

INPUT-TO-STATE STABILIZING SUB-OPTIMAL NONLINEAR MPC ALGORITHMS WITH AN APPLICATION TO DC-DC CONVERTERS

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Abstract: This paper focuses on the synthesis of computationally friendly sub-optimal nonlinear Model Predictive Control (MPC) algorithms with guaranteed robust stability. The input-to-state stability framework is employed to analyze the robustness of the resulting MPC closed-loop systems. Two new sub-optimal nonlinear MPC schemes are proposed, based on a contraction argument and an artificial Lyapunov function, respectively. The developed theory is illustrated by applying it to control a Buck-Boost DC-DC converter. *Copyright ©2006 IFAC*

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1. INTRODUCTION

One of the most studied properties of Nonlinear Model Predictive Control (NMPC) is the stability of the controlled system. Perhaps the most embraced stabilization method is the so-called terminal cost and constraint set approach, see, for example, the survey (Mayne *et al.*, 2000) for an overview. This method uses the value function of the MPC cost as a candidate Lyapunov function for the closed-loop system and achieves stability via a particular terminal cost and an additional constraint on the terminal state, i.e. the predicted state at the end of the prediction horizon. Its advantage consists in the fact that initial feasibility of the NMPC optimization problem implies feasibility all the way and the finite horizon MPC cost is proven to

be a good approximation of the infinite horizon MPC cost. However, these properties are only guaranteed under the standing assumption that the global optimum of the MPC optimization problem is attained online, at each sampling instant. Clearly, when dealing with nonlinear prediction models and hard constraints, it is difficult if not impossible to guarantee this assumption in practice, where numerical solvers usually provide (in the limited computational time available) a feasible, sub-optimal input sequence, rather than a globally optimal one. Such a sub-optimal input sequence needs to have certain properties to still guarantee robust stability of the MPC closed-loop system. Therefore, in practice, there is a need for sub-optimal MPC algorithms based on simpler optimization problems, which can be solved faster, and that still have an a priori robust stability guarantee.

An important result regarding sub-optimal NMPC was presented in (Scokaert *et al.*, 1999), where it is shown that feasibility of the NMPC optimization problem rather than optimality is sufficient for stability. In

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(Scokaert *et al.*, 1999), stability is achieved without requiring optimality, by forcing the MPC value function to decrease at each sampling-instant, which can be expressed in terms of an additional constraint. However, when nonlinear prediction models are used, this constraint becomes highly nonlinear and difficult to implement from a computational point of view, as the MPC value function depends on the whole sequence of unknown predicted future inputs. Feasibility is guaranteed for the nominal case in (Scokaert *et al.*, 1999) by adding a terminal equality or inequality constraint.

This paper investigates the possibility of designing Input-to-State Stabilizing (ISS) (Jiang and Wang, 2001), but computationally friendly sub-optimal MPC algorithms. We propose to achieve this goal via new, simpler stabilizing constraints, that can be implemented as a finite number of linear inequalities. Two sub-optimal NMPC algorithms are presented. The first one is based on a contraction argument, i.e. we prove that, if the norm of the state of the closed-loop system is sufficiently decreasing at each sampling instant, then ISS is guaranteed. The second NMPC scheme resorts to an ∞ -norm based artificial Lyapunov function, which only depends on the measured state and the first element of the sub-optimal sequence of predicted future inputs. A method for computing this function off-line for a linear approximation of the nonlinear model is also given. A case study on the control of DC-DC converters is included to illustrate the potential of the developed theory for practical applications.

Compared to (Scokaert *et al.*, 1999), we do not guarantee that initial feasibility implies feasibility all the way for the proposed algorithms. However, note that we consider perturbed systems. In this case, feasibility all the way is not guaranteed for the algorithms of (Scokaert *et al.*, 1999) either. From a computational point of view, we obtain faster NMPC algorithms, as our stabilizing constraints can be written as a finite number of linear inequalities. Moreover, we also provide a robust stability guarantee in terms of ISS, which ensures a bound on the norm of the MPC closed-loop system state.

Notation and basic definitions

Let \mathbb{R} , \mathbb{R}_+ , \mathbb{Z} and \mathbb{Z}_+ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{\geq c_1}$ to denote the set $\{k \in \mathbb{Z}_+ \mid k \geq c_1\}$ for some $c_1 \in \mathbb{Z}_+$. Let $\|\cdot\|$ denote the ∞ -norm for shortness. For a matrix $Z \in \mathbb{R}^{m \times n}$ let $\|Z\| := \sup_{x \neq 0} \frac{\|Zx\|}{\|x\|}$ denote its corresponding induced matrix norm. For a sequence $\{z_p\}_{p \in \mathbb{Z}_+}$ with $z_p \in \mathbb{R}^l$ let $\|\{z_p\}_{p \in \mathbb{Z}_+}\| := \sup\{\|z_p\| \mid p \in \mathbb{Z}_+\}$. Let $z_{[k]} \in \{\mathbb{R}^l\}^{k+1}$ denote the truncation of $\{z_p\}_{p \in \mathbb{Z}_+}$ at time $k \in \mathbb{Z}_+$, i.e. $z_{[k],p} = z_p$, $p \leq k$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\partial\mathcal{S}$ the boundary of \mathcal{S} , by $\text{int}(\mathcal{S})$ its interior and by $\text{cl}(\mathcal{S})$ its closure. A polyhedron (or a polyhedral set)

in \mathbb{R}^n is a set obtained as the intersection of a finite number of open and/or closed half-spaces. A function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class \mathcal{K} if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class \mathcal{KL} if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and $\lim_{k \rightarrow \infty} \beta(s, k) = 0$.

2. INPUT-TO-STATE STABILITY PRELIMINARIES

Consider the discrete-time perturbed nonlinear system described by

$$x_{k+1} = G(x_k, w_k), \quad k \in \mathbb{Z}_+, \quad (1)$$

where $x_k \in \mathbb{R}^n$ is the state, $w_k \in \mathbb{R}^l$ is an unknown disturbance input and $G : \mathbb{R}^n \times \mathbb{R}^l \rightarrow \mathbb{R}^n$ is a nonlinear, possibly discontinuous function. For simplicity of notation, we assume that the origin is an equilibrium in (1) for zero disturbance, meaning that $G(0, 0) = 0$. For system (1), we now introduce the notion of input-to-state stability (Jiang and Wang, 2001).

Definition 1. Let \mathbb{X} and \mathbb{W} be subsets of \mathbb{R}^n and \mathbb{R}^l , respectively, with $0 \in \text{int}(\mathbb{X})$. We call system (1) ISS for initial conditions in \mathbb{X} and disturbances in \mathbb{W} if there exist a \mathcal{KL} -function $\beta(\cdot, \cdot)$ and a \mathcal{K} -function $\gamma(\cdot)$ such that, for each $x_0 \in \mathbb{X}$ and all $\{w_p\}_{p \in \mathbb{Z}_+}$ with $w_p \in \mathbb{W}$ for all $p \in \mathbb{Z}_+$, it holds that the corresponding state trajectory satisfies $\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|w_{[k-1]}\|)$ for all $k \in \mathbb{Z}_{\geq 1}$.

Theorem 2. Let \mathbb{W} be a subset of \mathbb{R}^l with $0 \in \text{int}(\mathbb{W})$ and let \mathbb{X} be a Robust Positive Invariant (RPI) set² for (1) and disturbances in \mathbb{W} with $0 \in \text{int}(\mathbb{X})$. Furthermore, let $\alpha_1(s) := as^\lambda$, $\alpha_2(s) := bs^\lambda$, $\alpha_3(s) := cs^\lambda$ for some $a, b, c, \lambda > 0$, $\sigma(\cdot) \in \mathcal{K}$ and let $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a function with $V(0) = 0$. Consider now the following inequalities:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad (2a)$$

$$V(G(x, w)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|w\|). \quad (2b)$$

If inequalities (2) hold for all $x \in \mathbb{X}$ and all $w \in \mathbb{W}$, then system (1) is ISS for initial conditions in \mathbb{X} and disturbances in \mathbb{W} . Moreover, the ISS property of Definition 1 holds with

$$\beta(s, k) \triangleq \alpha_1^{-1}(2\rho^k \alpha_2(s)), \quad \gamma(s) \triangleq \alpha_1^{-1} \left(\frac{2\sigma(s)}{1-\rho} \right), \quad (3)$$

where $\rho := \frac{c}{b} \in [0, 1)$.

The proof of Theorem 2, including how the specific $\beta(\cdot, \cdot)$ and $\gamma(\cdot)$ functions given in (3) are obtained, is given in (Lazar *et al.*, 2005a). Note that, the conditions (2) imply Lyapunov asymptotic stability when the disturbance input converges to zero (Jiang and Wang, 2001).

² Meaning that for all $x \in \mathbb{X}$ it holds that $G(x, w) \in \mathbb{X}$ for all $w \in \mathbb{W}$.

Definition 3. A function $V(\cdot)$ that satisfies the hypothesis of Theorem 2 is called an *ISS Lyapunov function*.

3. SUB-OPTIMAL NMPC ALGORITHMS

We consider nominal and perturbed discrete-time nonlinear systems of the form:

$$x_{k+1} = f(x_k) + g(x_k)u_k, \quad k \in \mathbb{Z}_+, \quad (4a)$$

$$\tilde{x}_{k+1} = f(\tilde{x}_k) + g(\tilde{x}_k)u_k + w_k, \quad k \in \mathbb{Z}_+, \quad (4b)$$

where $x_k, \tilde{x}_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $w_k \in \mathbb{W} \subset \mathbb{R}^n$ are the state, the input and an additive disturbance, respectively, and $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ are nonlinear functions with $f(0) = 0$. In the sequel we will consider the case when sub-optimal NMPC is used to generate the control input u_k in (4). We assume that the state and the input vectors are constrained for both systems (4a) and (4b), in a compact subset \mathbb{X} of \mathbb{R}^n and a compact subset \mathbb{U} of \mathbb{R}^m , respectively, which contain the origin in their interior. For a fixed $N \in \mathbb{Z}_{\geq 1}$, let $\mathbf{x}_k(x_k, \mathbf{u}_k) \triangleq (x_{1|k}, \dots, x_{N|k})$ denote the state sequence generated by the nominal system (4a) from initial state $x_{0|k} \triangleq x_k$ and by applying an input sequence $\mathbf{u}_k \triangleq (u_{0|k}, \dots, u_{N-1|k})$. Let $F: \mathbb{R}^n \rightarrow \mathbb{R}_+$ with $F(0) = 0$ and $L: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_+$ with $L(0, 0) = 0$ be mappings. At time $k \in \mathbb{Z}_+$ let $x_k \in \mathbb{X}$ be given. The basic NMPC scenario consists in minimizing at each sampling instant $k \in \mathbb{Z}_+$ a finite horizon cost function of the form

$$J(x_k, \mathbf{u}_k) \triangleq F(x_{N|k}) + \sum_{i=0}^{N-1} L(x_{i|k}, u_{i|k}), \quad (5)$$

with prediction model (4a), over all input sequences \mathbf{u}_k , subject to state and input constraints.

Let $\mathbb{X}_f(N) \subseteq \mathbb{X}$ denote the set of *feasible states* with respect to the above optimization problem, i.e. the set of all states for which there exists a sequence of inputs that satisfies the input constraints and results in a predicted state trajectory that satisfies the state constraints. Then,

$$V_{\text{MPC}}: \mathbb{X}_f(N) \rightarrow \mathbb{R}_+, \quad V_{\text{MPC}}(x_k) \triangleq \inf_{\mathbf{u}_k} J(x_k, \mathbf{u}_k)$$

is the MPC value function corresponding to the cost (5). If there exists an optimal sequence of controls $\mathbf{u}_k^* \triangleq (u_{0|k}^*, u_{1|k}^*, \dots, u_{N-1|k}^*)$ that minimizes (5), the infimum above is a minimum and $V_{\text{MPC}}(x_k) = J(x_k, \mathbf{u}_k^*)$. Then, an *optimal* MPC control law is defined as $u^{\text{MPC}}(x_k) \triangleq u_{0|k}^*$, $k \in \mathbb{Z}_+$. Stability of the resulting MPC closed-loop system is usually guaranteed by adding a particular constraint on the terminal state $x_{N|k}$, see, for example, the survey (Mayne *et al.*, 2000).

As mentioned in the introduction, in practice, the available solvers provide only a feasible, sub-optimal sequence of inputs $\bar{\mathbf{u}}_k \triangleq (\bar{u}_{0|k}, \bar{u}_{1|k}, \dots, \bar{u}_{N-1|k})$ and the control applied to the plant, i.e. $\bar{u}_{0|k}$, is a *sub-optimal* MPC control. The resulting value function is then $\bar{V}(x_k) \triangleq J(x_k, \bar{\mathbf{u}}_k)$. The stability of the resulting

MPC closed-loop system may be unclear now, or may even be lost. Next, we present two sub-optimal NMPC algorithms that still guarantee stability a priori.

3.1 A contraction approach

In this paper we consider ∞ -norm based MPC costs, i.e. $F(x) \triangleq \|Px\|$ and $L(x, u) \triangleq \|Qx\| + \|R_u u\|$, where $P \in \mathbb{R}^{p \times n}$, $Q \in \mathbb{R}^{q \times n}$ and $R_u \in \mathbb{R}^{r_u \times m}$ are assumed to be known matrices that have full-column rank. To set-up the sub-optimal MPC algorithm we assume³ that a Lipschitz condition holds for the dynamics $f(\cdot), g(\cdot)$ in the sense that there exist $\mathcal{L}_f, \mathcal{L}_g > 0$ such that

$$\|f(x) + g(x)u\| \leq \mathcal{L}_f \|x\| + \mathcal{L}_g \|u\|, \quad \forall x \in \mathbb{X}, \forall u \in \mathbb{U}. \quad (6)$$

We also assume that all the controls in the sequence of predicted future inputs satisfy the regularity condition

$$\|u_{i|k}\| \leq \theta_i \|x_{0|k}\|, \quad i = 0, \dots, N-1, \quad (7)$$

for some constants $\theta_i > 0$. Since the control laws $u_{i|k}$ are not known explicitly, to ensure that (7) holds we will choose the constants θ_i a priori and impose (7) as an additional constraint to the NMPC optimization problem. Then, using (6) and (7) successively, one can easily establish a class \mathcal{K} upper bound on $J(x, \mathbf{u})$ for any $x \in \mathbb{X}$ and feasible \mathbf{u} , i.e. $J(x, \mathbf{u}) \leq \alpha_2(\|x\|)$ with $\alpha_2(s) \triangleq \mathcal{C}(\mathcal{L}_f, \mathcal{L}_g, \theta, N)s$, where $\mathcal{C}(\mathcal{L}_f, \mathcal{L}_g, \theta, N) > 0$ is a constant that depends on $\mathcal{L}_f, \mathcal{L}_g, \theta \triangleq (\theta_0, \dots, \theta_{N-1})$ and N . Since Q has full-column rank, there exists a $\xi_Q > 0$ such that $\|Qx\| > \xi_Q \|x\|$ for all $x \in \mathbb{X}$. Then, it holds that $J(x, \mathbf{u}) \geq \alpha_1(\|x\|)$ for all $x \in \mathbb{X}$ and any \mathbf{u} , where $\alpha_1(s) \triangleq \xi_Q s$. Let $\tau \in (0, 1)$ be a known constant.

Algorithm 1.

Step 1: At time $k \in \mathbb{Z}_+$ measure the state x_k , let $x_{0|k} := x_k$ and minimize the cost (5) subject to:

$$x_{i+1|k} = f(x_{i|k}) + g(x_{i|k})u_{i|k}, \quad i = 0, \dots, N-1, \quad (8a)$$

$$\alpha_2(\|f(x_{0|k}) + g(x_{0|k})u_{0|k}\| - (1 - \tau)\alpha_1(\|x_{0|k}\|)) \leq 0, \quad (8b)$$

$$x_{i|k} \in \mathbb{X}, \quad i = 1, \dots, N, \quad (8c)$$

$$u_{i|k} \in \mathbb{U}, \quad i = 0, \dots, N-1, \quad (8d)$$

$$\|u_{i|k}\| \leq \theta_i \|x_{0|k}\|, \quad i = 0, \dots, N-1. \quad (8e)$$

Step 2: Let $\bar{\mathbf{u}}_k$ be a feasible sequence of inputs calculated at Step 1. Apply to the perturbed system (4b) the control input $\bar{u}^{\text{MPC}}(x_k) := \bar{u}_{0|k}$.

Theorem 4. Let $\mathbb{X}_f(N)$ be the set of states $x \in \mathbb{X}$ for which the optimization problem in Step 1 of Algorithm 1 is feasible and let $\tilde{\mathbb{X}}_f(N) \subseteq \mathbb{X}_f(N)$ be a RPI set for system (4b) in closed-loop with $\bar{u}^{\text{MPC}}(\cdot)$ with $0 \in \text{int}(\tilde{\mathbb{X}}_f(N))$. Then, the perturbed system (4b) in closed-loop with $\bar{u}^{\text{MPC}}(\cdot)$ is ISS for initial conditions in $\tilde{\mathbb{X}}_f(N)$ and disturbances in \mathbb{W} .

³ Note that the continuity assumption is required *only* for the contraction based sub-optimal NMPC algorithm.

PROOF. The proof consists in showing that $\bar{V}(x_k) = J(x_k, \bar{\mathbf{u}}_k)$ is an ISS Lyapunov function. Let \mathcal{C} denote $\mathcal{C}(\mathcal{L}_f, \mathcal{L}_g, \theta, N)$ for shortness. By construction and from constraint (8e) we have that $\bar{V}(\cdot)$ satisfies (2a) for all $x \in \tilde{\mathbb{X}}_f(N)$ with $\alpha_1(\|x\|) = \xi_Q\|x\|$ and $\alpha_2(\|x\|) = \mathcal{C}\|x\|$. From constraint (8b) we have that for all $x \in \tilde{\mathbb{X}}_f(N)$ and any feasible $\bar{\mathbf{u}}$ ($\|\cdot\|$ denotes the ∞ -norm):

$$\begin{aligned} & \bar{V}(f(x) + g(x)\bar{u}^{\text{MPC}}(x) + w) - \bar{V}(x) \\ & \leq \alpha_2(\|f(x) + g(x)\bar{u}^{\text{MPC}}(x) + w\|) - \alpha_1(\|x\|) \\ & \leq \alpha_2(\|f(x) + g(x)\bar{u}^{\text{MPC}}(x)\| + \|w\|) - \alpha_1(\|x\|) \\ & \leq \alpha_2(\|f(x) + g(x)\bar{u}^{\text{MPC}}(x)\|) + \alpha_2(\|w\|) - \alpha_1(\|x\|) \\ & \leq -\alpha_3(\|x\|) + \sigma(\|w\|), \end{aligned}$$

where $\alpha_3(s) \triangleq \tau\alpha_1(s) = \tau\xi_Q s$ and $\sigma(s) \triangleq \alpha_2(s) = \mathcal{C}s$. The statement then follows from Theorem 2. \square

The drawback of Algorithm 1 is that the gain of $\alpha_2(\cdot)$, i.e. $\mathcal{C}(\mathcal{L}_f, \mathcal{L}_g, \theta, N)$, is a strictly increasing function of N , which implies that for long prediction horizons, the contractive constraint (8b) may become very conservative. Moreover, the constant $\mathcal{C}(\mathcal{L}_f, \mathcal{L}_g, \theta, N)$ also depends on the ∞ -norm of P , Q and R_u . Hence, one cannot freely choose the MPC cost weights, e.g. following performance motivations, since a large ∞ -norm may imply a large ISS gain for the closed-loop system, via $\sigma(\cdot)$, $\alpha_1(\cdot)$ and (3).

3.2 An artificial Lyapunov function approach

In practice it would be desirable that the design of the MPC cost, i.e. choosing $F(\cdot)$, $L(\cdot, \cdot)$ and N , is separated from guaranteeing stability, so that the MPC cost can be tuned for best performance. A possible solution to achieve this goal is to resort to an artificial Lyapunov function, which is designed independently of the MPC cost function. In this section, an ∞ -norms artificial Lyapunov function is employed to derive an ISS sub-optimal NMPC algorithm.

Consider the candidate ISS Lyapunov function $V(x) \triangleq \|P_V x\|$, where $P_V \in \mathbb{R}^{p_v \times n}$ is a full-column rank matrix. Let $Q_V \in \mathbb{R}^{q_v \times n}$ be a known matrix with full-column rank. The sub-optimal NMPC algorithm is now formulated as follows.

Algorithm 2.

Step 1: At time $k \in \mathbb{Z}_+$ measure the state x_k , let $x_{0|k} := x_k$ and minimize the cost (5) subject to:

$$x_{i+1|k} = f(x_{i|k}) + g(x_{i|k})u_{i|k}, \quad i = 0, \dots, N-1, \quad (9a)$$

$$\begin{aligned} & \|P_V(f(x_{0|k}) + g(x_{0|k})u_{0|k})\| - \|P_V x_{0|k}\| \\ & \leq -\|Q_V x_{0|k}\|, \end{aligned} \quad (9b)$$

$$x_{i|k} \in \mathbb{X}, \quad i = 1, \dots, N, \quad (9c)$$

$$u_{i|k} \in \mathbb{U}, \quad i = 0, \dots, N-1. \quad (9d)$$

Step 2: Let $\bar{\mathbf{u}}_k$ be a feasible sequence of inputs calculated at Step 1. Apply to the perturbed system (4b) the control input $\bar{u}^{\text{MPC}}(x_k) := \bar{u}_{0|k}$.

Theorem 5. Let $\mathbb{X}_f(N)$ be the set of states $x \in \mathbb{X}$ for which the optimization problem in Step 1 of Algorithm 2 is feasible and let $\tilde{\mathbb{X}}_f(N) \subseteq \mathbb{X}_f(N)$ be a RPI set for system (4b) in closed-loop with $\bar{u}^{\text{MPC}}(\cdot)$ with $0 \in \text{int}(\tilde{\mathbb{X}}_f(N))$. Then, the perturbed system (4b) in closed-loop with $\bar{u}^{\text{MPC}}(\cdot)$ is ISS for initial conditions in $\tilde{\mathbb{X}}_f(N)$ and disturbances in \mathbb{W} .

PROOF. The proof consists in showing that $V(x_k) = \|P_V x_k\|$ is an ISS Lyapunov function for system (4b) in closed-loop with $\bar{u}^{\text{MPC}}(\cdot)$. Since P_V has full-column rank, there exist $c_2 \geq c_1 > 0$ such that $c_1\|x\| \leq \|P_V x\| \leq c_2\|x\|$ for all x . Hence, $V(\cdot)$ satisfies condition (2a) for $\alpha_1(\|x\|) \triangleq c_1\|x\|$ and $\alpha_2(\|x\|) \triangleq c_2\|x\|$. From constraint (9b) and using the triangle inequality, we have that for all $x \in \tilde{\mathbb{X}}_f(N)$ and any feasible $\bar{\mathbf{u}}$:

$$\begin{aligned} & V(f(x) + g(x)\bar{u}^{\text{MPC}}(x) + w) - V(x) \\ & = \|P_V(f(x) + g(x)\bar{u}^{\text{MPC}}(x) + w)\| - \|P_V x\| \\ & \leq \|P_V(f(x) + g(x)\bar{u}^{\text{MPC}}(x))\| + \|P_V w\| - \|P_V x\| \\ & \leq -\|Q_V x\| + \|P_V w\| \leq -\alpha_3(\|x\|) + \sigma(\|w\|), \end{aligned}$$

where $\alpha_3(s) \triangleq \xi_{Q_V} s$ ($\xi_{Q_V} > 0$ is such that $\|Q_V x\| \geq \xi_{Q_V}\|x\|$ for all x) and $\sigma(s) \triangleq c_2 s$. The statement then follows from Theorem 2. \square

Next, we present a method for computing the ∞ -norm based artificial Lyapunov function $V(\cdot)$ off-line. Let

$$x_{k+1} = Ax_k + Bu_k, \quad k \in \mathbb{Z}_+, \quad (10)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, be a linear approximation of (4a) around $(0, 0)$. We assume that there exists a neighborhood $\mathcal{N} \subset \mathbb{X}$ of the origin where $Ax + Bu \approx f(x) + g(x)u$ for all $x \in \mathcal{N}$ and all $u \in \mathbb{U}$. For a given full-column rank matrix Q_V , to compute the matrix P_V , we consider a linear state-feedback $u_k = Kx_k$, $K \in \mathbb{R}^{m \times n}$, $k \in \mathbb{Z}_+$, and we make use of the following result.

Lemma 6. Suppose that a full-column rank matrix P_V and a gain K satisfy

$$1 - \|P_V(A + BK)P_V^{-L}\| - \|Q_V P_V^{-L}\| \geq 0, \quad (11)$$

where $P_V^{-L} \triangleq (P_V^T P_V)^{-1} P_V^T$ is the left Moore-Penrose inverse of P_V . Then, it holds that $\|P_V(A + BK)x\| - \|P_V x\| \leq -\|Q_V x\|$ for all x and, the function $V(x) = \|P_V x\|$ is an ISS Lyapunov function for the closed-loop system $x_{k+1} = (A + BK)x_k + w_k$.

The proof of Lemma 6 is a particular case of the proof of a more general result presented in (Lazar *et al.*, 2005b) and is omitted here due to space limitations. We also refer the reader to (Lazar *et al.*, 2005b) for ways to find a solution to inequality (11). Note that, due to the use of an artificial Lyapunov function, the weights of the MPC cost function and the length of the prediction horizon can now be freely chosen to achieve physical performance requirements. The value

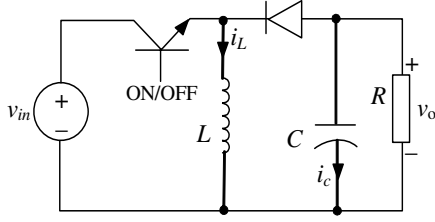


Fig. 1. A schematic view of a Buck-Boost converter.

function of the MPC cost is no longer used as an ISS Lyapunov function and the norms of P , Q and R_u no longer influence the ISS gain of the closed-loop system.

Remark 7. In practice, the constraint⁴ $x_{1|k} \in \mathbb{X} \sim \mathbb{W}$ is usually added to the optimization problem to ensure that the closed-loop system state, i.e. $\tilde{x}_{k+1} = x_{1|k} + w_k$, $k \in \mathbb{Z}_+$, does not violate the state constraints at time $k+1$ for any disturbance in \mathbb{W} .

Remark 8. When the sets \mathbb{X} , \mathbb{U} (and \mathbb{W}) are polyhedral, which is often the case in practice, the constraints (8b)-(8e), as well as the constraints (9b)-(9d), can be written as a finite number of linear inequalities since the measured state $x_{0|k}$ is known and $\|\cdot\|$ denotes the ∞ -norm. Moreover, since the nonlinear system (4) is affine with respect to the input, for $N = 1$, the optimization problem that has to be solved at Step 1 of Algorithm 1 or Algorithm 2 can be formulated as a single linear program.

4. APPLICATION TO DC-DC CONVERTERS

In this section we illustrate Algorithm 2 on a Buck-Boost DC-DC converter power circuit. See Figure 1 for a schematic representation of an ideal circuit (i.e. neglecting the parasite components). The following nonlinear averaged model of the converter, which was developed in (Lazar and De Keyser, 2004) by applying the theory of (Kassakian *et al.*, 1992), is used to obtain a prediction model:

$$x_{k+1}^m = \begin{bmatrix} x_{1,k}^m + \frac{T}{L}x_{2,k}^m - \frac{T}{L}(x_{2,k}^m - V_{in})u_k^m \\ -\frac{T}{C}x_{1,k}^m + \frac{T}{C}x_{1,k}^m u_k^m + (1 - \frac{T}{C})x_{2,k}^m \end{bmatrix}, \quad k \in \mathbb{Z}_+, \quad (12)$$

where $x_k^m = [x_{1,k}^m \ x_{2,k}^m]^\top \in \mathbb{R}^2$ and $u_k^m \in \mathbb{R}$ are the state and the input, respectively. x_1^m (i_L) represents the current flowing through the inductor, x_2^m (v_o) the output voltage and u^m represents the duty cycle (i.e. the fraction of the sampling period during which the transistor is kept ON). The sampling period is $T = 0.65$ milliseconds. The parameters of the circuit are the inductance $L = 4.2\text{mH}$, the capacitance $C = 2200\mu\text{F}$, the load resistance $R = 165\Omega$ and the source input voltage v_{in} , with nominal value $V_{in} = 15\text{V}$. The control objective is to reach a desired steady state value of

the output voltage, i.e. x_2^{ss} , as fast as possible and with minimum overshoot. From x_2^{ss} one can obtain the steady state duty cycle and inductor current as follows:

$$u^{ss} = \frac{x_2^{ss}}{x_2^{ss} - V_{in}}, \quad x_1^{ss} = \frac{x_2^{ss}}{R(u^{ss} - 1)}. \quad (13)$$

Furthermore, the following physical constraints must be fulfilled at all times $k \in \mathbb{Z}_+$:

$$x_{1,k}^m \in [0.01, 5], \quad x_{2,k}^m \in [-20, 0], \quad u_k^m \in [0.1, 0.9]. \quad (14)$$

To implement Algorithm 2, we first perform the following coordinate transformation on (12):

$$x_{1,k} = x_{1,k}^m - x_1^{ss}, \quad x_{2,k} = x_{2,k}^m - x_2^{ss}, \quad u_k = u_k^m - u^{ss}. \quad (15)$$

We obtain the following system description

$$x_{k+1} = \begin{bmatrix} x_{1,k} + \alpha x_{2,k} + (\beta - \frac{T}{L}x_{2,k})u_k \\ (\frac{T}{C}x_{1,k} + \gamma)u_k + (1 - \frac{T}{RC})x_{2,k} + \delta x_{1,k} \end{bmatrix}, \quad (16)$$

where the constants α , β , γ and δ depend on the fixed steady state value x_2^{ss} as follows $\alpha = \frac{T}{L}(1 - \frac{x_2^{ss}}{x_2^{ss} - V_{in}})$, $\beta = \frac{T}{L}(V_{in} - x_2^{ss})$, $\gamma = \frac{T}{RCV_{in}}x_2^{ss}(x_2^{ss} - V_{in})$ and $\delta = \frac{T}{C}(\frac{x_2^{ss}}{x_2^{ss} - V_{in}} - 1)$. Using (15) and (13), the constraints given in (14) can be converted to:

$$x_{1,k} \in [\underline{b}^{x_1}, \bar{b}^{x_1}], \quad x_{2,k} \in [\underline{b}^{x_2}, \bar{b}^{x_2}], \quad u_k \in [\underline{b}^u, \bar{b}^u], \quad (17)$$

where $\underline{b}^{x_1} = 0.01 - \frac{1}{RV_{in}}x_2^{ss}(x_2^{ss} - V_{in})$, $\bar{b}^{x_2} = -20 - x_2^{ss}$, $\underline{b}^u = 0.1 - \frac{x_2^{ss}}{x_2^{ss} - V_{in}}$, $\bar{b}^{x_1} = 5 - \frac{1}{RV_{in}}x_2^{ss}(x_2^{ss} - V_{in})$, $\bar{b}^{x_2} = -x_2^{ss}$ and $\bar{b}^u = 0.9 - \frac{x_2^{ss}}{x_2^{ss} - V_{in}}$. The control objective can now be formulated as to stabilize (16) around the equilibrium $(0, 0)$ while fulfilling the constraints given in (17). Next, to compute an ∞ -norm based artificial Lyapunov function via Lemma 6, we linearize system (16) around the equilibrium $(0, 0)$ (for zero input $u_k = 0 \in [\underline{b}^u, \bar{b}^u]$). The linearized equations are:

$$\Delta x_{k+1} = A\Delta x_k + B\Delta u_k, \quad (18)$$

where Δx_k and Δu_k represent “small” deviations from the equilibrium $(0, 0)$ and zero input $u_k = 0$, respectively. The matrices A and B are given by

$$A \triangleq \left. \frac{\partial f}{\partial x} \right|_{x=0, u=0} = \begin{bmatrix} 1 & \alpha \\ \delta & 1 - \frac{T}{RC} \end{bmatrix}, \quad B \triangleq \left. \frac{\partial f}{\partial u} \right|_{x=0, u=0} = \begin{bmatrix} \beta \\ \gamma \end{bmatrix}.$$

For the linear model corresponding to a steady state output voltage $x_2^{ss} = -4\text{V}$ (which yields $u^{ss} = 0.2105$ and $x_1^{ss} = 0.0307\text{A}$), by applying the method of (Lazar *et al.*, 2005b) to find the matrix P_V and the feedback gain K satisfying (11) for $Q_V = 0.001 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we have obtained the solution $P_V = \begin{bmatrix} 0.9197 & -0.6895 \\ -0.5815 & 1.8109 \end{bmatrix}$ and $K = [-0.4648 \ 0.4125]$. The MPC cost matrices have been chosen as follows, to ensure a good performance: $P = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $R_u = 0.1$.

To assess the real-time applicability of the developed theory for this type of very fast system with a sampling period well below one millisecond, we chose $N = 1$ and we formulated the optimization problem in Step 1 of Algorithm 2 as a Linear Programming (LP) problem.

⁴ $\mathbb{X} \sim \mathbb{W} := \{x \in \mathbb{R}^n \mid x + \mathbb{W} \subseteq \mathbb{X}\}$ is the Pontryagin difference of the sets \mathbb{X} and \mathbb{W} .

The resulting LP problem has 3 optimization variables and 18 constraints.

In one simulation, we tested first the start-up behavior (see Figure 2 column one) and then, after reaching the desired operating point, we tested the disturbance rejection (see Figure 2 column two). The dynamics were simultaneously affected by an asymptotically decreasing additive disturbance of the form $w = [\frac{1}{k} \ 0]^T$ and a 50% drop of the load (i.e. $R=82.5\Omega$) for $k = 80, 81, \dots, 180$ (or from time instant 0.052 until time instant 0.117 - in seconds). For $k > 180$ the disturbance was set equal to zero and the load was set to its nominal value (i.e. $R=165\Omega$) to show that the closed-loop system is ISS, i.e. that asymptotic stability is recovered when the disturbance input vanishes. The trajectories over the time interval $[0 \ 0.1495]$ (in seconds, or 230 sampling periods) of the state and sub-optimal NMPC control input are plotted in Figure 2. Moreover, in Figure 2 (first plot in the second column) one can observe that during the disturbance rejection phase of the simulation, the output voltage is well within the operating margin required in industry for DC-DC converters, i.e. $\pm 3\%$ of the desired operating value.

Note that, although the simulations were performed for the transformed system (16), we chose to plot all variables in the original coordinates corresponding to system (12), which have more physical meaning.

The LP problem equivalent to the sub-optimal NMPC optimization problem in Step 1 of Algorithm 2 was always solved⁵ within the allowed sampling interval, with an worst case CPU time over 20 runs of 0.6314 milliseconds. The very good closed-loop performance obtained for $N = 1$ collaborated with the computational time estimate is encouraging for further development of the real-time application of the presented theory to control DC-DC power converters, especially using faster platforms, such as Digital Signal Processors (DSP).

5. CONCLUSIONS

Two new computationally friendly sub-optimal NMPC algorithms with an a priori input-to-state stability guarantee were presented. The first one employs a contraction constraint on the norm of the closed-loop system state, while the second algorithm uses an ∞ -norm based artificial Lyapunov function. For both NMPC schemes, the input-to-state stabilization constraints can be written as a finite number of linear inequalities. A case study on the control of a Buck-Boost DC-DC power converter that includes preliminary real-time numerical data was presented to illus-

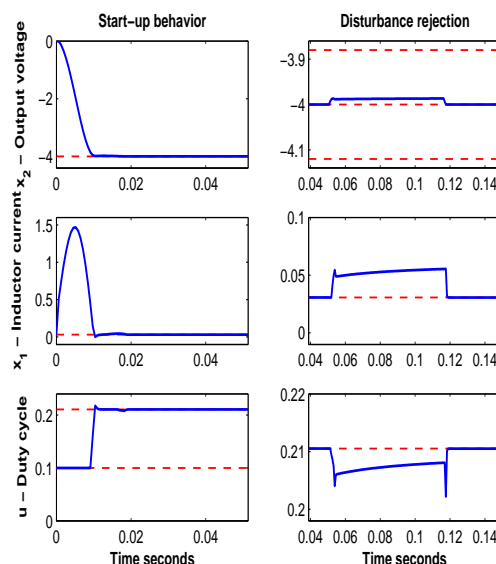


Fig. 2. State trajectories and sub-optimal NMPC input histories for $N = 1$ - blue solid lines, desired steady state values, input constraint (in first column, bottom plot) and industrial operating margins for DC-DC converters ($\pm 3\%$ of the desired output voltage, in second column, first plot) - red dashed lines.

trate the potential of the developed theory for real-time applications.

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⁵ The simulation platform was Matlab 7.0.4 (R14) (CDD Dual Simplex LP solver) running on a Linux Fedora Core 5 operating system powered by an Intel Pentium 4 with a 3.2 GHz CPU.