

# Transient performance improvement of linear systems using a split-path nonlinear integrator

S.J.L.M. van Loon, B.G.B. Hunnekens, W.P.M.H. Heemels, N. van de Wouw, H. Nijmeijer

**Abstract**— In this paper, we introduce the split-path nonlinear integrator (SPANI) as a novel nonlinear filter designed to improve the *transient* performance of linear systems in terms of overshoot. In particular, this nonlinear controller targets the well-known trade-off induced by integral action, which removes steady-state errors due to constant external disturbances, but deteriorates transient performance in terms of increased overshoot. The rationale behind the proposed SPANI filter is to ensure that the integral action has, at all times, the same sign as the closed-loop error signal, which, as we will show, enables a reduction in overshoot (i.e., improves transient performance). The resulting closed-loop dynamics can be described by a continuous-time switched dynamical system, for which we will provide sufficient conditions for stability. Furthermore, we illustrate the effectiveness, the design and the tuning of the proposed controller in simulation examples.

## I. INTRODUCTION

In classical linear control theory, it is well-known that Bode's gain-phase relationship causes a hard limitation on achievable performance trade-offs in linear time-invariant (LTI) feedback control systems, see e.g., [1], [2]. The interdependence between gain and phase is often in conflict with the desired performance specification set by the control engineer. For example, it is impossible to add integral action into a feedback control system, typically included to achieve zero steady-state errors, without introducing the negative effect of phase lag as well. It was this fundamental gain-phase relationship for LTI systems that motivated W. C. Foster and co-workers in 1966 to develop the split-path nonlinear (SPAN) filter, in which they intended to design the gain and phase characteristics separately [3]. In this paper, we focus on enhancing *transient* performance of linear motion systems in terms of overshoot by proposing a variation and extension to the SPAN filter, which will be called the split-path nonlinear integrator (SPANI).

A number of other nonlinear control strategies, designed for improving the transient performance of LTI systems, has been proposed in the literature. One concept is reset control, which was first introduced in 1958 [4] and allows the controller to reset (a subset of) its state(s) if certain conditions are satisfied. Especially in the last two decades, it has regained attention in both theoretically orientated research, see e.g., [5]–[7], as well as in applications [8]. In [9], potential benefits

of hybrid control for linear systems have been discussed, and in particular, it has been shown that a switched integral controller can improve the transient performance. Another concept is variable gain integral control [10], which limits the integral action if the error exceeds a certain threshold, thereby limiting, in turn, the amount of overshoot. A similar objective is achieved in [11], in which a sliding mode controller with saturated integrator is studied. In [12], the concept of composite nonlinear feedback is employed to improve the transient response of second-order LTI systems.

All aforementioned nonlinear control strategies have in common that closed-loop stability cannot be verified anymore using the linear-based Nyquist stability theorem. Hence, the importance of the development of other testable stability conditions is evident. Despite this, none of the works thus far that considered SPAN filters, e.g., [3], [13], provided such results. In this paper, however, we propose the first testable Lyapunov-based stability conditions for a feedback control system including the proposed SPANI controller.

Despite its potency to outperform linear controllers, see e.g., [3], [13], the SPAN filter did not receive much attention until recently. In [13], the authors showed that a controller with SPAN filter can outperform an LTI controller with respect to overshoot to a step response. In this paper, we aim to achieve the same objective by our newly proposed variant/extension to the SPAN filter. Furthermore, we would like to emphasize that, similarly as in [10], the linear components of our proposed controller configuration can be designed using well-known loop-shaping techniques, which enhances the applicability to real-life control problems.

Summarizing, the contributions of this paper are as follows. Firstly, a novel nonlinear SPANI filter is proposed that has the ability to improve the transient performance of linear systems in terms of overshoot. Secondly, we will show that the feedback control configuration with add-on SPANI controller can be modeled as a continuous-time switched dynamical system, for which sufficient Lyapunov-based stability conditions are provided. Moreover, the potential of the SPANI filter to improve upon transient behavior as well as its intuitive design will be highlighted throughout the paper by means of simulations.

The remainder of the paper is organized as follows. In Section II, we introduce and motivate our proposed SPANI filter. Subsequently, in Section III, we introduce a switched systems model of our closed-loop system and in Section IV, we provide stability conditions for this system in terms of linear matrix inequalities (LMIs). Finally, we illustrate the effectiveness of the proposed nonlinear control strategy using

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The authors are with the Eindhoven University of Technology, Department of Mechanical Engineering, P.O. Box 513, NL 5600 MB Eindhoven, the Netherlands, {s.j.l.m.v.loon, b.g.b.hunnekens, m.heemels, n.v.d.wouw, h.nijmeijer}@tue.nl

a simulation example of a motor-load system in Section V and draw conclusions in Section VI.

### A. Nomenclature

The following notational conventions will be used. Let  $\mathbb{R}$  denote the set of real numbers. We call a matrix  $P \in \mathbb{R}^{n \times n}$  positive definite and write  $P \succ 0$ , if  $P = P^\top$  and  $x^\top P x > 0$  for all  $x \neq 0$ , similarly, we call  $P \prec 0$  negative definite. For brevity, we write symmetric matrices of the form  $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$  as  $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ .

## II. SPLIT-PATH NONLINEAR INTEGRATOR

In Section II-A, we will briefly revisit the original SPAN filter, and, based on these historical developments, introduce a new variation/extension to this filter: the SPANI filter. Additionally, in Section II-B, a description of the complete feedback control system will be given. Finally, we end this section with an illustrative example in order to demonstrate the effectiveness of the newly proposed controller configuration.

### A. Introduction and motivation of the SPANI filter

Originally, the key motivation behind the development of the SPAN filter was to obtain a filter in which the gain and phase could be designed independently [3]. To achieve this, the input signal, being the closed-loop error  $e$ , is divided into two separate branches which outputs are multiplied in order to form the control input  $u_s$ , as schematically depicted in Fig. 1. The lower branch contains a sign element, which destroys all magnitude information as its output is either  $\pm 1$ , thereby retaining all *phase* information. The opposite holds for the upper branch, as it contains an absolute value element; hence, all sign information is lost but the *magnitude* information is retained. Moreover, both branches contain a linear filter  $\mathcal{H}_i(s)$ ,  $i \in \{1, 2\}$ ,  $s \in \mathbb{C}$ . In both [3] and [13], the authors use filters of the form  $\mathcal{H}_1(s) = 1/(s + \tau_1)$  (low-pass filter) and  $\mathcal{H}_2(s) = (s + \tau_2)/(s + \tau_3)$  (lead filter), with the aim to add phase lead without magnitude amplification.

In this paper, we use the concept of the SPAN filter to obtain a new nonlinear controller with the goal to improve the *transient* performance of linear motion systems, which is quantified in terms of overshoot to step responses of the closed-loop system. For that purpose, we select a linear integrator for  $\mathcal{H}_1(s)$ , i.e.,  $\mathcal{H}_1(s) = \omega_i/s$ , and take  $\mathcal{H}_2(s) = 1$ . This nonlinear filter will be denoted by the *split-path nonlinear integrator* (SPANI), and is schematically represented in the dashed blue rectangle in Fig. 2. The rationale behind the design of this SPANI filter can be best understood by considering a step response (or the response to a step disturbance) on a system containing integral control. In order to achieve a zero steady-state error, the integrator integrates the error  $e$  over time resulting in build-up of integral buffer. As soon as the error  $e$  becomes zero, i.e.,  $e = 0$ , the integrator still has the integrated error stored in its state. Due to the phase lag introduced by the integrator, it takes some time to empty this buffer, causing the error to overshoot. In contrast to a linear integrator, the SPANI enforces, due to the absolute value and sign element, see Fig. 2, the integral

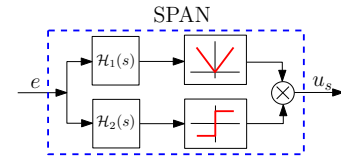


Fig. 1. Schematic representation of the SPAN filter.

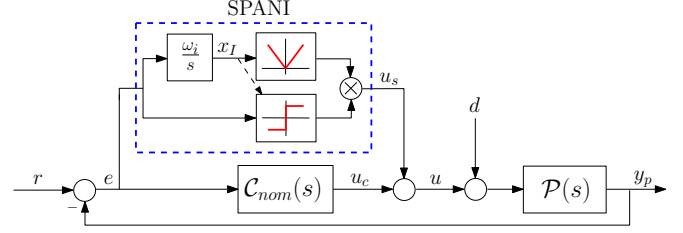


Fig. 2. Feedback loop with plant  $\mathcal{P}(s)$ , linear controller  $\mathcal{C}_{nom}(s)$  and the SPANI controller.

action to take the same sign as the error signal. This results in non-smooth behavior as soon as  $e = 0$ , i.e., at that specific time instant an instantaneous switch of the sign of the integral action takes place, inducing a reduction in overshoot.

### B. Description of the control system

The feedback configuration used in this paper is shown in Fig. 2. In this figure,  $e := r - y_p$  denotes the tracking error between the reference signal  $r$  and the output of the plant  $y_p$ ,  $d$  denotes an unknown, bounded input disturbance and  $u := u_c + u_s$  the total control input, which consists of the control input  $u_c$  of the linear controller  $\mathcal{C}_{nom}(s)$  and of the control input  $u_s$  of the SPANI. The linear closed-loop part consists of a single-input-single-output (SISO) LTI plant  $\mathcal{P}(s)$ ,  $s \in \mathbb{C}$ , and a nominal stabilizing SISO LTI (loop-shaped) controller  $\mathcal{C}_{nom}(s)$ ,  $s \in \mathbb{C}$ .

The nonlinear part, being the SPANI filter, is placed in parallel to  $\mathcal{C}_{nom}(s)$  in order to obtain maximum effect of discontinuities in the SPANI output  $u_s$ , i.e., they are not fed through the nominal controller  $\mathcal{C}_{nom}(s)$ . The state (and output) of the integrator  $\mathcal{C}_I(s) = \omega_i/s$ , with gain  $\omega_i$ , is defined by  $x_I \in \mathbb{R}$ . The *sign*-function in the lower branch of the SPANI, see Fig. 2, is defined as

$$\text{sign}(e, x_I) = \begin{cases} 1 & \text{if } e > 0, \\ 1 & \text{if } e = 0 \text{ and } x_I \geq 0, \\ -1 & \text{if } e = 0 \text{ and } x_I < 0, \\ -1 & \text{if } e < 0. \end{cases} \quad (1)$$

In essence, (1) reflects a normal *sign*-function, except at  $e = 0$ , where we use  $\pm 1$  depending on the sign of  $x_I$ , such that  $u_s = +x_I$  for  $e = 0$  (the dependence of the sign-function on  $x_I$  is denoted by the dashed arrow in Fig. 2).

**Remark II.1** Foster and co-workers did not present any modeling results in [3]. Perhaps, as a consequence, no formal definition of the *sign*-function, see Fig. 1, was provided in [3]. One of the contributions of this paper is, however, to present a modeling framework of the closed-loop control

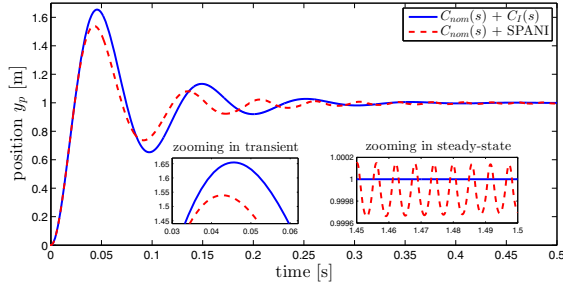


Fig. 3. Transient unit step response simulations of the controlled motion system with and without SPANI controller.

system with SPANI filter amenable to stability analysis, and, as a result, a formal definition of the *sign*-function, see (1), is required.

### C. Illustrative example

Let us consider a basic example to illustrate the potential of the control configuration in Fig. 2, concerning a mass-spring-damper system, with plant  $\mathcal{P}(s) = 1/(ms^2 + bs + k)$ , mass  $m = 0.01$  kg, damping coefficient  $b = 0.03$  Ns/m and spring constant  $k = 1$  N/m, being controlled by an input force  $F$  and the output  $y_p$  corresponding to the position of the mass. Two types of controllers are designed, a linear one and a nonlinear one, both capable to regulate the system to the reference  $r = 1$ . The linear controller, designed using classical loop-shaping techniques, consists of a stabilizing linear controller  $\mathcal{C}(s) = \mathcal{C}_{nom}(s) + \mathcal{C}_I(s)$ , with  $\mathcal{C}_{nom}(s) = 22.4(\frac{1}{35}s + 1)/(\frac{1}{100}s + 1)$  being the nominal controller without integral action, and a linear integrator  $\mathcal{C}_I(s) = \omega_i/s$ , with gain  $\omega_i = 12$  rad/s. In the nonlinear controller configuration, we use the same nominal controller  $\mathcal{C}_{nom}(s)$  and replace the linear integrator  $\mathcal{C}_I(s)$  by the SPANI, using the same gain  $\omega_i = 12$  rad/s, to arrive at a feedback control configuration as shown in Fig. 2.

The unit step response simulations are depicted in Fig. 3. Let us first focus on the transient response in terms of overshoot. As can be concluded from the figure, the response of the control configuration with SPANI filter clearly has the least amount of overshoot, thereby illustrating the potential of the proposed configuration. Note that up to the crossing  $y_p = 1$ , i.e., the time instant where the error  $e$  changes sign for the first time, the responses are identical (equal rise-time). The reduction in overshoot is achieved by the instantaneous change of sign of the control input  $u_s$  of the SPANI as soon as the error  $e$  crosses zero.

Let us now focus on the steady-state behavior in Fig. 3. This reveals that, contrary to the linear control configuration, the system with SPANI does not track the step reference asymptotically ( $e := r - y_p \not\rightarrow 0$  as  $t \rightarrow \infty$ ), but instead shows undesirable steady-state oscillations. In the sections below, we will explain more about this undesired behavior and present a slight modification to the SPANI filter in order to eliminate such undesired steady-state behavior while retaining an improved transient response in terms of overshoot.

## III. SWITCHED SYSTEM MODELING

In this paper, see Fig. 2, we study SISO LTI plants of the form

$$\mathcal{P} : \begin{cases} \dot{x}_p = A_p x_p + B_p u + B_p d \\ y_p = C_p x_p, \end{cases} \quad (2)$$

where  $x_p \in \mathbb{R}^{n_p}$  denotes the state of the plant,  $d \in \mathbb{R}$  denotes an unknown but bounded disturbance and  $u \in \mathbb{R}$  the control input. The nominal SISO LTI controller is given by

$$\mathcal{C}_{nom} : \begin{cases} \dot{x}_c = A_c x_c + B_c e \\ u_c = C_c x_c + D_c e, \end{cases} \quad (3)$$

in which  $x_c \in \mathbb{R}^{n_c}$  denotes the state of the nominal part of the controller and  $e := r - y_p \in \mathbb{R}$  the closed-loop error, with reference  $r \in \mathbb{R}$ . The SPANI, see Fig. 2 (or Fig. 5(a)), contains a linear integrator with gain  $\omega_i \in \mathbb{R}$ . Due to the absolute value and *sign*-function (1), the output  $u_s$  of the SPANI is forced to take the same sign as the error signal  $e$ . Hence, we can model the SPANI controller as a switched system with dynamics

$$\text{SPANI} : \begin{cases} \dot{x}_I = \omega_i e \\ u_s = \begin{cases} +x_I & \text{if } ex_I \geq 0 \\ -x_I & \text{if } ex_I < 0, \end{cases} \end{cases} \quad (4)$$

where  $x_I \in \mathbb{R}$  denotes the state of the integrator in the SPANI controller. For this SPANI controller, the situation where the ‘default’ integrator is active ( $u_s = +x_I$ ) corresponds to  $ex_I \geq 0$  and the situation where the integrator swaps sign ( $u_s = -x_I$ ) corresponds to  $ex_I < 0$ , see Fig. 4(a) for a representation in the  $(e, x_I)$ -plane.

Now, we propose a slight modification to the switching condition of the SPANI filter, which is given as follows

$$\text{SPANI} : \begin{cases} \dot{x}_I = \omega_i e \\ u_s = \begin{cases} +x_I & \text{if } x_I(\epsilon x_I + e) \geq 0 \\ -x_I & \text{if } x_I(\epsilon x_I + e) < 0, \end{cases} \end{cases} \quad (5)$$

for some (typically small)  $\epsilon > 0$  associated with tilting of one of the switching boundaries, see Fig. 4(b) for a representation in the  $(e, x_I)$ -plane and Fig. 5(b) for a schematic representation in a block diagram. This modification indeed prevents the oscillatory behavior, observed in Fig. 3, from occurring. This can be intuitively explained as follows. Consider Fig. 4 and focus first on the default SPANI, i.e., Fig. 4(a). Note that the equilibrium point, with  $(e, x_I) = (e^*, x_I^*)$ , is located as represented in the figure, i.e., exactly on the switching plane (note that this is the case since  $e^* = 0$  is enforced by the integral action, and typically, e.g., if constant disturbances are present, it takes integral action to achieve this ( $x_I^* \neq 0$ )). Hence, small variations around this equilibrium may cause the dynamics to switch, resulting in an instantaneous change of sign of  $u_s$ . By introducing the tilting parameter  $\epsilon$ , we are able to achieve that the equilibrium is located strictly inside the set where  $x_I(\epsilon x_I + e) \geq 0$ , see Fig. 4(b). We will show in Section IV that this results in asymptotic tracking of constant

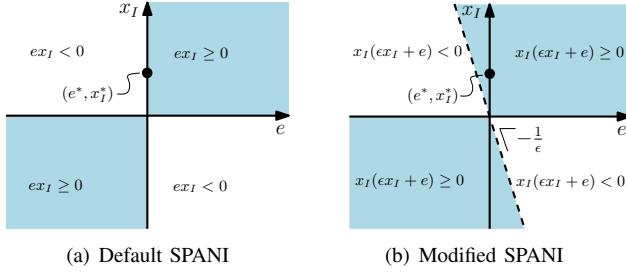


Fig. 4. Schematic representation of the  $(e, x_I)$ -plane for the feedback configuration with both the default and the modified SPANI filter. Note that the switching boundaries belong to the blue areas in both cases.

references, even in the presence of constant disturbances, and that this prevents the oscillatory behavior in Fig. 3.

To obtain a complete closed-loop model of the feedback configuration in Fig. 2 (though with the modified SPANI as in Figure 5(b)), we use the interconnections  $e := r - y_p$  and  $u := u_c + u_s$ , combine (2), (3) and (5), and define the augmented state-vector  $x := [x_p^\top x_c^\top x_I^\top]^\top \in \mathbb{R}^n$ , where  $n = n_p + n_c + 1$ , to arrive at a continuous-time switched dynamical model given by

$$\dot{x} = \begin{cases} \bar{A}_1 x + \bar{B}_r r + \bar{B}_d d & \text{if } x_I(\epsilon x_I + e) \geq 0 \\ \bar{A}_2 x + \bar{B}_r r + \bar{B}_d d & \text{if } x_I(\epsilon x_I + e) < 0, \end{cases} \quad (6a)$$

$$(6b)$$

with output  $y_p = \bar{C}x$ , and where

$$\bar{A}_1 := \begin{bmatrix} A_p - B_p D_c C_p & B_p C_c & +B_p \\ -B_c C_p & A_c & 0 \\ -\omega_i C_p & 0 & 0 \end{bmatrix}, \quad (7a)$$

$$\bar{A}_2 := \begin{bmatrix} A_p - B_p D_c C_p & B_p C_c & -B_p \\ -B_c C_p & A_c & 0 \\ -\omega_i C_p & 0 & 0 \end{bmatrix}, \quad (7b)$$

$$\bar{B}_r := \begin{bmatrix} B_p D_c \\ B_c \\ \omega_i \end{bmatrix}, \quad \bar{B}_d := \begin{bmatrix} B_p \\ 0 \\ 0 \end{bmatrix}, \quad \bar{C} := \begin{bmatrix} C_p^\top \\ 0 \\ 0 \end{bmatrix}^\top. \quad (7c)$$

Observe that the switched dynamical system (6), (7) contains a lot of structure, i.e., the closed-loop system matrices  $\bar{A}_1$  and  $\bar{A}_2$  are almost identical, except for the minus sign in the upper right element in  $\bar{A}_2$ . By proper design, the linear controller  $\mathcal{C}_{nom}(s) + C_I(s)$ , see Fig. 2, is stabilizing, and as a result, the matrix  $\bar{A}_1$  is always Hurwitz. Due to the sign change in its upper right element, this generally does not hold for the matrix  $\bar{A}_2$ . In fact, due to the wrong sign of the integral action  $\bar{A}_2$  will in general not be Hurwitz. When observing the  $(e, x_I)$ -plane in Fig. 4(b), the set where the stable dynamics (6a) are active corresponds with the blue areas and the set where the unstable dynamics (6b) are active corresponds to the white areas.

#### IV. STABILITY ANALYSIS

In this section, LMI-based conditions are presented to verify stability of the closed-loop dynamics. Moreover, we revisit the example of Section II-C, in which we will show that, firstly, the oscillatory behavior is prevented by the proposed modification to the SPANI (5), and secondly, that the stability conditions can be used to determine a proper choice for the tilting parameter  $\epsilon > 0$ .

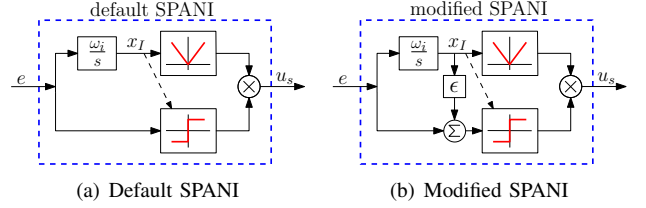


Fig. 5. Schematic representation of both the default SPANI filter, see (4), and the modified SPANI filter, see (5).

#### A. Stability conditions

In this section, we consider constant (step) references  $r(t) = r_c$ ,  $t \in \mathbb{R}_{\geq 0}$ , and constant disturbances  $d(t) = d_c$ ,  $t \in \mathbb{R}_{\geq 0}$ , and present sufficient conditions to verify global exponential stability (GES) of the (unique) equilibrium point  $x^*$  of (6), (7) that satisfies

$$\bar{A}_1 x^* + \bar{B}_r r_c + \bar{B}_d d_c = 0. \quad (8)$$

Note that, from (8) (or (5)) it follows that (due to the integral action)  $e^* = 0$  in the equilibrium  $x^*$ , such that the equilibrium conforms to the  $\bar{A}_1$  dynamics.

**Remark IV.1** Note that sliding motions, see e.g., [14], [15], will typically not exist in the control configuration studied in this paper when considering motion systems. However, to cover the cases in which they do occur, we will explicitly take the possibility of sliding modes into account in deriving the stability conditions.

The following theorem poses sufficient conditions under which GES of the equilibrium  $x^*$  can be guaranteed for the switched dynamical system (6), (7). Consequently, under these conditions the exact tracking of the constant reference  $r_c$ , and disturbance rejection of the constant disturbance  $d_c$ , is guaranteed. Hereto, let us introduce a few definitions. A matrix  $Q$  is given by

$$Q := \begin{bmatrix} \bar{A}_2^\top P + P \bar{A}_2 & P \bar{A}_d \bar{A}_1^{-1} \bar{B}_r & P \bar{A}_d \bar{A}_1^{-1} \bar{B}_d \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}, \quad (9)$$

with  $\bar{A}_d := \bar{A}_1 - \bar{A}_2$ . Furthermore, a matrix  $\bar{R}$  is given by

$$\bar{R} := \left[ \begin{array}{cc|c|c|c} 0 & 0 & -\frac{1}{2} C_p^\top & -\frac{1}{2} \gamma_r C_p^\top & -\frac{1}{2} \gamma_d C_p^\top \\ * & 0 & 0 & 0 & 0 \\ * & * & \epsilon & \epsilon \gamma_r & \epsilon \gamma_d \\ \hline * & * & * & \epsilon \gamma_r^2 & \epsilon \gamma_r \gamma_d \\ * & * & * & * & \epsilon \gamma_d^2 \end{array} \right], \quad (10)$$

for scalars  $\gamma_r = -[O_{1 \times n_p} \ O_{1 \times n_c} \ 1] \bar{A}_1^{-1} \bar{B}_r$  and  $\gamma_d = -[O_{1 \times n_p} \ O_{1 \times n_c} \ 1] \bar{A}_1^{-1} \bar{B}_d$ , related to the integral state in equilibrium i.e.,  $x_I^* = \gamma_r r_c + \gamma_d d_c$ . Let us motivate these scalars by considering the transfer function from  $r, d$  to  $x_I$ , see Fig. 2, which is given by  $x_I(s) = \frac{\omega_i}{s + s\mathcal{P}(s)\mathcal{C}_{nom}(s) + \mathcal{P}(s)\omega_i} r(s) - \frac{\mathcal{P}(s)\omega_i}{s + s\mathcal{P}(s)\mathcal{C}_{nom}(s) + \mathcal{P}(s)\omega_i} d(s)$ , such that in the equilibrium it holds that  $x_I^* = \frac{1}{\mathcal{P}(0)} r_c - d_c$ . Hence,  $\gamma_r = \frac{1}{\mathcal{P}(0)}$  and  $\gamma_d = -1$ . Finally, let the matrix  $M$  be given by

$$M := \begin{bmatrix} I_{n \times n} & O_{n \times 1} \\ O_{2 \times n} & \begin{bmatrix} \gamma_r \\ \gamma_d \end{bmatrix} \end{bmatrix}. \quad (11)$$

**Theorem IV.2** Consider the switched system (6), (7) for some  $\epsilon > 0$ , in which we assume that  $\bar{A}_1$  is a Hurwitz matrix. If there exist a positive definite matrix  $P = P^\top$  and a constant  $\alpha \geq 0$  satisfying

$$\bar{A}_1^\top P + P \bar{A}_1 \prec 0 \quad (12)$$

$$M^\top (Q - \alpha \bar{R}) M \prec 0, \quad (13)$$

then the equilibrium point  $x^*$  of system (6), (7) satisfying  $e^* = r_c - C_p x_p^* = 0$ , is globally exponentially stable for any constant reference  $r_c$  and any constant disturbance  $d_c$ , even in the presence of sliding modes.

*Proof:* The proof is omitted due to space limitations, but can be found in [16]. ■

**Remark IV.3** Note that by increasing  $\epsilon$ , the stable  $\bar{A}_1$  dynamics is active in a larger region of the state-space, see Fig. 4. In fact, for  $\epsilon = \infty$ , the linear, and stabilizing, controller  $C_{nom}(s) + C_I(s)$  is active in the entire state-space. This fact gives rise to the intuition that stability of the closed-loop system can be guaranteed by choosing  $\epsilon$  large enough. From a overshoot-reduction point of view, however, a small  $\epsilon$  is favorable. Hence, a trade-off between stability and performance arises and Theorem IV.2 is instrumental in making this trade-off.

### B. Motivating example revisited

Let us again consider the illustrative example of Section II-C. In this example, we demonstrated that our proposed SPANI filter (4) has the potential to increase transient performance with respect to overshoot, but that its steady-state behavior was worsened compared to a linear feedback control configuration due to the occurrence of oscillations. In this section, we will compare both responses of Section II-C to the response obtained with a modified SPANI filter (5), in which  $\epsilon$  is selected as  $\epsilon = 0.04$ . In fact,  $\epsilon = 0.04$  is the lower bound for which the conditions of Theorem IV.2 yield feasible results<sup>1</sup>.

The dash-dotted black line in the upper plot of Fig. 6 shows the response using the modified SPANI filter (5). Clearly, the response asymptotically converges to its desired value. However, this is achieved at the expense of a (slight) increase in overshoot, as is shown on the ‘zoom’ part of the transient response in the upper plot of Fig. 6. The bottom figure in Fig. 6 shows the integral action  $u_s$  of the controllers, which reveals in the upper zoom part that the swap in integral action occurs later in time for the modified SPANI filter, causing an increase in overshoot compared to the case in which  $\epsilon = 0$ . On the other hand, the steady-state oscillations in  $u_s$  are the highest in amplitude (and non-vanishing) for the default SPANI filter, see again the bottom figure of Fig. 6.

<sup>1</sup>From simulations it follows that any  $\epsilon > 0$  yields an asymptotically stable response. This discrepancy between the simulations and the LMI-based stability conditions of Theorem IV.2 might be relaxed (in terms of conservatism) by considering piecewise quadratic Lyapunov functions instead.

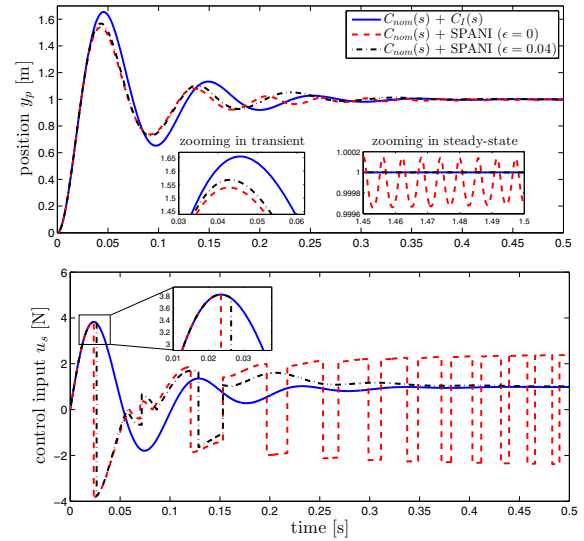


Fig. 6. Motivating example revisited. The upper figure shows the response  $y_p$  and the bottom figure the integral action  $u_s$  to a step reference input.

Therefore, we may conclude that taking the parameter  $\epsilon$  larger than zero is necessary from a stability point of view, although it also limits the potential transient performance profits. It is therefore up to the user to balance this trade-off in the most optimal sense, in which the conditions of Theorem IV.2 may serve as a guideline (see also Section V and Remark IV.3).

## V. SIMULATION EXAMPLE

In this section, the proposed nonlinear control strategy will be applied to a simulation model of a 4th-order motion system and compared with the results obtained by linear feedback control. The motion system under study consists of a non-collocated motor-load system, which resembles a realistic industrial motion control setting where sensors and actuators are typically placed at different locations. In addition, the system is subject to an external force disturbance, which motivates the use of integral control in order to obtain a zero steady-state error, see e.g., [10].

### A. Plant model and linear controller design

The plant dynamics are described by the following 4th-order LTI model

$$\mathcal{P}(s) = \frac{4.746 \cdot 10^8}{s^4 + 4.22s^3 + 1.364 \cdot 10^5 s^2}, \quad s \in \mathbb{C}, \quad (14)$$

which will form the basis for controller design. Using manual loop-shaping techniques, a nominal stabilizing controller  $C_{nom}(s)$  without integral action is designed, consisting of a lead filter, a notch filter, and a 2nd-order low-pass filter, leading to

$$C_{nom}(s) = \frac{1.399 \cdot 10^{-7} s^3 + 2.715 \cdot 10^{-6} s^2}{3.989 \cdot 10^{-14} s^5 + 5.915 \cdot 10^{-11} s^4 \cdots + 0.01911s + 0.3} \cdots \frac{\cdots}{+5.329 \cdot 10^{-8} s^3 + 2.612 \cdot 10^{-5} s^2 + 0.007958s + 1}, \quad (15)$$



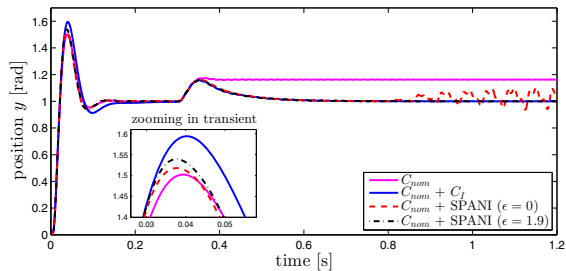


Fig. 7. Response to a step reference input for the linear feedback system with no integral action (solid magenta), with integral action (solid blue), and for the configuration with SPANI, using (4) (dashed red) and (5) (with  $\epsilon = 1.9$ ) (dash-dotted black).

in which  $s \in \mathbb{C}$ . The linear integrator  $C_I(s) = \omega_i/s$  is designed in parallel to  $C_{nom}(s)$  with gain  $\omega_i = 1.8\pi$ . The overall linear controller design results in satisfactory stability margins and a bandwidth around 10 Hz for the open-loop transfer function  $O_{L,I}(s) = \mathcal{P}(s)(C_{nom}(s) + C_I(s))$ .

As discussed before in Section II, once a stabilizing controller structure has been designed, incorporating a SPANI filter into the controller configuration follows straightforwardly by replacing  $C_I(s)$  by the SPANI, see Fig. 2, using the same gain  $\omega_i$ . The results presented in Theorem IV.2 can be used to determine a favorable choice for  $\epsilon$ . Using a balanced state-space representation of (6), (7), and a line search over  $\epsilon$ , one can show that the conditions of Theorem IV.2 yield feasible results for any  $\epsilon \geq 1.9$ . Hence, in the remainder of this example, we select  $\epsilon = 1.9$ .

### B. Transient performance comparison

Let us compare the transient performance of a feedback control system with and without SPANI. In order to do so, we study the response subject to a step reference of  $r(t) = r_c = 1$  rad, applied at the system at  $t = 0$  s, and a step disturbance  $d_c = 0.1$  V acting on the system at  $t = 0.3$  s.

First the response for the two linear controllers is considered, see Fig. 7. Clearly, the use of an integrator increases the overshoot of the system's closed-loop response (blue versus magenta). However, the configuration with only the nominal controller  $C_{nom}(s)$  is not capable of removing the steady-state errors due to constant force disturbances  $d_c$ .

Consider now the nonlinear controllers, which both show a reduction in overshoot compared to the linear controller with integral action  $C_I(s)$ . Clearly, the response of the configuration with 'default' SPANI ((5) with  $\epsilon = 0$ ) shows the least amount of overshoot. However, as we can conclude from Fig. 7, selecting  $\epsilon = 0$  leads to unstable behavior. This proves the usefulness of our presented stability conditions in Theorem IV.2, as these allow us to compute a lower bound on  $\epsilon$  such that closed-loop stability of (6), (7) is guaranteed. By using the configuration with SPANI in which we select  $\epsilon = 1.9$  (dash-dotted black), we see a reduction in overshoot compared to  $C_{nom}(s) + C_I(s)$ , while the system asymptotically converges to  $y_p = r_c = 1$  rad, even with the presence of a constant force disturbance  $d = d_c = 0.1$  [V].

## VI. CONCLUSIONS

In this paper, we introduced the split-path nonlinear integrator (SPANI) as a novel variation/extension to a nonlinear filter, that was originally introduced in the late 1960s. The SPANI is especially designed for transient performance improvement of linear systems. In particular, we focussed on the transient performance improvement in terms of overshoot to step responses, while being able to achieve zero steady-state errors in the presence of constant disturbances. The potential of the proposed SPANI controller was demonstrated by means of simulations. Moreover, it was shown that the feedback control configuration with SPANI can be modeled as a continuous-time switched dynamical system, for which sufficient Lyapunov-based stability conditions have been provided in terms of LMIs. These conditions proved to be useful in the design of the SPANI, which on itself, appeared to be easy accessible for control engineers as all the individual components of the proposed nonlinear controller can be obtained using classical loop-shaping techniques.

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