leading to global solutions, see Heemels et al. (2000b), Lootsma et al. (1999) for details.

4. Filippov versus forward solution

However, there is an open question: *how are the conditions of Theorem* 1 *related to the uniqueness* (*left or right*) *of the solutions of the closed loop system when the solutions are understood in the sense of Filippov*? As mentioned in the introduction, a first indication of a possible source causing problems stems from an example constructed by Filippov (Filippov, 1988, p. 116), which exhibits non-uniqueness of Filippov solutions due to a specific type of Zeno behaviour excluded in the forward solution concept. However, besides the fact that Filippov's example contains multiple relays, it also does not satisfy the multiple relay "positivity" conditions for well-posedness as given in Lootsma et al. (1999). Hence, Filippov's example is not a counterexample for the conjecture that the conditions of Theorem 1 also suffice for uniqueness of Filippov solutions. To our best knowledge, there is also no other example present in literature showing that the "positivity" conditions are not sufficient for Filippov uniqueness as well and this fact motivates our study.

4.1. Relative degree one

First, we consider the case of relative degree one. In this case the right uniqueness of Filippov solutions indeed follows from the positivity of the first Markov parameter.

Theorem 2. *Consider a linear SISO system* (1) *of relative degree one in closed loop with the feedback* (2). *Then the Filippov solution is right unique for all initial conditions if and only if the first Markov parameter CB is positive.*

Proof. 1. ⇐. A linear SISO system of relative degree one can be represented (possibly after a coordinate change) as follows:

$$\dot{x}_1 = ax_1 + c^\top z + \mathrm{d}u,$$
$$\dot{z} = Fz + Dx_1, \tag{7}$$
$$y = x_1,$$

with $x_1, u, y \in \mathbb{R}^1$, $z \in \mathbb{R}^{n-1}$ and matrices a, c, d, F, D are of corresponding dimensions. This representation is used in Utkin (1978) and a general case of nonlinear systems is treated in Byrnes and Isidori (1991). The representation (7) is usually referred to as the *normal form*. Since the first Markov parameter is positive, $d = CB > 0$.

The right uniqueness of all solutions for system (7)–(2) follows from Theorem 2.10.1 (by observing that $(\xi - \zeta)(\mathrm{sgn}(\xi) - \mathrm{sgn}(\zeta)) \geqslant 0$) and also from Theorem 2.10.2 in Filippov (1988).

2. ⇒. From Theorem 1 negativity of $CB = d$ implies the existence of multiple forward solutions (for each mode $A1$, $A2$ and $A3$ one trajectory) starting in the origin. As local forward solutions are also local Filippov solutions, the result follows. □

Note that $CB = 0$ complies with relative degree larger than one and no statements are provided in Theorem 2 for this situation.

4.2. Relative degree two

The next result shows that uniqueness of Filippov solutions is guaranteed if the first Markov parameter is positive in case of relative degree two. Consider the set $\mathcal{O} := \{x \mid Cx = 0, CAx = 0, |CA^2x| = CAB\}$. Let $\Omega = \mathbb{R}^n \setminus \mathcal{O}$.

Theorem 3. *Consider a linear SISO system* (1) *of relative degree two in closed loop with the feedback* (2). *Then the Filippov solution of* (1)–(2) *is unique for any initial conditions from* Ω *if and only if the second Markov parameter* CAB *is positive.*

Proof. 1. ⇐. A linear SISO system of relative degree two can be represented in the following *normal form* (Byrnes & Isidori, 1991):

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = ax_1 + bx_2 + c^\top z + \mathrm{d}u, \tag{8}$$
$$\dot{z} = Fz + Dx_1 + Hx_2,$$

$$y = x_1, \tag{9}$$

where $x_1, x_2, y, u \in \mathbb{R}^1$, $z \in \mathbb{R}^{n-2}$ and a, b, c, d, F, D, H are of corresponding dimensions. Since the second Markov parameter is positive, it holds that $d = CAB > 0$. We will denote $\mathrm{col}(x_1, x_2, z)$ by x for brevity. In new coordinates $\mathcal{O} = \{x \mid x_1 = 0, x_2 = 0, |c^\top z| = d\}$.

We start with the right uniqueness. If $x_{10} \neq 0$ the uniqueness follows from the standard uniqueness theorems as we are at a non-zero distance away for the switching surface $Cx=0$. If $x_{10}=0, x_{20}=\dot{x}_1(0,x_0) \neq 0$ the trajectory intersects the switching surface transversely and the uniqueness follows from Filippov's argument (see 2.10 in Filippov, 1988). So, the only case left to consider is $x_{10}=x_{20}=0$. Additionally, if $|c^\top z_0| > d$, the trajectory intersects the switching surface transversely and the uniqueness again follows. Consider first the case where $n \geqslant 3$ (it is the zero dynamics are non-trivial and z has dimension larger or equal to 1). Then it is sufficient to prove right uniqueness for the initial conditions $x_{10}=0, x_{20}=0, |c^\top z_0| < d$.

Observe that along any trajectory of the system (1)–(2) it holds that (see Filippov, 1988, p. 155)

$$\frac{\mathrm{d}}{\mathrm{d}t}|x_1(t,x_0)| = \left.\frac{\mathrm{d}}{\mathrm{d}h}|x_1(t,x_0)+hx_2(t,x_0)|\right|_{h=0} = x_2(t,x_0)\,\mathrm{sgn}\,x_1(t,x_0)$$

for *almost* all $t \geqslant 0$.

Consider the following Lipschitz-continuous function:

$$V(x_1,x_2,z) = \mathrm{d}|x_1| - x_1 c^\top z + \tfrac{1}{2}x_2^2. \tag{10}$$

Given $\varepsilon \in (0,d)$, $\mu > 0$, consider the set

$$\Omega_{\varepsilon,\mu} = \{(x_1,x_2,z)^\top \,\|c^\top z| \leqslant d-\varepsilon, |c^\top Fz| \leqslant \mu\}.$$

On this set it follows that $V(x_1,x_2,z) \geqslant \varepsilon|x_1| + x_2^2/2 \geqslant 0$. Then, in $\Omega_{\varepsilon,\mu}$ for almost all $t \geqslant 0$,

$$\dot{V}(x(t,x_0)) \leqslant bx_2^2 + \alpha x_1 x_2 + \beta x_1^2 + \mu|x_1|,$$

where $\alpha = |a - c^\top H|, \beta = |c^\top D|$.

Using the inequalities

$$\xi_1^2 \leqslant |\xi_1|, \quad \xi_1\xi_2 \leqslant (\xi_1^2+\xi_2^2)/2 \leqslant (|\xi_1|+\xi_2^2)/2$$

which are valid for real ξ_1, ξ_2 with $|\xi_1| \leqslant 1$, one can see that if $|x_1| \leqslant 1$ in the set $\Omega_{\varepsilon,\mu}$ it follows that

$$\dot{V} \leqslant \left(b + \frac{\alpha}{2}\right)x_2^2 + \left(\frac{\alpha}{2} + \beta + \mu\right)|x_1| \leqslant \delta(\varepsilon,\mu)V$$

for some positive $\delta(\varepsilon,\mu)$. In other words, the absolutely continuous non-negative function of time

$$V(x_1(t,x_0), x_2(t,x_0), z(t,x_0))\mathrm{e}^{-\delta(\varepsilon,\mu)t}$$

does not increase whenever the solution lies in the set of the form $\Omega_V \subset \Omega_{\varepsilon,\mu}$, $\Omega_V := \{x \in \Omega_{\varepsilon,\mu} \mid V(x) \leqslant C,\ |x_1| \leqslant 1\}$ for some positive C. Since $V(0,0,z)=0$ this means that if $x_{10}=0, x_{2,0}=0, z_0 \in \Omega_{\varepsilon,\mu}$ then

$$V(x_1(t,x_0), x_2(t,x_0), z(t,x_0)) = 0$$

for arbitrary non-negative t for which $z \in \Omega_\varepsilon$. Since ε can be arbitrarily small it means that on the set $x_1=0$, $x_2=0, |c^\top z| < d$ a second order sliding mode occurs and no solutions can leave the interval $x_1=0, x_2=0, |c^\top z| < d$. The evolution of the system on this set is determined by a linear equation $\dot{z} = Fz$ and hence the right uniqueness follows.

Now we have to prove the right uniqueness for $n=2$ (no zero dynamics). In this case the system equations have the form

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= ax_1 + bx_2 + \mathrm{d}u,\end{aligned} \tag{11}$$

$$y = x_1. \tag{12}$$

As mentioned above, for this system it is sufficient to prove right uniqueness only for zero initial conditions. Considering the positive definite function $V = \mathrm{d}|x_1| + x_2^2/2$ and using the previous argument we obtain the inequality $\dot{V} \leqslant \delta V$ for some $\delta > 0$ from which the right uniqueness of the zero solution follows.

To prove the left uniqueness it suffices to reverse the time and to consider the behaviour of the solutions of the time–reversed system, which also has a positive leading Markov parameter and is consequently right unique as well!

2. $\Rightarrow$. This implication follows from Theorem 1. □

Note that in the proof of the theorem (to show the left uniqueness of solutions) we use an argument based on time reversal.

Note that in case the conditions of Theorem 2 or 3 hold, a typical kind of Zeno behaviour (within the class of Filippov solutions) can be excluded for the relay system (1)–(2). Indeed, left accumulations of relay switching times do not happen, as we know that the right unique Filippov solution must be equal to the unique forward solution which has the special property that left accumulations cannot occur. Moreover, in the case of relative degree 2, we can even exclude right accumulations of the switching times.

Remark 1. It is not difficult to see that the arguments applied in the sufficiency parts of the previous results can still be used for affine nonlinear control systems of relative degree one or two in a closed loop with negative relay feedback written in the normal form (Byrnes & Isidori, 1991). In other words uniqueness of the Filippov solutions can be deduced (at least locally in time) from the *first-order approximation* of the nonlinear system which has a normal form with relative degree one or two.

4.3. Relative degree three

Up to this point one may conjecture that a statement similar to Theorem 1 is true for the Filippov solutions as well. However, the following counterexample which is inspired by Johansson (1997), Johansson et al. (1999b) proves that this is not true in general.

Consider the following system:

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u, \qquad (13)$$

$$y = (1\ 0\ 0)x,$$

$$u = -\text{sgn}\, y, \qquad (14)$$

with $x = (x_1, x_2, x_3)^\top$.

This system is a triple integrator in closed loop with negative relay feedback. System (13) has relative degree three and its third Markov parameter is 1, so system (13)–(14) has a *unique* forward solutions for arbitrary initial conditions (Theorem 1). Our goal is to show that if one accepts Filippov's concept of solutions, then there are *infinitely* many solutions of (13)–(14) starting at the origin.

Theorem 4. *There are infinitely many Filippov solutions of system* (13)–(14) *starting at the origin.*

The proof of this result is based on the following lemma that is derived from Johansson (1997), Johansson et al. (1999c). For the sake of completeness, the proof of the result is included.

Lemma 1. *Any forward solution of the system* (13)–(14) *starting at* $t = 0$ *in the set*

$$\Omega = \{x \mid x_1 = 0, x_2 > 0\}$$

is right unique and undergoes an infinite number of switches with intervals between them of length $t_1, t_2, \ldots$ *for which the following estimate is true*:

$$t_k > (\sqrt{2} + 1)^{k-1} t_1. \qquad (15)$$

Proof. Assume that a trajectory of system (13), (14) passes the switch plane Ω at $t = 0$ with

$$x(0, x_0) = x_0 = (0, x_{20}, x_{30})^\top \in \Omega.$$

Thus

$$\text{sgn}\, y(0_+) = 1, \qquad u(0_+) = -1.$$

Then, until the next switch,

$$\begin{aligned} x_1(t, x_0) &= x_{20}t + x_{30}t^2/2 - t^3/6, \\ x_2(t, x_0) &= x_{20} + x_{30}t - t^2/2, \qquad (16) \\ x_3(t, x_0) &= x_{30} - t. \end{aligned}$$

Note that a switch will occur and let $t = t_1$ be the time instant of the first switch and let x_0' be the state at the switch (note $x_{10}' = 0$ and $x_{20}' < 0$). Since $x_1(t_1, x_0) = 0$ one has

$$6x_{20} + 3x_{30}t_1 - t_1^2 = 0. \qquad (17)$$

This equation can be always solved for positive t_1. At $t = t_1$

$$\begin{aligned} x_{20}' &:= x_2(t_1, x_0) = x_{20} + x_{30}t_1 - t_1^2/2, \\ x_{30}' &:= x_3(t_1, x_0) = x_{30} - t_1. \end{aligned} \qquad (18)$$

Then, until the second switch which occurs at $t = t_1 + t_2$

$$\begin{aligned} x_1(t, x_0') &= x_{20}'t + x_{30}'t^2/2 + t^3/6, \\ x_2(t, x_0') &= x_{20}' + x_{30}'t + t^2/2, \qquad (19) \\ x_3(t, x_0') &= x_{30}' + t. \end{aligned}$$

Again a new switch will occur at, say, time $t_1 + t_2$. Since $x_1(t_2, x_0') = 0$ one has

$$6x_{20}' + 3x_{30}'t_2 + t_2^2 = 0. \qquad (20)$$

As before, this equation is uniquely solvable for positive t_2 since $x_{20}' < 0$ and therefore, $t_2 < \infty$. Substituting (18) into (20) yields

$$6(x_{20} + x_{30}t_1 - t_1^2/2) + 3(x_{30} - t_1)t_2 + t_2^2 = 0. \qquad (21)$$

Eqs. (17) and (21) can be solved with respect to x_{20} and x_{30} which gives

$$x_{20} = t_1 \frac{t_2^2 - 2t_1t_2 - t_1^2}{6(t_1 + t_2)}.$$

However, as we assumed $x_{20} > 0$, we must have that

$$t_2 > (\sqrt{2} + 1)t_1. \qquad (22)$$

A similar computation can be performed starting from an initial condition in

$$\Omega' = \{x \mid x_1 = 0, x_2 < 0\}.$$

Repeated evaluation of (22) now gives (15). Note that the solutions starting from Ω are (locally) right unique in Filippov's sense as $x_{10} = 0$, $x_{20} > 0$ implies that $y(t) > 0$, $u(t) = -1$ on $(0, t_1)$. Similar reasoning can be applied to $x_{10}' = 0$, $x_{20}' < 0$ to obtain (local) right uniqueness on $[t_1, t_1 + t_2]$. Repeating the arguments leads to right uniqueness on $[0, \infty)$ (note that there is no finite accumulation point of the relay switching times). □

Proof of Theorem 4. Given $\varepsilon > 0$, consider the set

$$S_\varepsilon^0 = \{x \in \mathbb{R}^3 \mid x_1 = 0, x_2 > 0, |x_2| + |x_3| \leqslant \varepsilon\}.$$

Given an arbitrary solution $x(t, x_0)$ with $x_0 \in S_\varepsilon^0$ and let $T(x_0) \geqslant 0$ be the time instant of the first intersection of the trajectory of $x(t, x_0)$ with the switching plane $x_1 = 0$ for a time instant larger than or equal to $t = 1$, i.e. $x_1(T(x_0), x_0) = 0$. Denote

$$t_{\max} = \sup_{z \in S_\varepsilon^0} T(z).$$

By definition, $t_{\max} \geqslant 1$. Moreover, we claim that $t_{\max}$ is bounded. This fact follows from the proof of Lemma 1 (see (17), (21)).

Consider an arbitrary converging sequence $\{x_0^i\}$, $i = 1, \ldots, \infty$, with $x_0^i \in S_\varepsilon^0$, $x_0^i \to 0$ as $i \to \infty$. Since the set of Filippov solutions defined on the finite interval $[0, t_{\max}]$ for the initial conditions from the compact set $\mathrm{cl}S_\varepsilon^0$ is compact, a subsequence of $x(t, x_0^i)$ converges to a solution $\xi_0(t) = x(t, \bar{x}_0)$, where $x(t, \bar{x}_0)$ is some Filippov solution of the closed loop system and $\bar{x}_0 \in \mathrm{cl}S_\varepsilon^0$ (see Section 2). Without loss of generality, we may assume that the sequence itself is convergent in the $\mathscr{C}[0, t_{\max}]$-topology. This implies that

$$\forall t \in [0, t_{\max}]\ x(t, x_0^i) \to \xi_0(t) \quad \text{as } i \to \infty.$$

Since $0 \in \mathrm{cl}S_\varepsilon^0$ and $x(0, x_0^i) = x_0^i \to 0$ as $i \to \infty$, it follows that $\xi_0(0) = \bar{x}_0 = 0$. Thus $\xi_0(t)$ is a Filippov solution of the closed loop system starting at the origin.

Our goal is to show that for the sequence $\{x_0^i\}$ of points in $\mathbb{R}^3$ there is a sequence of times $\{q^i\}$, $1 \leqslant q^i \leqslant t_{\max}$ such that

$$\liminf_{i\to\infty} \|x(q^i, x_0^i)\| > 0. \tag{23}$$

Observe that if (23) is true, then $\xi_0 \not\equiv 0$ and hence, we constructed a non-zero solution starting from the origin.

Given i, denote by t_j^i the length of the intervals between successive switches for the solution $x(t, x_0^i)$. Denote by n^i the minimal number of switches for the solution $x(t, x_0^i)$, $t \in [0, t_{\max}]$ necessary to satisfy the following inequality:

$$\tilde{q}^i := \sum_{j=1}^{n^i} t_j^i \geqslant 1. \tag{24}$$

By definition, $\tilde{q}^i \leqslant t_{\max}$ for all i.

Let $r = 1 + \sqrt{2}$. By virtue of Lemma 1, we have

$$t_j^i < t_{n^i}^i \left(\frac{1}{r}\right)^{n^i - j}, \quad j < n^i$$

and hence from (24)

$$1 \leqslant \sum_{j=1}^{n^i} t_j^i < t_{n^i}^i \sum_{j=0}^{n^i - 1} r^{-j}.$$

Thus

$$t_{n^i}^i > \left[\frac{1}{\sum_{j=0}^{n^i-1} r^{-j}}\right] > \left[\frac{1}{\sum_{j=0}^{\infty} r^{-j}}\right] = 2 - \sqrt{2}.$$

This value is bounded away from zero for any i. From the third equation of the closed loop system it follows that $\dot{x}_3 = 1$ or $\dot{x}_3 = -1$ in the interval $[\tilde{q}^i - t_{n^i}^i]$. Hence there exists $\{q^i\}$ with $\tilde{q}^i - t_{n^i}^i \leqslant q^i \leqslant \tilde{q}^i$ such that

$$\liminf_{i\to\infty} |x_3(q^i, x_0^i)| > 0. \qquad \square$$

Hans Schumacher pointed out that if $x(t)$ is a solution to (13)–(14), then also the functions $x_a(t) := a^{-3}x(at)$ will be Filippov solutions for all $a > 0$. As we constructed a *non-zero* solution from the origin which is not a homogeneous function of degree 3 (i.e. does not satisfy $a^3 x(t) = x(at)$, which can be seen from the explicit expressions in (16) and (19)), this immediately shows that there are infinitely many Filippov solutions starting in the origin.

The constructed non-zero Filippov solutions with zero initial state start with a left-accumulation point of relay switching times and hence are not forward solutions as used in Lootsma et al. (1999). Note that any small perturbation of the zero initial condition immediately gives rise to a solution moving away from the origin (see Lemma 1) in which the switching times increase exponentially according to (15).

Remark 2. Recently, in Imura and van der Schaft (2000) it was proposed to utilize a new solution concept, called extended Carathéodory solution for systems of the form (3) with discontinuous right-hand side. Loosely speaking, extended Carathéodory solutions are Filippov solutions without sliding modes (as given by mode A3 in Section 3) and without left accumulations of relay switching times. For the following piece-wise linear bi-modal system:

$$\begin{aligned} &\text{mode 1}: \ \dot{x} = A_1 x \quad \text{if } y = Cx \geqslant 0, \\ &\text{mode 2}: \ \dot{x} = A_2 x \quad \text{if } y = Cx \leqslant 0, \end{aligned} \tag{25}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^1$ and the matrices A_1, A_2 and C of corresponding dimensions, necessary and sufficient conditions of the existence and uniqueness of Carathéodory solutions were found. Using the result of Imura and van der Schaft (2000) one can check that for the reformulation of the triple integrator example (13)–(14) in the form (25) with

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{26}$$

$$C = (1\ 0\ 0\ 0) \tag{27}$$

has no solutions in the extended Carathéodory sense starting from $(0, 0, 0, 1)^\top$. At the same time, using the triple integrator example, one can notice that there are infinitely many Filippov solutions and a unique forward solution starting from the same initial conditions.

5. Conclusions

In this paper we compared different solution concepts and the corresponding well-posedness results for a linear time-invariant system coupled to an ideal relay, namely Filippov solutions and forward solutions. One advantage of the forward solution concept is that it is immediately extendable to broader classes of hybrid dynamical systems (e.g. linear complementarity systems (van der Schaft & Schumacher, 1996, 1998; Heemels et al., 2000b, 1999) as it adopts a multiple mode view on the dynamical behaviour, while the Filippov solution does not. Another major advantage of the forward solution is that the (available) well-posedness conditions for linear relay systems (also for the multiple relay case) are relatively easy to check. This is partly because the possibility of left accumulations of relay switching times (a particular kind of Zeno behaviour) is excluded. Based on a simple observation that any LTI system in closed loop with a relay feedback is related to a bimodal system, we briefly compared the well-posedness results for such systems also in the extended Carathéodory context.

In this paper we actually showed that if the underlying linear system has relative degree one or two, the "positivity" conditions for forward well-posedness (Lootsma et al., 1999) given by the positivity of the leading Markov parameter are still sufficient and even necessary for (right) uniqueness of Filippov solutions. In particular, the existence of (left-)accumulation points of switching times can be excluded a priori on the basis of these conditions, which facilitates simulation and analysis of these systems. However, for higher relative degrees this is unfortunately not the case as is proven by the triple integrator example. By analogy, bimodal systems of an order at least four can have multiple Filippov solutions with left-accumulation points, one forward solution and no extended Carathéodory solution. The system (13)–(14) (with one relay and of relative degree three) demonstrates the following drawbacks of the forward (and extended Carathéodory) solutions:

(1) *Reversing time.* If $x(t)$ is a forward solution to $\dot{x}=f(x)$, then the reversed-time trajectory $x(-t)$ is *not* necessarily a forward solution to $\dot{x}=-f(x)$.

(2) *Compactness of the set of solutions.* The set of forward solutions defined on $\alpha \leqslant t \leqslant \beta$ with initial conditions from a given compact set is *not* compact with respect to $\mathscr{C}[\alpha,\beta]$ (uniform) topology and hence, with respect to any weaker topology. This property immediately follows from the triple integrator example (13)–(14) for which we proved the existence of multiple Filippov solutions originating from the origin. At the same time, Theorem 1 predicts the uniqueness of the forward solutions. Then using compactness property of the Filippov solutions (see Section 2) it is possible to build a fundamental sequence of forward solutions converging in $\mathscr{C}[\alpha,\beta]$-topology (and, therefore, in any weaker topology) to a Filippov solution, which in turn is not a forward solution since it starts from the origin and is not identically zero.

(3) *Continuous dependence on the initial conditions.* Uniqueness of a forward solution does *not* necessarily imply continuous dependence on the initial data. This property again follows from the example (13)–(14): the unique forward solution (see Theorem 1) starting at the origin does not depend continuously on the initial conditions.

It is worth mentioning that in the Introduction to the monograph (Filippov, 1988) discussing possible approaches to define a solution for discontinuous systems Filippov claimed that compactness of the solutions is a mandatory property for any possible solution concept. Unfortunately, the forward and extended Carathéodory solutions (and also executions or runs of hybrid automata as defined in Johansson et al. (1999a) do not possess this property due to the absence of left-accumulation points. So, depending on the solution concept you choose certain questions become easier or harder to solve: for forward solutions it is relatively easy to show well-posedness, but the three properties mentioned above do not hold for forward solutions and consequently, other analysis problems might become more difficult.

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