Stability analysis of networked control systems with periodic protocols and uniform quantizers

S.J.L.M. van Loon * M.C.F. Donkers * N. van de Wouw * W.P.M.H. Heemels *

* *Mechanical Engineering Department, Eindhoven University of Technology, Netherlands, (e-mail: s.j.l.m.v.loon@tue.nl)*

Abstract: The presence of a communication network in a control loop induces imperfections, such as quantization effects, packet dropouts, time-varying transmission intervals, time-varying transmission delays and communication constraints. The objectives of this paper are to provide a unifying modeling framework that incorporates all these imperfections, and to present novel techniques for the stability analysis for these networked control systems (NCSs). We focus on linear plants, linear controllers and periodic protocols, which leads to a modeling framework for NCSs based on discrete-time switched linear uncertain systems. Using an overapproximated system in the form of a polytopic model with additive norm-bounded uncertainty, we use LMI-based techniques to analyze the input-to-state stability (ISS) of the obtained NCS models with respect to the norm-bounded additive disturbances on plant and controller signals induced by quantization. These ISS conditions will be used to assess closed-loop stability and performance for periodic communication protocols and uniform quantizers, although the framework allows for extensions towards other types of protocols and other types of quantizers as well. We illustrate the effectiveness of the developed theory on a benchmark example of a batch reactor.

1. INTRODUCTION

Networked control systems (NCSs) are feedback control systems, in which the control loops are closed over a shared communication network. Compared to a traditional control system, in which the sensors, controllers and actuators are connected through dedicated point-to-point connections, NCSs offer advantages, such as, e.g., increased flexibility and maintainability of the system, and reduced wiring. However, NCSs also introduce new challenges that need to be overcome before the advantages they offer can be fully exploited. Generally speaking, NCSs are subject to network-induced communication imperfections and constraints that can be categorized into five types: (i) quantization errors, (ii) packet dropouts, (iii) time-varying sampling/transmission intervals, (iv) time-varying transmission delays and (v) communication constraints, requiring network protocols. All these networked-induced imperfections may degrade the closed-loop performance of an NCS, or even worse, may cause instability. Therefore, it is important to investigate how these effects influence closed-loop stability of the NCS. Because in any NCS all the previously mentioned imperfections can be present, it is important to develop a *unified framework* that allows to study the joint presence of all the network-induced effects. However, most of the available literature on NCS considers only some of the network-induced phenomena. For instance, Donkers et al. [2011], Heemels et al. [2010] considered (iii)-(v) simultaneously, Gao et al. [2008] focuses on type (i), (ii) and (iv) phenomena, Hetel et al. [2008], Naghshtabrizi et al. [2010], van de Wouw et al. [2010] study (iii) and (iv) simultaneously, and Nešić and Liberzon [2009] considers (i), (iii) and (v). Up to now, the only result that considers all these imperfections simultaneously, although under some restrictions, is Heemels et al. [2009].

In this paper, we focus on linear plants and controllers and study the stability of the corresponding NCSs in the presence of (i), (iii)-(v) types of network-induced phenomena, where the type (ii) network-induced phenomenon, i.e., packet dropouts, can also be accommodated for by modeling them as prolongation of the transmission interval, see Remark 1 below. To obtain the methods for stability analysis, this paper extends the work of Donkers et al. [2011] by including quantization, which requires different techniques to analyze the closed-loop stability, as we will show below. The difference between Heemels et al. [2009] and the work presented in this paper is that Heemels et al. [2009] exploits an emulation-based approach for the design of continuous-time controllers and uses a modeling framework based on hybrid systems, while we take an approach based on discrete-time switched linear uncertain systems. As was shown in Donkers et al. [2011], the latter approach can lead to less conservative results (e.g., in terms of allowable bounds on delays and sampling intervals), in the linear context, and allows the controller to be given in continuous-time as well as in discrete-time. Given these advantages of the switched linear systems approach, it is of interest to extend the work of Donkers et al. [2011] to incorporate quantization-induced disturbances as one of the stability and performance limiting factors in a *unified modeling and analysis framework* for NCSs. This forms indeed the objective of this paper. In doing so, we focus on one of the most common types of (memoryless) quan-

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tizers used in practice, being uniform quantizers, see, e.g., Delchamps [1990], although the same modeling framework allows to consider other types of quantizers as well, e.g., logarithmic quantizers and 'zoom' quantizers as introduced in Elia and Mitter [2001] and Brockett and Liberzon [2000], respectively, see e.g., Remark 6 below.

1.1 Nomenclature

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_{\geqslant 0}$ denote the set of integers, non-negative integers, real numbers and nonnegative real numbers, respectively. We denote a bock-diagonal matrix, with the entries $A_1, \ldots, A_n$ on the diagonal, as $\operatorname{diag}(A_1, \ldots, A_n)$ and $A^\top \in \mathbb{R}^{m\times n}$ denotes the transposed of matrix $A \in \mathbb{R}^{n\times m}$. For a vector $x \in \mathbb{R}^n$, we denote by x^i the i-th component and $\|x\| := \sqrt{x^\top x} = \sqrt{\sum_i |x^i|^2}$ its Euclidean norm. For a symmetric matrix A, we denote by $\lambda_{\max}(A)$, $\lambda_{\min}(A)$ the maximum and minimum eigenvalue of A, respectively. We denote by $\|A\| := \sqrt{\lambda_{\max}(A^\top A)}$ the spectral norm of a matrix A. Furthermore, for a discrete-time signal $z : \mathbb{N} \to \mathbb{R}^n$, the ℓ_∞-norm is defined as $\|z\|_{\ell_\infty} = \sup_{k\in\mathbb{N}} \|z_k\|$. Furthermore, we define the sets of signals with a finite ℓ_∞-norm as $\ell_\infty := \{z : \mathbb{N} \to \mathbb{R}^n \mid \|z\|_{\ell_\infty} < \infty\}$. We sometimes write symmetric matrices of the form $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$, as $\begin{bmatrix} A & B \\ \star & C \end{bmatrix}$.

2. NCS MODEL AND PROBLEM STATEMENT

In this section, we introduce the model describing NCSs subject to quantization, communication constraints and varying transmission intervals and delays. We will later comment on how dropouts can be included as well (see Remark 1 below). The NCS that we consider in this paper is schematically depicted in Fig. 1, where ZOH denotes a zero-order hold function that transforms the discrete-time control input $\hat{u}(t_k)$ to a continuous-time control input $\hat{u}(t)$, and $\mathcal{Q}_y$ and $\mathcal{Q}_u$ represent the uniform quantizers of the plant output and control input, respectively. The plant is given by a linear time-invariant (LTI) continuous-time model of the form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x^p(t) = A^p x^p(t) + B^p \hat{u}(t) \\ \quad y(t) = C^p x^p(t), \end{cases} \tag{1}$$

where $x^p \in \mathbb{R}^{n_p}$ denotes the state of the plant, $\hat{u} \in \mathbb{R}^{n_u}$ the control variable available at the actuator, $y \in \mathbb{R}^{n_y}$ the (measured) output of the plant and $t \in \mathbb{R}_{\geqslant 0}$ time. The LTI controller is assumed to be given in discrete-time by

$$\begin{cases} x^c_{k+1} = A^c x^c_k + B^c \hat{y}_k \\ u(t_k) = C^c x^c_k + D^c \hat{y}(t_k), \end{cases} \tag{2}$$

where $x^c \in \mathbb{R}^{n_c}$ denotes the state of the controller, $\hat{y} \in \mathbb{R}^{n_y}$ a 'networked' version of the output of the plant available at the controller, $u \in \mathbb{R}^{n_u}$ denotes the controller output and t_k are the transmission instants. The 'networked' signals $\hat{y} : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_y}$ and $\hat{u} : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_u}$ will be taken left-continuous, meaning that $\lim_{t\uparrow s} \hat{y}(s) = \hat{y}(t)$ and $\lim_{t\uparrow s} \hat{u}(s) = \hat{u}(t)$ for all $s \in \mathbb{R}_{\geqslant 0}$. At transmission instant t_k, $k \in \mathbb{N}$, (parts of) the outputs of the plant $y(t_k)$ and controller $u(t_k)$ are sampled and quantized, after which they are transmitted over the network. We assume that this data arrives after a delay τ_k at instant $r_k := t_k + \tau_k$, called the arrival instant. This is illustrated in Fig. 2. The states of the controller x^c_{k+1} are updated after the most recently received output of the plant $\hat{y}$ is updated, i.e., by

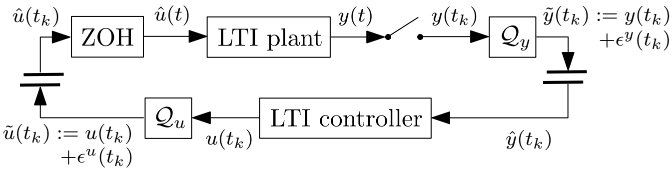

Fig. 1. Schematic overview of the NCS.

using $\hat{y}_k := \lim_{t\downarrow r_k} \hat{y}(t)$. Note that this update of x^c_{k+1} has to be performed in the time interval $(r_k, t_{k+1}]$. Although we consider a discrete-time controller, the framework presented in this paper also allows the controller to be given in continuous-time, see Remark 2 below.

Let us now explain in more detail the consequences of sampling, quantization and communication constraints. To do so, let us consider the case where the plant is equipped with n_y sensors and n_u actuators that are grouped into N nodes. At each transmission instant t_k, $k \in \mathbb{N}$, one node, denoted by $\sigma_k \in \{1, \ldots, N\}$, gets access to the network and transmits its current values. These transmitted values are received and implemented on the controller or the plant at arrival instant r_k. The sensor(s)/actuator(s), corresponding to the node that is allowed access to the network, collect their values from a sampled and quantized measurement of the corresponding entries of $y(t_k)$ and $u(t_k)$. This quantization process introduces a so-called quantization-induced error in both $y(t_k)$ and $u(t_k)$, denoted by $\epsilon^y(t_k)$ and $\epsilon^u(t_k)$, respectively, which are also illustrated in Fig. 1 and Fig. 2. The quantized signals are denoted by $\tilde{y}(t_k) = y(t_k) + \epsilon^y(t_k)$ and $\tilde{u}(t_k) = u(t_k) + \epsilon^u(t_k)$. We will make the quantization-induced error precise for uniform quantizers below. Furthermore, we assume that a transmission only occurs after the previous transmission has arrived, i.e., $t_{k+1} > r_k \geqslant t_k$, for all $k \in \mathbb{N}$. In other words, we consider the case where the delays τ_k are smaller than the transmission intervals h_k, i.e., $\tau_k < h_k$ for all $k \in \mathbb{N}$. After each transmission and reception, the values in $\hat{y}$ and $\hat{u}$ are updated with the latest received data, while the other values in $\hat{y}$ and $\hat{u}$ remain the same. This leads to the following constrained data exchange:

$$\begin{cases} \hat{y}(t) = \Gamma^y_{\sigma_k}(y(t_k) + \epsilon^y(t_k)) + (I - \Gamma^y_{\sigma_k})\hat{y}(t_k) \\ \hat{u}(t) = \Gamma^u_{\sigma_k}(u(t_k) + \epsilon^u(t_k)) + (I - \Gamma^u_{\sigma_k})\hat{u}(t_k) \end{cases} \tag{3}$$

for all $t \in (r_k, r_{k+1}]$, which models all the network effects, i.e., sampling, quantization, delays, scheduling and the ZOH. In (3), $\Gamma_{\sigma_i} := \operatorname{diag}(\Gamma^y_{\sigma_i}, \Gamma^u_{\sigma_i})$, $i = \{1, \ldots, N\}$, are diagonal matrices given by

$$\Gamma_i = \operatorname{diag}(\gamma_{i,1}, \ldots, \gamma_{i,n_y+n_u}). \tag{4}$$

In (4), the elements $\gamma_{i,j}$, with $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, n_y\}$, are equal to one, if plant output y^j is in node i and are zero elsewhere, and elements $\gamma_{i,j+n_y}$, with $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, n_u\}$, are equal to one, if controller output u^j is in node i and are zero elsewhere.

The value of $\sigma_k \in \{1, \ldots, N\}$ in (3) indicates which node is given access to the network at transmission instant t_k, $k \in \mathbb{N}$. Indeed, (3) reflects that the values in $\hat{y}$ and $\hat{u}$ are updated just after r_k, with the corresponding transmitted values at time t_k, while the others remain unaltered. In this paper, we consider the case where both the transmission intervals $h_k := t_{k+1} - t_k$, $k \in \mathbb{N}$, as well as the transmission delays $\tau_k := r_k - t_k$, $k \in \mathbb{N}$, are time-varying, as also indicated in Fig. 2. We assume that the variations in the transmission

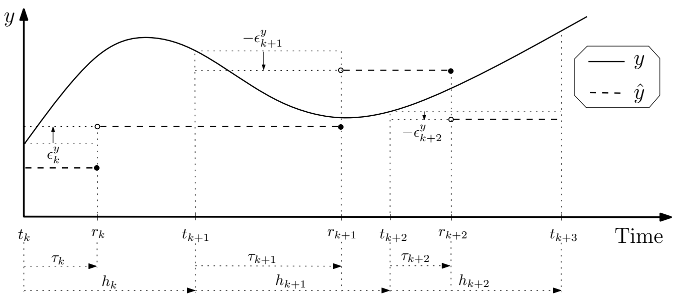

Fig. 2. Illustration of a typical evolution of y and $\hat{y}$ for a quantized control system.

intervals and delays are bounded and contained in the sets $[\underline{h}, \overline{h}]$ and $[\underline{\tau}, \overline{\tau}]$, respectively, with $0 < \underline{h} \leqslant \overline{h}$ and $0 \leqslant \underline{\tau} \leqslant \overline{\tau}$. Since we assume that delays are smaller than the transmission interval, we have that $(h_k, \tau_k) \in \Theta$, for all $k \in \mathbb{N}$, where

$$\Theta := \{(h,\tau) \in \mathbb{R}^2 \mid h \in [\underline{h}, \overline{h}],\ \tau \in [\underline{\tau}, \min\{h, \overline{\tau}\})\}. \quad (5)$$

Remark 1. The inclusion of packet dropouts can be realized by modeling them as prolongations of the transmission interval, see e.g., Donkers et al. [2011], Heemels et al. [2010], Nešić and Teel [2004]. To do so, let us assume that there is a bound $\bar{\delta}_d \in \mathbb{N}$ on the maximum number of successive packet dropouts. The stability bounds derived below are then still valid for $(h_k, \tau_k) \in \Theta'$, for all $k \in \mathbb{N}$, where

$$\Theta' := \{(h,\tau) \in \mathbb{R}^2 \mid h \in [\underline{h}, \overline{h}'],\ \tau \in [\underline{\tau}, \min\{h, \overline{\tau}\})\}, \quad (6)$$

in which $\overline{h}' := \frac{\overline{h}}{\bar{\delta}_d + 1}$.

2.1 The NCS as a Time-Varying Switched System

In this paper, we take a discrete-time modeling approach, and to derive a discrete-time model description, we first define the errors induced by the communication network and the quantizer as

$$\begin{cases} e^y(t) := \hat{y}(t) - y(t) \\ e^u(t) := \hat{u}(t) - u(t), \end{cases} \quad (7)$$

for all $t \in \mathbb{R}_{\geqslant 0}$. The discrete-time switched uncertain system can now be obtained by describing the evolution of the states between t_k and $t_{k+1} = t_k + h_k$. To do so, we define $x^p_k := x^p(t_k)$, $y_k := y(t_k)$, $u_k := u(t_k)$, $\tilde{y}_k := \tilde{y}(t_k)$, $\tilde{u}_k := \tilde{u}(t_k)$, $\hat{u}_k := \lim_{t \downarrow r_k} \hat{u}(t)$, $e^y_k := e^y(t_k)$, $e^u_k := e^u(t_k)$, $\epsilon^y_k := \epsilon^y(t_k)$ and $\epsilon^u_k := \epsilon^u(t_k)$. Since both $\hat{y}$ and $\hat{u}$, as in (3), are left-continuous piecewise constant signals, we can write $\hat{y}_{k-1} = \lim_{t \downarrow r_{k-1}} \hat{y}(t) = \hat{y}(r_k) = \hat{y}(t_k)$ and $\hat{u}_{k-1} = \lim_{t \downarrow r_{k-1}} \hat{u}(t) = \hat{u}(r_k) = \hat{u}(t_k)$. As (3) and (7) yield $\hat{u}_{k-1} = u_k + e^u_k$ and $\hat{u}_k - \hat{u}_{k-1} = \Gamma^u_{\sigma_k}(\epsilon^u_k - e^u_k)$, we can write the exact discretization of (1) as follows:

$$x^p_{k+1} = e^{A^p h_k} x^p_k + \int_0^{h_k} e^{A^p s} ds B^p (u_k + e^u_k) + \int_0^{h_k - \tau_k} e^{A^p s} ds B^p \Gamma^u_{\sigma_k} (\epsilon^u_k - e^u_k). \quad (8)$$

The complete NCS model is obtained by combining (3), (7) and (8) and introducing

$$\bar{x}_k := \begin{bmatrix} x^{p\top}_k & x^{c\top}_k & e^{y\top}_k & e^{u\top}_k \end{bmatrix}^\top, \quad \bar{\epsilon}_k := \begin{bmatrix} \epsilon^{y\top}_k & \epsilon^{u\top}_k \end{bmatrix}^\top,$$
$$\bar{z}_k := \begin{bmatrix} y^\top_k & u^\top_k \end{bmatrix}^\top, \quad (9)$$

which results in the discrete-time model (10), as shown on the top of the next page, in which $\tilde{A}_{\sigma_k, h_k, \tau_k} \in \mathbb{R}^{n_x \times n_x}$, $\tilde{B}_{\sigma_k, h_k, \tau_k} \in \mathbb{R}^{n_x \times n_z}$, $\tilde{C}_{\sigma_k} \in \mathbb{R}^{n_z \times n_x}$, with $n_x := n_p + n_c + n_y + n_u$, $n_z := n_y + n_u$ and

$$A_\rho := \operatorname{diag}(e^{A^p \rho}, A^c), \qquad B := \begin{bmatrix} 0 & B^p \\ B^c & 0 \end{bmatrix}, \quad (11a)$$

$$C := \operatorname{diag}(C^p, C^c), \qquad D := \begin{bmatrix} I & 0 \\ D^c & I \end{bmatrix}, \quad (11b)$$

$$E_\rho := \operatorname{diag}(\textstyle\int_0^\rho e^{A^p s} ds, I), \quad \rho \in \mathbb{R}. \quad (11c)$$

Remark 2. In this paper, we consider the case where the controller is given in discrete-time, see (2). However, the same type of model (10) also allows the controller to be given in continuous-time. In principle, this can be done by using different matrices definitions (11) in (10), as given in Donkers et al. [2011].

2.2 Periodic Protocols as Switching Function

Based on the previous modeling steps, the NCS is described by a parameter-varying discrete-time switched linear system (10) that is subject to an unknown disturbance $\bar{\epsilon}$ induced by quantization. In this framework, the scheduling protocol is considered as a switching function determining σ_k for each $k \in \mathbb{N}$. In this paper, we consider the class of periodic protocols, although quadratic protocols, with the well-known Try-Once-Discard (TOD) protocol as a special case, could be accommodated for in the same framework as well, see van Loon et al. [2012] for details. A periodic protocol is a protocol that satisfies for some $\tilde{N} \in \mathbb{N}$

$$\sigma_{k+\tilde{N}} = \sigma_k, \quad \text{for all } k \in \mathbb{N}. \quad (12)$$

$\tilde{N}$ is then called the period of the protocol. Actually, the well-known RR protocol belongs to this class and is defined by

$$\{\sigma_1, \ldots, \sigma_N\} = \{1, 2, \ldots, N\}, \quad (13)$$

and period $\tilde{N} = N$, i.e., during each period of the protocol every node has access to the network exactly once and in a fixed sequence.

2.3 Uniform Quantizers

The quantizer introduces a quantization-induced error $\bar{\epsilon}$ that can be considered as a disturbance in the NCS model (10). In this paper, we assume that each plant output and control input is quantized separately according to the mapping $q^i : \mathbb{R} \to \mathcal{Q}^i$, $i \in \{1, \ldots, n_z\}$, in which $\mathcal{Q}^i$ is a finite or countable subset of $\mathbb{R}$. We study the uniform quantizer, which is defined by

$$q^i(\bar{z}^i_k) = \zeta_i \left\lfloor \frac{\bar{z}^i_k}{\zeta_i} \right\rceil, \quad (14)$$

where $\zeta_i > 0$, $i \in \{1, \ldots, n_z\}$ denotes the step size and $\lfloor \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ is the rounding function that rounds off towards the nearest integer. In case $\frac{\bar{z}^i_k}{\zeta_i}$ in (14) is exactly in between two integers, its value is rounded up towards the nearest positive integer if $\frac{\bar{z}^i_k}{\zeta_i} > 0$, and is rounded down towards the nearest negative integer if $\frac{\bar{z}^i_k}{\zeta_i} < 0$. This quantizer results in a quantization error for each input/output signal $\bar{z}^i_k$, $i \in \{1, \ldots, n_z\}$, satisfying

$$|\bar{\epsilon}^i_k| = |\bar{z}^i_k - q^i(\bar{z}^i_k)| \leqslant \frac{\zeta_i}{2}, \quad (15)$$

for all $k \in \mathbb{N}$. Note that this type of quantizer introduces a bounded but nonvanishing quantization error, which

$$\begin{cases} \bar{x}_{k+1} = \underbrace{\begin{bmatrix} A_{h_k} + E_{h_k}BDC & E_{h_k}BD - E_{h_k-\tau_k}B\Gamma_{\sigma_k} \\ C(I - A_{h_k} - E_{h_k}BDC) & I - D^{-1}\Gamma_{\sigma_k} + C(E_{h_k-\tau_k}B\Gamma_{\sigma_k} - E_{h_k}BD) \end{bmatrix}}_{=:\tilde{A}_{\sigma_k,h_k,\tau_k}} \bar{x}_k + \underbrace{\begin{bmatrix} E_{h_k-\tau_k}B\Gamma_{\sigma_k} \\ D^{-1}\Gamma_{\sigma_k} - CE_{h_k-\tau_k}B\Gamma_{\sigma_k} \end{bmatrix}}_{=:\tilde{B}_{\sigma_k,h_k,\tau_k}} \bar{\epsilon}_k \\ \bar{z}_k = \underbrace{[DC \;\; I - D^{-1}]}_{=:\tilde{H}_{\sigma_k}} \bar{x}_k \end{cases} \tag{10}$$

prohibits asymptotic stabilization of the NCS. However, we can provide conditions that will guarantee the solutions to remain bounded and to converge to a vicinity of the origin. Hence, instead of (true) stability, only practical stability of the NCS can be achieved. In Section 5, we will quantify the steady-state performance in terms of an ultimate bound (UB) on the response.

2.4 Input-to-State Stability of the NCS

The problem studied in this paper is to analyse practical stability of the NCS given by (1), (2), (3) and (7), with protocol (12), and quantizer (14), where the time-varying and uncertain transmission intervals and transmission delays are taken from the set Θ defined in (5). The stability analysis will be based on studying exponential input-to-state stability (EISS) of the system (10). We will show in Section 5 how these stability conditions guarantee (practical) stability for the corresponding NCSs with the quantizer studied in this paper.

Let us now formally define EISS, in which we exploit the linearity properties of the closed-loop NCS model (10).

Definition 3. Jiang and Wang [2001] System (10) with switching sequence satisfying (12) is said to be exponentially input-to-state stable (EISS) with respect to $\bar{\epsilon} \in \ell_\infty$, if there exist $c \geqslant 0$, $\gamma_{\text{ISS}} > 0$ and $0 \leqslant \lambda < 1$ such that for any initial condition $\bar{x}_0 \in \mathbb{R}^{n_x}$, any sequence of transmission intervals $(h_0, h_1, \ldots)$, and any sequence of transmission delays $(\tau_0, \tau_1, \ldots)$, with $(h_k, \tau_k) \in \Theta$, for all $k \in \mathbb{N}$, it holds that

$$\|\bar{x}_k\| \leqslant c\lambda^k \|\bar{x}_0\| + \gamma_{\text{ISS}} \sup_{s \in [0,k-1]} \|\bar{\epsilon}_s\|. \tag{16}$$

3. OBTAINING A CONVEX OVERAPPROXIMATION

In the previous section, we obtained an NCS model in the form of a discrete-time switched uncertain linear system, as given by (10). However, the development of efficient stability analysis techniques for (10) directly is obstructed by the fact that the uncertainties appear in an exponential fashion in both $\tilde{A}_{\sigma_k,h_k,\tau_k}$ and $\tilde{B}_{\sigma_k,h_k,\tau_k}$. Therefore, we apply a procedure that overapproximates system (10) by a polytopic system with a norm-bounded additive uncertainty of the form

$$\bar{x}_{k+1} = \left(\textstyle\sum_{l=1}^{L} \alpha_k^l \bar{A}_{\sigma_k,l} + \bar{B}\Delta_k \bar{C}_{\sigma_k}\right) \bar{x}_k + \left(\textstyle\sum_{l=1}^{L} \alpha_k^l \bar{E}_{\sigma_k,l} + \bar{B}\Delta_k \bar{F}_{\sigma_k}\right) \bar{\epsilon}_k, \tag{17}$$

where $\bar{A}_{\sigma,l} \in \mathbb{R}^{n_x \times n_x}$, $\bar{B} \in \mathbb{R}^{n_x \times q}$, $\bar{C}_\sigma \in \mathbb{R}^{q \times n_x}$, $\bar{E}_{\sigma,l} \in \mathbb{R}^{n_x \times n_z}$, $\bar{F}_\sigma \in \mathbb{R}^{q \times n_z}$, for $\sigma \in \{1, \ldots, \tilde{N}\}$ and $l \in \{1, \ldots, L\}$, with L the number of vertices of the polytope. The vector $\alpha_k = [\alpha_k^1 \, \ldots \, \alpha_k^L]^\top \in \mathcal{A}$, $k \in \mathbb{N}$, is time-varying with

$$\mathcal{A} = \left\{\alpha \in \mathbb{R}^L \,\middle|\, \textstyle\sum_{l=1}^{L} \alpha^l = 1,\, \alpha^l \geqslant 0 \,\forall\, l \in \{1, \ldots, L\}\right\} \tag{18}$$

and $\Delta_k \in \boldsymbol{\Delta}$, $k \in \mathbb{N}$, where $\boldsymbol{\Delta}$ is a norm-bounded set of matrices in $\mathbb{R}^{q \times q}$ that describes the additive uncertainty. The system in (17) is an overapproximation of (10), in the sense that for all $\sigma \in \{1, \ldots, \tilde{N}\}$, it holds that

$$\left\{ \left[\tilde{A}_{\sigma,h,\tau} \;\; \tilde{B}_{\sigma,h,\tau}\right] \middle| (h,\tau) \in \Theta \right\} \subseteq \left\{ \textstyle\sum_{l=1}^{L} \alpha^l \left[\bar{A}_{\sigma,l} \;\; \bar{E}_{\sigma,l}\right] + \bar{B}\Delta \left[\bar{C}_\sigma \;\; \bar{F}_\sigma\right] \,\middle|\, \alpha \in \mathcal{A},\, \Delta \in \boldsymbol{\Delta} \right\}. \tag{19}$$

Overapproximating the uncertain system (10) by (17), in the sense of (19), is convenient for analysis, as obtaining EISS with a certain upper bound on the ISS-gain γ_{ISS} of the system (17), implies that the system (10) is EISS with the same upper bound on the ISS-gain γ_{ISS} as well.

In this paper, we will employ an overapproximation technique based on gridding and norm-bounding, described in Donkers et al. [2011], to construct a polytopic system with a norm-bounded additive uncertainty of the form of (17), satisfying (19). For the sake of brevity, we not discuss this overapproximation technique and refer to [Donkers et al., 2011, Procedure III.1] for a detailed description of the construction of all individual matrices in (17).

4. INPUT-TO-STATE STABILITY OF SWITCHED SYSTEMS WITH PARAMETRIC UNCERTAINTY

In the previous sections, we obtained a switched discrete-time uncertain NCS model (10) and introduced an overapproximation technique to embed system (10) in a switched polytopic system with norm-bounded uncertainty as in (17), satisfying (19). In this section, we will employ this overapproximated system to develop conditions that guarantee EISS, given a periodic protocol and a set Θ as in (5). In the following lemma, we state general sufficient conditions in terms of dissipation inequalities that guarantee EISS.

Lemma 4. Consider the system (17) with switching function (12) and uncertainty set $\boldsymbol{\Delta}$. Suppose there exist some positive scalars $\alpha_1, \alpha_2, \alpha_3, \kappa$ and a function $V : \mathbb{R}^{n_x} \times \mathbb{N} \to \mathbb{R}^{n_x}$ such that

$$\alpha_1 \|\bar{x}_k\|^2 \leqslant V(\bar{x}_k, k) \leqslant \alpha_2 \|\bar{x}_k\|^2 \tag{20a}$$

$$V(\bar{x}_{k+1}, k+1) - V(\bar{x}_k, k) \leqslant -\alpha_3 \|\bar{x}_k\|^2 + \kappa \|\bar{\epsilon}_k\|^2 \tag{20b}$$

for all $\bar{x}_k \in \mathbb{R}^{n_x}$, $\bar{\epsilon}_k \in \mathbb{R}^{n_z}$ and all $k \in \mathbb{N}$, with $\bar{x}_{k+1}$ given by (17) for some $\alpha_k \in \mathcal{A}$ and some $\Delta_k \in \boldsymbol{\Delta}$. Then, the system (17) is EISS with respect to bounded disturbances, i.e., satisfies (16), where $c = \sqrt{\frac{\alpha_2}{\alpha_1}}$, $\lambda = \sqrt{1 - \frac{\alpha_3}{\alpha_2}} \in [0, 1)$ and

$$\gamma_{\text{ISS}} = \sqrt{\frac{\alpha_2 \kappa}{\alpha_1 \alpha_3}}. \tag{21}$$

A function that satisfies (20) is called an EISS-Lyapunov function.

Proof: The proof follows from Theorem 2.5 of Lazar et al. [2008], adapted for linear systems. □

Below, we will provide sufficient conditions in terms of LMIs under which system (17), and thus the NCS model (10) (due to (19)), with a given periodic protocol satisfying (12), is EISS. To do so, let us introduce the set of matrices $\mathcal{R}$ given by

$$\mathcal{R} = \{\text{diag}(r_1 I_1, \ldots, r_K I_K, r_{K+1} I_1, \ldots, r_{2K} I_K, r_{2K+1} I_1, \ldots, r_{3K} I_K) \,|\, r_i > 0 \text{ for } i \in \{1, \ldots, 3K\}\}, \quad (22)$$

in which K is related to the total number of real Jordan blocks of A^p, see Donkers et al. [2011].

We will now analyze EISS of the system (10) for the class of periodic protocols (12). Let us introduce positive definite matrices P_i, $i \in \{1, \ldots, \tilde{N}\}$, and a time-dependent periodic candidate EISS-Lyapunov function, for $k \in \mathbb{N}$, of the form

$$V(\bar{x}_k, k) = \bar{x}_k^\top P_{k \bmod \tilde{N}} \bar{x}_k, \quad (23)$$

where k mod $\tilde{N}$ denotes k modulo $\tilde{N}$, which is the remainder of the division of k by $\tilde{N}$.

Theorem 5. Assume that there exist positive definite matrices P_i, $i \in \{1, \ldots, \tilde{N}\}$, matrices $R_{i,l} \in \mathcal{R}$, $i \in \{1, \ldots, \tilde{N}\}$ and $l \in \{1, \ldots, L\}$, and positive scalars α_3, κ, satisfying

$$\begin{bmatrix} P_i & 0 & 0 & \bar{A}_{\sigma_i,l}^\top P_{i+1} & \bar{C}_{\sigma_i}^\top R_{i,l} & \alpha_3 I \\ \star & \kappa I & 0 & \bar{E}_{\sigma_i,l}^\top P_{i+1} & \bar{F}_{\sigma_i}^\top R_{i,l} & 0 \\ \star & \star & R_{i,l} & \bar{B}^\top P_{i+1} & 0 & 0 \\ \star & \star & \star & P_{i+1} & 0 & 0 \\ \star & \star & \star & \star & R_{i,l} & 0 \\ \star & \star & \star & \star & \star & \alpha_3 I \end{bmatrix} \succeq 0, \quad (24)$$

where $P_{\tilde{N}+1} := P_1$, for all $i \in \{1, \ldots, \tilde{N}\}$, $l \in \{1, \ldots, L\}$. Then, the system (10) with protocol (12), is EISS with an upper bound on the ISS-gain γ_{ISS} given by (21), in which $\alpha_1 = \min_{i \in \{1,\ldots,\tilde{N}\}} \lambda_{\min}(P_i)$ and $\alpha_2 = \max_{i \in \{1,\ldots,\tilde{N}\}} \lambda_{\max}(P_i)$.

Proof: For the proof, see van Loon et al. [2012]. □

5. STABILITY AND PERFORMANCE FOR UNIFORM QUANTIZERS

In the previous section, we derived conditions for guaranteeing EISS of the switched discrete-time uncertain system (10) with protocol (12) in terms of LMIs. With these results, we can analyse stability and performance for the NCS with uniform quantizers discussed in Section 2.3.

The bound on the quantization-induced disturbance of the i-th component of $\bar{\epsilon}_k$, given a uniform quantizer, is given by (15). Using the fact that $\|\bar{\epsilon}_k\|^2 = \sum_{i=1}^{n_z} \|\bar{\epsilon}_k^i\|^2$, we arrive at a bound on $\bar{\epsilon}_k$ given by $\|\bar{\epsilon}_k\| \leqslant \sqrt{\sum_{i=1}^{n_z} \left(\frac{\zeta_i}{2}\right)^2}$. Using both this bound on $\bar{\epsilon}_k$ and assuming EISS of (10) as in Definition 3 (which can be verified by using the conditions in Theorem 5), we arrive at an UB for the solutions $\bar{x}_k$ of (10) as $k \to \infty$, given by

$$\limsup_{k\to\infty} \|\bar{x}_k\| \leqslant \gamma_{\text{ISS}} \sup_{s\in\mathbb{N}} \|\bar{\epsilon}_s\| \leqslant \gamma_{\text{ISS}} \sqrt{\sum_{i=1}^{n_z} \left(\frac{\zeta_i}{2}\right)^2}, \quad (25)$$

that depends on both the ISS-gain γ_{ISS} as well as the quantization density of all input/output signals, where the ISS gain γ_{ISS} can be expressed as follows

$$\gamma_{\text{ISS}} = \sqrt{\frac{\max_{i\in\{1,\ldots,\tilde{N}\}} \lambda_{\max}(P_i)\kappa}{\min_{i\in\{1,\ldots,\tilde{N}\}} \lambda_{\min}(P_i)\alpha_3}}, \quad (26)$$

in which $\alpha_3, \kappa > 0$ and $P_i > 0$ satisfy the conditions of Theorem 5.

As indicated by (25), practical stability of the system (10) is achieved with respect to the quantization step-sizes ζ_i, $i \in \{1, \ldots, n_z\}$, since it holds that selecting $\zeta_i \to 0$, for all $i \in \{1, \ldots, n_z\}$, implies that $\limsup_{k\to\infty} \|\bar{x}_k\| \to 0$. However, for any fixed ζ_i, the solutions $\bar{x}_k$ to system (10) have a certain UB. To obtain the smallest upper bound on the UB in (25), we aim at minimizing γ_{ISS} subject to the LMIs (24), with free variables P_i, and positive scalars α_3 and κ. Note that, minimizing the γ_{ISS} is a nonlinear minimization problem, which is in general not straightforward to solve. However, one efficient technique to solve this problem is discussed next.

Minimizing the upper bound on the ISS gain of the system (10), with protocol (12) is achieved by fixing $\alpha_3 > 0$, and putting a bound on both $\min_{i\in\{1,\ldots,\tilde{N}\}} \lambda_{\min}(P_i)$ and $\max_{i\in\{1,\ldots,\tilde{N}\}} \lambda_{\max}(P_i)$, by using the additional constraints $\underline{\omega} I \leqslant P_i \leqslant \overline{\omega} I$, for some $\underline{\omega}, \overline{\omega} > 0$. Consequently, we have $\alpha_1 = \min_{i\in\{1,\ldots,\tilde{N}\}} \lambda_{\min}(P_i) \geqslant \underline{\omega}$ and $\alpha_2 = \max_{i\in\{1,\ldots,\tilde{N}\}} \lambda_{\max}(P_i) \leqslant \overline{\omega}$. Subsequently, we minimize over κ.

Remark 6. The modeling and analysis framework presented in this paper also allows to include two other classes of quantizers, namely, logarithmic and 'zoom quantizers', as introduced in Elia and Mitter [2001] and Brockett and Liberzon [2000], respectively. However, for the sake of brevity, we refer to van Loon et al. [2012] regarding details.

6. ILLUSTRATIVE EXAMPLE

In this section, we illustrate the presented theory using a well-known benchmark example in the NCS literature, see e.g., Heemels et al. [2010], Nešić and Teel [2004], Walsh et al. [2002], consisting of a linearized model of a batch reactor controlled by a continuous-time proportional and integral controller. We refer to the aforementioned references for details regarding both the linearized model as well as the controller.

We assume that the controller is given in continuous-time which, as already indicated in Remark 2, requires only slight modifications of the matrices (11) as given in Donkers et al. [2011]. Furthermore, we consider an NCS setup where the controller is collocated with the corresponding actuators. Hence, only the two plant outputs ($n_z = 2$) are communicated via a network, where we assume that both these signals are quantized using a uniform quantizer with step size $\zeta_i = 10^{-3}$, $i = \{1, 2\}$. Furthermore, we assume that the maximum number of successive dropouts $\overline{\delta}_d$ (see Remark 1) is zero, that $\underline{\tau} = 0$ and we take $\underline{h} = 10^{-3}$. Using [Donkers et al., 2011, Procedure III.1], we obtain a convex overapproximation of the NCS model (10).

For the NCS with uniform quantizers, the objective is to obtain the smallest upper bound on the UB (25) by minimizing the ISS-gain (21) of (10) as described in Section 5. Hereto, we exploit the convex overapproximation of (10) and check for numerous combinations of $\overline{h}$ and $\overline{\tau}$ if the LMIs of Theorem 5 are feasible, while minimizing γ_{ISS}. This

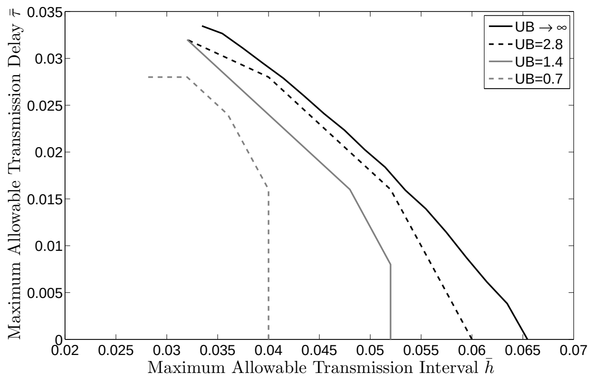

Fig. 3. Numerical results of minimization of $\gamma_{\rm ISS}$, resulting in various upper bounds on the UB.

provides information on an upper bound on $\gamma_{\rm ISS}$ depending on the bounds on the transmission intervals and delays $\overline{h}$ and $\overline{\tau}$, respectively. Using this information in (25), together with the quantization step sizes of all plant outputs $\zeta_i = 10^{-3}$, $i = \{1, 2\}$, we can compute an upper bound on the UB on the states. The results for this example are depicted in Fig. 3, where we selected $\alpha_3 = 10^{-3}$, $\underline{\omega} = 10^{-1}$ and $\overline{\omega} = 50$. The solid black line represents the quantization-free tradeoff curve between the maximum allowable transmission interval $\overline{h}$ and the maximum allowable delay $\overline{\tau}$ guaranteeing global exponential stability (GES). This curve forms a boundary for an NCS with uniform quantizers as it requires an infinite UB to reach this curve. Notice that the gradient of the UB becomes steeper the closer we approach the curve for UB $\to \infty$. This means that, for this example with quantization step sizes $\zeta_i = 10^{-3}$, $i = \{1, 2\}$, we can have a significantly smaller upper bound on the UB by allowing only a slight reduction in $\overline{h}$ and/or $\overline{\tau}$.

7. CONCLUSIONS

In this paper, we analyze the stability and performance of Networked Control Systems (NCSs) that are subject to quantization effects, time-varying transmission intervals, time-varying delays, communication constraints and discussed how dropouts could be included as well. The analysis is performed using a modeling framework for NCSs based on discrete-time switched linear uncertain systems, in which the communication sequence was determined by a periodic protocol. To handle the exponential uncertainty in the system matrices caused by the presence of varying transmission intervals and delays, a procedure that yield a convex overapproximation has been used. Exploiting the resulting overapproximated systems, we derived LMI-based conditions that guarantee exponential input-to-state stability (EISS) of the NCS with respect to quantization-induced disturbance inputs and showed how these conditions can be used to compute an ultimate bound on the state response for uniform quantizers. The application of the newly derived theory to a benchmark example showed that the developed theory can be used to make tradeoffs between the various network properties, such as bounds on transmission intervals and delays, the quantization properties and control criteria, such as ultimate bounds. Consequently, this work provides designers of NCSs with new and computationally efficient tools to support their multi-disciplinary design choices.

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