53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

Improved $\mathcal{L}_2$-gain analysis for a class of hybrid systems with applications to reset and event-triggered control

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Abstract— **In this paper, we consider a special class of hybrid systems with periodic time-triggered jump conditions, and in which the jump map has a piecewise linear character. This hybrid systems class forms a relevant field of study as different control applications can be modeled in this hybrid system framework, including reset control and event-triggered control systems. After showing the unifying modeling character of this class of dynamical systems, we are interested in analyzing stability and $\mathcal{L}_2$-gain properties and we present novel conditions to do so which are significantly less conservative than the existing ones in literature. The effectiveness of the proposed modeling and analysis techniques is illustrated by means of a reset control example.**

I. INTRODUCTION

Hybrid systems [1], combine continuous dynamics, often called flow dynamics and represented by ordinary differential equations on the one hand, and discrete dynamics, which are sometimes captured through jump dynamics and represented by instantaneous jumps/resets of states on the other hand. In this paper, we are interested in a particular class of hybrid systems with periodic time-triggered jump conditions and piecewise linear (PWL) jump maps. This class of hybrid systems finds its use in a broad spectrum of control applications including reset control [2]–[7], and event-triggered control [8]–[11], as we will highlight in this paper. In particular, we show that both application classes can be modeled in the considered hybrid modeling framework.

Besides showing the unifying modeling character of the studied class of hybrid systems, we are also interested in the stability and $\mathcal{L}_2$-gain analysis of these dynamical systems. The latter is an important performance measure for many situations, and has already attracted quite some attention in the literature [1], [11], [12] and [3], [4], [6], [13] for related classes of hybrid systems. Especially the work in [11], [12] focused on the hybrid systems class with periodic time-triggered jump conditions and exploited common quadratic timer-dependent Lyapunov/storage functions based on solutions to Riccati differential equations, see e.g., [14]. This analysis led to conditions based on linear matrix inequalities (LMIs) for obtaining and upper bounds on the $\mathcal{L}_2$-gain. In fact, in this paper we employ a similar analysis but instead of using a *common quadratic* timer-dependent Lyapunov/storage function, we propose to use more versatile timer-dependent piecewise quadratic (PWQ) Lyapunov/storage functions, thereby providing improved conditions for $\mathcal{L}_2$-gain estimates compared to the existing ones in literature. In contrast to the standard use of PWQ Lyapunov functions [15], due to the presence of both flow and jump dynamics, and the timer dependence of the Lyapunov/storage function for the $\mathcal{L}_2$-gain analysis, new proof techniques are needed. In particular, the proof of our main result is based on using *trajectory-dependent* Lyapunov/storage functions in the sense that the functions do not only depend on the actual value of the state, but also on (future) disturbance values. In order to show that the realized conditions result in better estimates of the $\mathcal{L}_2$-gain than the existing ones [11], we will provide a numerical example that indeed illustrates the realized improvements.

Summarizing, the contribution of this paper is twofold. First, we will show that the presented hybrid systems framework covers a broad variety of control applications, thereby demonstrating the unifying character of the hybrid systems class under study. The second contribution is formed by providing improved $\mathcal{L}_2$-gain estimates compared to the existing ones in literature.

The remainder of the paper is organized as follows. In Section II, we introduce a general representation of the hybrid modeling framework that we study in this paper, and provide the problem formulation. In Section III, we show how two control applications can be modeled in this unifying framework. The main result on improved conditions to analyze the stability and $\mathcal{L}_2$-gain properties of the hybrid system under study is presented in Section IV, and the effectiveness of the conditions is demonstrated using a numerical example in Section V. Finally, we end with conclusions in Section VI.

A. Nomenclature

The following notational conventions will be used. Let $\mathbb{N}$, $\mathbb{R}$ denote the set of non-negative integers and real numbers, respectively. We call a matrix $P \in \mathbb{R}^{n\times n}$ positive definite and write $P \succ 0$, if $P = P^\top$ and $x^\top Px > 0$ for all $x \neq 0$. Similarly, we call $P \in \mathbb{R}^{n\times n}$ negative definite, and write $P \prec 0$, when $P = P^\top$ and $x^\top Px < 0$ for all $x \neq 0$. We use

The work of Bas van Loon is supported by the Dutch Technology Foundation (STW) under project "HyperMotion: Hybrid Control for Performance Improvement of Linear Motion Systems" (no. 10953), and the work of Maurice Heemels is supported by the VICI grant "Wireless control systems: A new frontier in automation" (no. 11382) (STW/NWO). The work of Andy Teel is supported in part by AFOSR FA9550-12-1 0127 and NSF ECCS-1232035.

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I_n to denote the identity matrix with dimensions $n \times n$. For brevity, we write symmetric matrices of the form $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$ sometimes as $\begin{bmatrix} A & B \\ \star & C \end{bmatrix}$. Furthermore, a function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{K}_\infty$ function if it is zero at zero, continuous, strictly increasing and unbounded, i.e., $\lim_{s\to\infty} \phi(s) = \infty$.

II. Hybrid Model Class and Problem Formulation

In this paper, we study the class of hybrid systems given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \xi \\ \tau \end{bmatrix} = \begin{bmatrix} A\xi + Bw \\ 1 \end{bmatrix}, \text{ when } \tau \in [0, h] \tag{1a}$$

$$\begin{bmatrix} \xi^+ \\ \tau^+ \end{bmatrix} = \begin{cases} \begin{bmatrix} J_1\xi \\ 0 \end{bmatrix}, & \text{when } \tau = h \text{ and } \xi^\top Q\xi > 0 \\ \begin{bmatrix} J_2\xi \\ 0 \end{bmatrix}, & \text{when } \tau = h \text{ and } \xi^\top Q\xi \leq 0 \end{cases} \tag{1b}$$

$$z = C\xi + Dw. \tag{1c}$$

The states of this hybrid system consist of $\xi \in \mathbb{R}^{n_\xi}$ and a timer variable $\tau \in \mathbb{R}_{\geq 0}$. The variable $w \in \mathbb{R}^{n_w}$ denotes the disturbance input and z the performance output. Moreover, A, B, C, D, J_1, J_2, Q are constant real matrices of appropriate dimensions and $h \in \mathbb{R}_{>0}$ is a positive timer threshold.

Interpreting the dynamics of (1), which is done in the sense of [1], reveals that (1) has *periodic* time-triggered jump conditions, i.e., jumps take place at times kh, $k \in \mathbb{N}$. Note that this guarantees amongst others that the hybrid system (1) produces global solutions, i.e., defined on $t \in [0, \infty)$, and Zeno behavior, see, e.g., [1], does not occur. Moreover, the jump map is a (possibly discontinuous) piecewise linear (PWL) map given by (1b), and in between the jumps the system flows according to the differential equations in (1a). This class of systems includes the closed-loop systems arising from reset control systems [5], and periodic event-triggered control (PETC) for linear systems [11], and many more. How the two mentioned applications can be modeled in this unifying framework is discussed in detail in Section III below.

Remark II.1 *The hybrid system* (1) *could be seen as a sampled-data or time-regularized version of the hybrid system*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\xi = A\xi + Bw, \quad \textit{when } \xi^\top Q\xi > 0 \tag{2a}$$

$$\xi^+ = J\xi, \quad \textit{when } \xi^\top Q\xi \leq 0. \tag{2b}$$

Indeed, if we take in (1) $J_1 = I_{n_\xi}$ *and* $J_2 = J$, *the resulting hybrid model can be seen as an implementation of the jump rule in* (2b) *but only verified at the times* $t_k = kh$, $k \in \mathbb{N}$. *Such regularizations are often used when* (2) *might exhibit Zeno behavior, see, e.g., [4], [5], [11], [13] (the occurrence of Zeno behavior in systems as in* (2) *has been investigated in [16]). Clearly, its 'sampled-data' version of the form* (1) *has not, which therefore has analysis and implementation advantages.*

Remark II.2 *Note that the hybrid system* (1) *has a PWL jump map with only two regions specified by* $\xi^\top Q\xi > 0$ *and* $\xi^\top Q\xi \leq 0$, *respectively. However, the modeling and analysis provided below can easily be extended to conewise linear jump maps with more than* 2 *regions.*

In this paper, we are, besides showing the (unifying) modeling character of the class of systems described by (1), also interested in the stability and $\mathcal{L}_2$-gain analysis of these systems.

Definition II.1 *The hybrid system* (1) *is said to be globally exponentially stable (GES), if there exist* $c > 0$ *and* $\rho > 0$ *such that for any initial condition* $\xi(0) = \xi_0 \in \mathbb{R}^{n_\xi}$ *all corresponding solutions to* (1) *with* $\tau(0) \in [0, h]$ *and* $w = 0$ *satisfy* $\|\xi(t)\| \leq ce^{-\rho t}\|\xi_0\|$ *for all* $t \in \mathbb{R}_{\geq 0}$. *In this case, we call* ρ *a (lower bound on the) decay rate.*

Definition II.2 *The hybrid system* (1) *is said to have an* $\mathcal{L}_2$-*gain from* w *to* z *smaller than or equal to* γ, *if there is a* $\mathcal{K}_\infty$ *function* $\delta : \mathbb{R}^{n_\xi} \to \mathbb{R}_{\geq 0}$ *such that for any* $w \in \mathcal{L}_2$, *any initial state* $\xi(0) = \xi_0 \in \mathbb{R}^{n_\xi}$, *and* $\tau(0) \in [0, h]$, *the corresponding solution to* (1) *satisfies* $\|z\|_{\mathcal{L}_2} \leq \delta(\xi_0) + \gamma\|w\|_{\mathcal{L}_2}$, *where* $\mathcal{L}_2$ *denotes the set of square-integrable functions and* $\|\cdot\|_{\mathcal{L}_2}$ *the corresponding* $\mathcal{L}_2$-*norm.*

Before presenting new techniques to analyze GES and the $\mathcal{L}_2$-gain, we will first shown the unifying modeling capabilities of the model class (1).

III. Unified Modeling Framework

In this section, we will consider two different control applications, namely, periodic event-triggered control and reset control systems, and show that they can be written in the hybrid system framework given by (1).

A. Periodic Event-Triggered Control Systems

The first domain of application is event-triggered control (ETC), see e.g., [8], [9], [11] for some recent approaches, and [10] for a recent overview. ETC is a control strategy that is designed to reduce the amount of computations and communications in a feedback control system by updating and communicating sensor and actuator data only when needed to guarantee stability or performance properties. The ETC strategy that we consider in this paper is recently proposed in [11] as a novel ETC strategy for linear systems that combines ideas from periodic sampled-data control and ETC, leading to so-called periodic event-triggered control (PETC) systems. In PETC, the event-triggering condition is verified periodically in time instead of continuously as in standard ETC [8], [9]. Hence, at every sampling interval it is decided whether or not new measurements and control signals need to be computed and transmitted.

In the PETC setting [11] that we consider in this paper, the plant is given by a continuous-time linear time-invariant (LTI) system of the form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x_p &= A_px_p + B_{pu}u + B_{pw}w \\ y &= C_px_p, \end{cases} \tag{3}$$

where $x_p \in \mathbb{R}^{n_p}$ denotes the state of the plant, $u \in \mathbb{R}^{n_u}$ the control input and $y \in \mathbb{R}^{n_y}$ the plant output. This plant (3) is

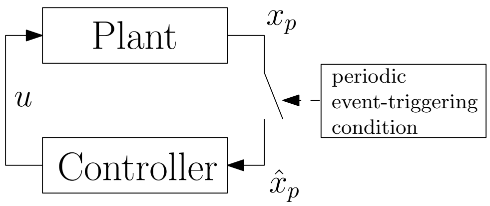

Fig. 1. Schematic representation of an event-triggered control system.

controlled in an event-triggered feedback fashion using the following state-feedback controller

$$u(t) = K\hat{x}_p(t), \quad \text{for } \; t \in \mathbb{R}_{\geq 0}, \tag{4}$$

where $\hat{x}_p \in \mathbb{R}^{n_p}$ is a left-continuous signal[1], given for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$, by

$$\hat{x}_p(t) = \begin{cases} x_p(t_k), & \text{when } \; \xi(t_k)^\top Q \xi(t_k) > 0, \\ \hat{x}_p(t_k), & \text{when } \; \xi(t_k)^\top Q \xi(t_k) \leq 0, \end{cases} \tag{5}$$

where $\xi := [x_p^\top \;\; \hat{x}_p^\top]^\top$ and t_k, $k \in \mathbb{N}$, are the sampling times, which are periodic in the sense that $t_k = kh$, $k \in \mathbb{N}$, with $h > 0$ the sampling interval. Fig. 1 shows a schematic representation of the PETC configuration that we consider in this paper. In this figure, $\hat{x}_p(t)$ denotes the most recently transmitted measurement of the state $x_p(t)$ to the controller. Whether or not $x_p(t)$ is transmitted is based on an event-triggering condition. In particular, if at time t_k it holds that $\xi^\top(t_k) Q \xi(t_k) > 0$, the current state $x_p(t_k)$ is transmitted to the controller and $\hat{x}_p$, and as a consequence u, are updated accordingly. If, however, $\xi^\top(t_k) Q \xi(t_k) \leq 0$, the current state information is not sent to the controller and $\hat{x}_p$ and u are kept the same for (at least) another sampling interval. In [11] it was shown that such quadratic event-triggering conditions form a relevant class of triggering conditions because many popular event triggering conditions can be written in this form. For instance, an event-triggering condition of the form

$$\|\hat{x}_p(t_k) - x_p(t_k)\| > \sigma \|x_p(t_k)\|, \tag{6}$$

with $\sigma > 0$, can be used to determine whether, at time t_k, it is required to transmit $x_p(t_k)$ to the controller, or that the latest sent value $\hat{x}_p(t_k)$ is still adequate. Clearly, condition (6) can be written in the quadratic form in (5) by taking

$$Q = \begin{bmatrix} (1-\sigma^2) I_{n_p} & -I_{n_p} \\ -I_{n_p} & I_{n_p} \end{bmatrix}. \tag{7}$$

The complete model of the PETC system can be captured in the hybrid system format of (1), by combining (3), (4) and (5), where we obtain

$$A = \begin{bmatrix} A_p & B_{pu}K \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_{pw} \\ 0 \end{bmatrix}, \quad J_1 = \begin{bmatrix} I_{n_p} & 0 \\ I_{n_p} & 0 \end{bmatrix},$$

and $J_2 = I_{n_\xi}$. In addition to the state feedback controller in (4), one can also use dynamic output-feedback PETC controllers and output-based event-triggering conditions [11] in a straightforward fashion.

[1] A signal $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is called left-continuous, if for all $t > 0$, $\lim_{s \uparrow t} x(s) = x(t)$.

B. Reset Control Systems

A second domain of applications of the class of hybrid systems that we consider in this paper is formed by reset control. Reset control is a discontinuous control strategy designed as a means to overcome the fundamental limitations of linear feedback by allowing to reset the controller state, or subset of states, whenever certain conditions on its input and output are satisfied, see e.g., [2]–[4]. In all afore-cited papers the reset condition is monitored continuously, while in [5] the authors proposed to verify the reset condition at discrete-time instances. In other words, at every sampling time $t_k = kh$, $k \in \mathbb{N}$, with sampling interval $h > 0$, it is decided whether or not a reset takes place. This periodic reset verification can indeed be modeled in the hybrid systems class of interest. In order to show this, we consider reset controllers to control systems of the form (3) of the type

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_c \\ \tau \end{bmatrix} = \begin{bmatrix} A_c x_c + B_c e \\ 1 \end{bmatrix}, \;\text{ when } \tau \in [0, h] \tag{8a}$$

$$\begin{bmatrix} x_c^+ \\ \tau^+ \end{bmatrix} = \begin{cases} \begin{bmatrix} x_c \\ 0 \end{bmatrix}, & \text{when } \tau = h \text{ and } \xi^\top Q \xi > 0 \\ \begin{bmatrix} R_c x_c \\ 0 \end{bmatrix}, & \text{when } \tau = h \text{ and } \xi^\top Q \xi \leq 0 \end{cases} \tag{8b}$$

$$u = C_c x_c + D_c e, \tag{8c}$$

where $x_c \in \mathbb{R}^{n_c}$ denotes the continuous state of the controller and x_c^+ its value after a reset, $R_c \in \mathbb{R}^{n_c \times n_c}$ is the reset matrix and $e := r - y \in \mathbb{R}$ is the error between the reference signal r and the output of the plant y. Note that although we consider continuous-time reset controllers of the type (8), the modeling framework in (8) also allows to study discrete-time reset controllers as well. This, however, is not considered here for brevity. The reset condition that we employ in this paper is based on the sign of the product between the error e and controller input $u \in \mathbb{R}$, and originally proposed in [13][2]. In particular, the reset controller (8) acts like a linear controller whenever its input e and output u have the same sign, i.e., $e^\top u > 0$, and it resets its output otherwise. This reset condition can be represented, for the case $r = 0$, in a general quadratic relation as in (8b), with

$$Q = \begin{bmatrix} C_p & 0 \\ -D_c C_p & C_c \end{bmatrix}^\top \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} C_p & 0 \\ -D_c C_p & C_c \end{bmatrix}. \tag{9}$$

Remark III.1 *Note that two well-known reset controllers, namely, the Clegg integrator and the first-order reset element (FORE), see e.g., [13] and the references therein, can be modeled as in* (8)*, if implemented in a periodic time-triggered manner, using*

$$\textit{Clegg integrator}: \; \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right], \tag{10}$$

$$\textit{FORE}: \; \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{c|c} -\beta & 1 \\ \hline 1 & 0 \end{array}\right], \tag{11}$$

in which $\beta \in \mathbb{R}$ denotes the single pole of the FORE.

[2] In [13], the reset condition is verified in continuous time, so not at discrete times $t_k = kh$, $k \in \mathbb{N}$ for some $h > 0$.

The interconnection of the reset control system (8) and plant (3) can be written in the hybrid system format of (1), with augmented state vector $\xi = [x_p^\top \ x_c^\top]^\top$, in which

$$A = \begin{bmatrix} A_p - B_{pu}D_cC_p & B_{pu}C_c \\ -B_cC_p & A_c \end{bmatrix},\ B = \begin{bmatrix} B_{pw} \\ 0 \end{bmatrix},\ J_2 = \begin{bmatrix} I_{n_p} & 0 \\ 0 & R_c \end{bmatrix},$$

and $J_1 = I_{n_\xi}$.

Remark III.2 *By showing that the PETC applications [11] and periodic reset control systems [5] fit in the same framework* (1)*, it is possible to use the stability conditions of [11] for PETC in the context of reset control [5], which actually provides less conservative stability conditions than used in [5], cf. [11, Theorem III.4] with [5, Proposition 4]. Transforming results from one application domain to another is one particular advantage of using a unifying modeling framework.*

Remark III.3 *Also specific classes of networked control systems as considered in e.g., [17], with quadratic protocols (such as the Try-Once-Discard protocol), can be modeled as hybrid systems of the form* (1)*, provided the transmission intervals are assumed constant and no transmission delays occur.*

IV. Stability and $\mathcal{L}_2$-Gain Analysis of the Hybrid System

In this section, we present improved conditions to analyze stability and performance of the hybrid system (1). As these conditions build upon results presented in [11, Section III.A], we first briefly recall this analysis, see [11] for more details.

A. Riccati-Based Analysis

In [11], an $\mathcal{L}_2$-gain analysis is performed on systems of the form (1), which is based on a Lyapunov/storage function [14] of the form

$$V(\xi, \tau) = \xi^\top P(\tau)\xi, \tag{12}$$

where $P : [0, h] \to \mathbb{R}^{n_\xi \times n_\xi}$ with $P(\tau) \succ 0$, for $\tau \in [0, h]$. The function $P : [0, h] \to \mathbb{R}^{n_\xi \times n_\xi}$ will be chosen such that it becomes a candidate storage function for the system (1) with the supply rate $\gamma^{-2}z^\top z - w^\top w$. In particular, we will select P to satisfy the Riccati differential equation (where we omitted τ for compactness of notation)

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}\tau}P = &-A^\top P - PA - 2\rho P - \gamma^{-2}C^\top C \\ &- (PB + \gamma^{-2}C^\top D)M(B^\top P + \gamma^{-2}D^\top C), \end{aligned} \tag{13}$$

provided the solution exists on $[0, h]$ for a desired convergence rate $\rho > 0$, in which $M := (I - \gamma^{-2}D^\top D)^{-1}$ is assumed to exist and to be positive definite, which means that $\gamma^2 > \lambda_{\max}(D^\top D)$. It is proven in [11] that this choice for the matrix function P yields

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V \leq -2\rho V - \gamma^{-2}z^\top z + w^\top w, \tag{14}$$

during flow (1a). Moreover, in [11], additional conditions were derived to guarantee that during jumps

$$V(J_1\xi, 0) \leq V(\xi, h), \quad \text{for all } \xi \text{ with } \xi^\top Q\xi > 0, \tag{15a}$$
$$V(J_2\xi, 0) \leq V(\xi, h), \quad \text{for all } \xi \text{ with } \xi^\top Q\xi \leq 0, \tag{15b}$$

implying that the storage function does not increase during the jumps (1b). These conditions were obtained by relating $P_0 := P(0)$ to $P_h := P(h)$. In order to do so, the Hamiltonian matrix

$$H := \begin{bmatrix} A + \rho I + \gamma^{-2}BMD^\top C & BMB^\top \\ -C^\top LC & -(A + \rho I + \gamma^{-2}BMD^\top C)^\top \end{bmatrix}, \tag{16}$$

was introduced in which $L := (\gamma^2 I - DD^\top)^{-1}$. Finally, the matrix exponential

$$F(\tau) := e^{-H\tau} = \begin{bmatrix} F_{11}(\tau) & F_{12}(\tau) \\ F_{21}(\tau) & F_{22}(\tau) \end{bmatrix} \tag{17}$$

was defined, which enables the computation of the explicit solution to the Riccati equation (13), see [11] for further details.

Assumption IV.1 $F_{11}(\tau)$ *is invertible for all* $\tau \in [0, h]$.

Let us also introduce the notation $\bar{F}_{11} := F_{11}(h)$, $\bar{F}_{12} := F_{12}(h)$, $\bar{F}_{21} := F_{21}(h)$ and $\bar{F}_{22} := F_{22}(h)$, and the matrix $\bar{S}$ that satisfies $\bar{S}\bar{S}^\top := -\bar{F}_{11}^{-1}\bar{F}_{12}$. The matrix $\bar{S}$ exists under Assumption IV.1, because this assumption will guarantee that the matrix $-\bar{F}_{11}^{-1}\bar{F}_{12}$ is positive semi-definite.

Theorem IV.1 *[11] Consider the impulsive system* (1) *and let* $\rho > 0$ *and* $\gamma > \sqrt{\lambda_{max}(D^\top D)}$ *be given. Assume that Assumption IV.1 holds and that there exist a matrix* $P_h \succ 0$*, and scalars* $\mu_i \geq 0$*,* $i \in \{1, 2\}$*, such that*

$$\begin{bmatrix} P_h + (-1)^i\mu_i Q & J_i^\top \bar{F}_{11}^{-\top}P_h\bar{S} & J_i^\top(\bar{F}_{11}^{-\top}P_h\bar{F}_{11}^{-1} + \bar{F}_{21}\bar{F}_{11}^{-1}) \\ \star & I - \bar{S}^\top P_h\bar{S} & 0 \\ \star & \star & \bar{F}_{11}^{-\top}P_h\bar{F}_{11}^{-1} + \bar{F}_{21}\bar{F}_{11}^{-1} \end{bmatrix} \succ 0. \tag{18}$$

Then, the hybrid system (1) *is GES with convergence rate* ρ *(when* $w = 0$*) and has an* $\mathcal{L}_2$*-gain from* w *to* z *smaller than or equal to* γ*.*

The conditions (18) guarantee indeed that (15) holds, and (14) and (15) together can be used to establish GES and an $\mathcal{L}_2$-gain smaller than or equal to γ.

B. Main result

The analysis in [11], as recalled above, is based on a *common* quadratic timer-dependent Lyapunov/storage function as in (12). The novelty in our improved conditions lies in the fact that we will use a more versatile timer-dependent piecewise quadratic Lyapunov/storage function, see e.g., [15], [18], based on the regions

$$\Omega_i := \Big\{\xi \in \mathbb{R}^{n_\xi} \,\Big|\, \xi^\top X_i\xi \geq 0\Big\}, \tag{19}$$

in which the symmetric matrices X_i, $i \in \{1, \ldots, N\}$, are such that $\{\Omega_1, \Omega_2, \ldots, \Omega_N\}$ forms a partition of $\mathbb{R}^{n_\xi}$, i.e., $\cup_{i=1}^N \Omega_i = \mathbb{R}^{n_\xi}$ and the intersection of Ω_i and Ω_j, $i, j \in \{1, \ldots, N\}$, is of zero measure. We assume that $\{\xi \in \mathbb{R}^{n_\xi} \mid \xi^\top Q\xi \leq 0\} \subseteq \bigcup_{i=1}^{N_1} \Omega_i$ and $\{\xi \in \mathbb{R}^{n_\xi} \mid \xi^\top Q\xi \geq 0\} \subseteq \bigcup_{i=N_1+1}^{N} \Omega_i$.

Note that the construction of a Lyapunov/storage function for the hybrid system (1) is less straightforward if compared to the more classical case of discrete-time and continuous-time piecewise affine systems (see [15], [18]). This is due to the presence of both flow and jump dynamics and the fact that the jumps do depend both on time and the state. To introduce the storage function we use, we need the following notation. We denote by $\bar{\xi}(t, \xi_0, w)$ the solution to $\dot{\xi} = A\xi + Bw$ at time t with $\xi(0) = \xi_0$ and input $w \in \mathcal{L}_2$. Given $w \in \mathcal{L}_2$ and some fixed $t \in \mathbb{R}_{\geq 0}$, we denote by w_t the time-shifted signal given by $w_t(s) = w(s+t)$ for $s \geq 0$. We propose now to use the timer-dependent PWQ storage function given by

$$W(\xi, \tau, w, t) = \xi^\top \bar{P}_i(\tau)\xi \quad \text{if} \quad \bar{\xi}(h-\tau, \xi, w_t) \in \Omega_i, \tag{20}$$

where $\bar{P}_i(\tau)$, for each $i \in \{1, \dots, N\}$, is a solution to the Riccati differential equation

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}\tau}\bar{P}_i &= -A^\top \bar{P}_i - \bar{P}_i A - 2\rho \bar{P}_i - \gamma^{-2}C^\top C \\ &\quad - (\bar{P}_i B + \gamma^{-2}C^\top D)M(B^\top \bar{P}_i + \gamma^{-2}D^\top C),\end{aligned} \tag{21}$$

for a desired convergence rate $\rho > 0$, satisfying $\bar{P}_i(h) = P_i$, where the positive definite matrices P_i, $i \in \{1, \dots, N\}$, are chosen according to solutions of the LMIs presented in Theorem IV.2 below. Interestingly, note that the value of $W(\xi, \tau, w, t)$ at time t (given τ) depends on ξ and $w|_{[t, t+h-\tau]}$ and thus on *future* disturbance values. As such, we have a trajectory/disturbance-dependent Lyapunov/storage function. Thus the trajectory-dependence of the Lyapunov/storage function deviates from the common results in the literature on dissipativity or $\mathcal{L}_2$-gain analysis as there the Lyapunov/storage function typically depends only on the current state (or sometimes time), but not on future values of the disturbances/state. As a consequence, the interpretation of W as a genuine Lyapunov/storage function is less natural. We merely use it as a function (or functional) to establish the desired $\mathcal{L}_2$-gain properties in the mathematical proof.

Theorem IV.2 *Let $\gamma > \sqrt{\lambda_{\max}(D^\top D)}$, $N_1 < N$, and Assumption IV.1 hold. Suppose that there exist matrices $P_i = P_i^\top$, $i \in \{1, \dots, N\}$, and scalars $\mu_{i,j} \geq 0$, $i, j \in \{1, \dots, N\}$, satisfying*

$$\begin{bmatrix} P_i - \mu_{i,j}X_i & J_1^\top \bar{F}_{11}^{-\top} P_j \bar{S} & J_1^\top(\bar{F}_{21}\bar{F}_{11}^{-1} + \bar{F}_{11}^{-\top} P_j \bar{F}_{11}^{-1}) \\ \star & I - \bar{S}^\top P_j \bar{S} & 0 \\ \star & \star & \bar{F}_{21}\bar{F}_{11}^{-1} + \bar{F}_{11}^{-\top} P_j \bar{F}_{11}^{-1} \end{bmatrix} \succ 0 \tag{22a}$$

for all $i \in \{N_1 + 1, \dots, N\}$, $j \in \{1, \dots, N\}$, and

$$\begin{bmatrix} P_i - \mu_{i,j}X_i & J_2^\top \bar{F}_{11}^{-\top} P_j \bar{S} & J_2^\top(\bar{F}_{21}\bar{F}_{11}^{-1} + \bar{F}_{11}^{-\top} P_j \bar{F}_{11}^{-1}) \\ \star & I - \bar{S}^\top P_j \bar{S} & 0 \\ \star & \star & \bar{F}_{21}\bar{F}_{11}^{-1} + \bar{F}_{11}^{-\top} P_j \bar{F}_{11}^{-1} \end{bmatrix} \succ 0 \tag{22b}$$

for all $i \in \{1, \dots, N_1\}$, $j \in \{1, \dots, N\}$, and

$$P_i \succ 0, \quad \textit{for all } i \in \{1, \dots, N\}. \tag{22c}$$

Then, the hybrid system (1) *is GES with convergence rate ρ (when $w = 0$) and has an $\mathcal{L}_2$-gain from w to z smaller than or equal to γ.*

Proof: The proof will exploit the Lyapunov/storage function as provided in (20) given a fixed initial state $\xi(0) = \xi_0 \in \mathbb{R}^{n_\xi}$, $w \in \mathcal{L}_2$ and $\tau(0) = 0$. Now we will prove three important facts.

(i) *Under Assumption IV.1, W as in* (20) *is a well-defined Lyapunov/storage function candidate for all $\tau \in [0, h]$.* Following the proof of Theorem III.2 of [11], and especially eqn. (63) of [11], we have that

$$\begin{aligned}\bar{P}_i(h-\tau) &= \bar{F}_{21}(\tau)\bar{F}_{11}^{-1}(\tau) + \bar{F}_{11}^{-\top}(\tau)\Big(\bar{P}_i(h) + \\ &\bar{P}_i(h)\bar{S}(\tau)\left(I - \bar{S}(\tau)^\top \bar{P}_i(h)\bar{S}(\tau)\right)^{-1}\bar{S}(\tau)^\top \bar{P}_i(h)\Big)\bar{F}_{11}^{-1}(\tau)\end{aligned} \tag{23}$$

for which we have that $\bar{P}_i(h) = P_i$, $i \in \{1, \dots, N\}$. Requiring $\bar{P}_i(h-\tau)$ to be well defined for all $\tau \in [0, h]$ is equivalent to the existence of $\left(I - S(\tau)^\top \bar{P}_i(h)S(\tau)\right)^{-1}$ for all $\tau \in [0, h]$, as indicated by (23). This can be established by following the reasoning in [11, Proof of Theorem III.2].

(ii) *During flow it holds that $\dot{W} \leq -2\rho W - \gamma^{-2}z^\top z + w^\top w$.* This is implied by the fact that each component storage function $\xi^\top \bar{P}_i(\tau)\xi$, $i \in \{1, \dots, N\}$, satisfies the Riccati differential equation (13) that implies the mentioned dissipation inequality during flow, see, e.g., [11]. It is important to observe that due to the particular construction of W in (20) it holds that for each $k \in \mathbb{N}$ there exists an $i \in \{1, 2, \dots, N\}$ such that for all $t \in (kh, (k+1)h)$ $W(\xi, \tau, w, t) = \xi^\top \bar{P}_i(\tau)\xi$. Hence, the value of i in (20) changes only during jumps.

(iii) *During jumps the Lyapunov/storage function W does not increase i.e.,*

$$W(J_1\xi, 0, w, t) \leq W(\xi, h, w, t), \quad \textit{for all } \xi \in \Omega_i, \tag{24}$$

with $i \in \{N_1 + 1, \dots, N\}$, and

$$W(J_2\xi, 0, w, t) \leq W(\xi, h, w, t), \quad \textit{for all } \xi \in \Omega_i, \tag{25}$$

with $i \in \{1, \dots, N_1\}$. This is implied by feasibility of the conditions of Theorem IV.2, see [11, Proof of Theorem III.2].

Combining the above three facts, and using $\mathcal{L}_2$-gain techniques as in [14], we can guarantee GES of (1) (in case $w = 0$), and that the $\mathcal{L}_2$-gain of (1) is smaller than or equal to γ. Let $w \in \mathcal{L}_2$ be given. From item (i), and following the reasoning in [11, Proof of Theorem III.2], we have that

$$c_1\|\xi\|^2 \leq W(\xi, \tau, w, t) \leq c_2\|\xi\|^2, \tag{26}$$

for some $0 < c_1 \leq c_2$ for all $\tau \in [0, h]$ and all $\xi \in \mathbb{R}^{n_\xi}$. Conditions (ii) and (iii) combined guarantee that

$$\begin{aligned}W(\xi(t), \tau(t), w, t) - W(\xi_0, 0, w, 0) &\leq \\ \int_0^t \Big[-\gamma^{-2}\|z\|^2 + \|w\|^2\Big]dt,\end{aligned} \tag{27}$$

which by using that $W(\xi(t), \tau(t), w, t) \geq 0$ and letting $t \to \infty$ gives

$$-W(\xi_0, 0, w, 0) \leq \int_0^\infty \Big[-\gamma^{-2}\|z\|^2 + \|w\|^2\Big]dt. \tag{28}$$

Due to the inequality (26), we can bound $W(\xi_0, 0, w, 0) \leq c_2\|\xi_0\|^2$, and thus obtain

$$\int_0^\infty \|z\|^2 dt \leq c_2\gamma^2\|\xi_0\|^2 + \gamma^2 \int_0^\infty \|w\|^2 dt. \tag{29}$$

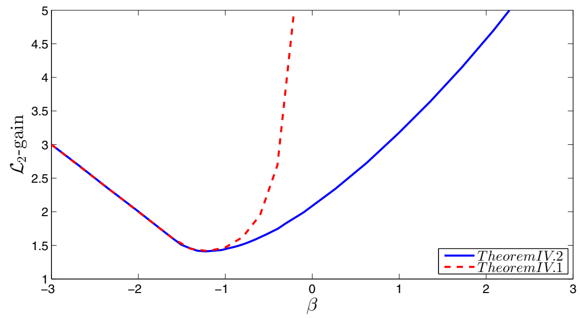

Fig. 2. $\mathcal{L}_2$-gain as function of the pole β of the FORE.

Consequently, we have that $\|z\|_{\mathcal{L}_2} \leq \gamma\sqrt{c_2}\|\xi_0\| + \gamma\|w\|_{\mathcal{L}_2}$. This completes the proof. ■

V. Example

In this section, we illustrate the improvement of the presented theory using a numerical example taken from [4]. In this example, the plant consists of an integrator system of the form (3) with

$$\left[\begin{array}{c|c|c|c} A_p & B_{pu} & B_{pw} & C_p \end{array}\right] = \left[\begin{array}{c|c|c|c} 0 & 1 & 1 & 1 \end{array}\right], \quad (30)$$

and $t_k = kh$, $k \in \mathbb{N}$, with sampling interval $h = 0.1$, which is controlled by a FORE of the form (8) and (11). The partition we use of the state-space into N number of sectors as in (19) is inspired by [3], [13] and based on defining the angles $\phi_i = [-\sin(\theta_i)\ \cos(\theta_i)]^\top$ for $\theta_i = \frac{i\pi}{N}$, $i \in \{0, 1, \ldots, N\}$, such that we can define the following sector matrices $S_i = \phi_i(-\phi_{i-1})^\top + \phi_{i-1}(-\phi_i)^\top$. This allows us to form the symmetric matrices X_i of (19) as follows

$$X_i = \begin{bmatrix} C_p & 0 \\ -D_cC_p & C_c \end{bmatrix}^\top S_i \begin{bmatrix} C_p & 0 \\ -D_cC_p & C_c \end{bmatrix}, \quad (31)$$

for all $i \in \{1, \ldots, N\}$. In the remainder of this example, we select $N_1 = 5$ and $N = 10$.

In Fig. 2, the $\mathcal{L}_2$-gain is represented as a function of the pole β of the FORE. The dashed line is obtained by the existing conditions in [11], see Theorem IV.1 using a common Lyapunov/storage function as in (12). The solid curve is obtained using the conditions of Theorem IV.2. From these curves it can be concluded that the results of Theorem IV.2 provide a significant improvement compared to the existing approach based on a common quadratic Lyapunov/storage function. In fact, for $\beta > 0$ the existing approach could not even establish a finite $\mathcal{L}_2$-gain, while the new approach presented here leads to such guarantees.

VI. Conclusions

In this paper, we have considered a particular class of hybrid systems with periodic time-triggered jump conditions and piecewise linear jump maps. The relevance of this framework is demonstrated by showing that a wide variety of control applications in the domains of reset control, event-triggered control, and networked control systems, can be captured in this framework. Interestingly, the unifying character of the framework can enable the transfer of results between the diverse application domains (see, e.g., Remark III.2). In addition, we provided improved conditions to analyze the stability and the $\mathcal{L}_2$-performance of the hybrid systems under study using trajectory-dependent Lyapunov/storage functions as a technical novelty. The new conditions include the existing ones as a special case, and, hence, will never provide worse estimates of the $\mathcal{L}_2$-gain. In fact, using a numerical example it was shown that these new conditions result in (significantly) better estimates for the $\mathcal{L}_2$-gain compared to the existing ones in the literature.

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