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On input-to-state stability of min–max nonlinear model predictive control

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Abstract

In this paper we consider discrete-time nonlinear systems that are affected, possibly simultaneously, by parametric uncertainties and other disturbance inputs. The min–max model predictive control (MPC) methodology is employed to obtain a controller that robustly steers the state of the system towards a desired equilibrium. The aim is to provide a priori sufficient conditions for robust stability of the resulting closed-loop system using the input-to-state stability (ISS) framework. First, we show that only input-to-state *practical* stability can be ensured in general for closed-loop min–max MPC systems; and we provide explicit bounds on the evolution of the closed-loop system state. Then, we derive new conditions for guaranteeing ISS of min–max MPC closed-loop systems, using a dual-mode approach. An example illustrates the presented theory.

Keywords: Min–max; Nonlinear model predictive control; Input-to-state stability

1. Introduction

One of the practically relevant problems in control theory is the robust regulation towards a desired equilibrium of discrete-time systems affected, possibly simultaneously, by time-varying parametric uncertainties and other disturbance inputs. In the case when hard constraints are imposed on state and input variables, the robust model predictive control (MPC) methodology provides a reliable solution for tackling this control problem, see, for example, [17] for an overview. The research related to robust MPC is focused on solving efficiently the corresponding optimization problems on one hand and guaranteeing (robust) stability of the controlled system, on the other hand. In this paper we are interested in stability issues and, therefore, we position our results only with respect to articles on (robust) stability of nonlinear MPC.

There are several ways for designing robust MPC controllers for perturbed nonlinear systems. One way is to rely on the inherent robustness properties of nominally stabilizing nonlinear MPC algorithms, e.g. as it was done in [21,14,11,2]. Another approach is to incorporate knowledge about the disturbances in the MPC problem formulation via open-loop worst case scenarios. This includes MPC algorithms based on tightened constraints, e.g. as the one of [12], and MPC algorithms, based on open-loop min–max optimization problems, see, for example, the survey [17]. To incorporate feedback to disturbances, the closed-loop or feedback min–max MPC problem set-up was introduced in [10] and further developed in [18,15,13,16]. The open-loop approach is computationally somewhat easier than the feedback approach, but the set of feasible states corresponding to the feedback min–max MPC optimization problem is usually much larger. Sufficient conditions for robust asymptotic stability of closed-loop (feedback) min–max MPC systems were presented in [18] under the assumption that the (additive) disturbance input converges to zero as the state converges to the origin.

Recently, input-to-state stability (ISS) [22,23,6] results for min–max nonlinear MPC were presented in [13,16]. In [13] it was shown that, in general, only input-to-state *practical* stability (ISpS) [3–5] can be a priori ensured for min–max nonlinear MPC. ISpS is a weaker property than ISS, as ISpS does not imply asymptotic stability for zero disturbance inputs. The reason for the absence of ISS in general is that the effect of a non-zero disturbance input is taken into account by the min–max

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MPC controller, even if the disturbance input vanishes in reality. Still, in the case when the disturbance input converges to zero, it is desirable that *asymptotic stability* is recovered for the controlled system. In [16], an $\mathcal{H}_\infty$ [1] strategy was used to modify the classical min–max MPC cost function [17] such that ISS is guaranteed for the closed-loop min–max MPC system. Furthermore, in [16] it was proven that a local upper bound on the min–max MPC value function, rather than a global one, is sufficient for ISS.

In this article we propose a new approach for designing min–max MPC schemes for nonlinear systems with guaranteed ISS. In contrast with [16], our results apply to the classical min–max MPC problem set-up, which is also employed in [18,13]. First, we develop novel ISpS conditions for min–max nonlinear MPC that allow us to derive explicit bounds on the evolution of the MPC closed-loop system state. Furthermore, we prove that these conditions actually imply that the state trajectory of the closed-loop system is ultimately bounded (UB) in a robustly positively invariant (RPI) set. Then, we use a dual-mode approach in combination with a new technique based on $\mathcal{KL}$-estimates of stability, e.g. see [8], to derive a priori sufficient conditions for ISS of min–max nonlinear MPC. This result is important because it unifies the properties of [13,18]. More specifically, it can be used to design robustly asymptotically stable min–max MPC closed-loop systems without a priori assuming that the disturbance input converges to zero as the state of the closed-loop system converges to the origin.

The paper is organized as follows. After introducing the notation in Section 1.1, the ISS framework is presented in Section 2. The min–max MPC problem set-up is briefly described in Section 3. The ISpS results for min–max nonlinear MPC are presented in Section 4 and the sufficient conditions for ISS of dual-mode min–max nonlinear MPC are given in Section 5. An illustrative example is worked out in Section 6. Conclusions are summarized in Section 7.

1.1. Notation and basic notions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{\geqslant c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ to denote the sets $\{k \in \mathbb{Z}_+ \mid k \geqslant c_1\}$ and $\{k \in \mathbb{Z}_+ \mid c_1 < k \leqslant c_2\}$, respectively, for some $c_1 \in \mathbb{Z}_+$, $c_2 \in \mathbb{Z}_{>c_1}$, and $\mathbb{Z}^N$ to denote the N-times Cartesian product $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$, for some $N \in \mathbb{Z}_{\geqslant 1}$. We use $\|\cdot\|$ to denote an arbitrary p-norm. With some abuse of notation we will use both $(z_0, z_1, \ldots)$ and $\{z_l\}_{l\in\mathbb{Z}_+}$ with $z_l \in \mathbb{R}^n$, $l \in \mathbb{R}_+$, to denote a sequence. For a sequence $\mathbf{z} := \{z_l\}_{l\in\mathbb{Z}_+}$ let $\|\mathbf{z}\| := \sup\{\|z_l\| \mid l \in \mathbb{Z}_+\}$ and let $\mathbf{z}_{[k]}$ denote the truncation of $\mathbf{z}$ at time $k \in \mathbb{Z}_+$, i.e. $\mathbf{z}_{[k]} = \{z_l\}_{l\in\mathbb{Z}_{[0,k]}}$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\text{int}(\mathcal{S})$ its interior. For any $r > 0$ define a ball of radius r as $\mathcal{B}_r := \{x \in \mathbb{R}^n \mid \|x\| \leqslant r\}$.

2. Input-to-state stability

In this section we present the ISS framework [22,23,6] for discrete-time autonomous nonlinear systems, which will be employed in this paper to study the behavior of perturbed nonlinear systems in closed-loop with min–max MPC controllers.

Consider the discrete-time autonomous perturbed nonlinear system described by

$$x_{k+1} = G(x_k, w_k, v_k), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{W} \subset \mathbb{R}^{d_w}$ and $v_k \in \mathbb{V} \subset \mathbb{R}^{d_v}$ are the state, unknown time-varying *parametric uncertainties* and other *disturbance inputs* (possibly additive), respectively, and, $G : \mathbb{R}^n \times \mathbb{R}^{d_w} \times \mathbb{R}^{d_v} \to \mathbb{R}^n$ is an arbitrary nonlinear, possibly discontinuous, function. In what follows we assume that $\mathbb{W}$ and $\mathbb{V}$ are bounded sets. Throughout the article let $\mathbf{w} := \{w_l \mid l \in \mathbb{Z}_+,\ w_l \in \mathbb{W}\}$ and $\mathbf{v} := \{v_l \mid l \in \mathbb{Z}_+,\ v_l \in \mathbb{V}\}$ denote some arbitrary sequences of disturbances.

Definition 2.1 (*RPI*). A set $\mathcal{P} \subseteq \mathbb{R}^n$ that contains the origin in its interior is called a *RPI set* for system (1) (with respect to $\mathbb{W}$ and $\mathbb{V}$) if for all $x \in \mathcal{P}$ it holds that $G(x, w, v) \in \mathcal{P}$ for all $w \in \mathbb{W}$ and all $v \in \mathbb{V}$.

Definition 2.2 (*UB*). System (1) is said to be *UB* in a set $\mathcal{P} \subset \mathbb{R}^n$ for initial conditions in $\mathbb{X} \subseteq \mathbb{R}^n$ (with respect to $\mathbb{W}$ and $\mathbb{V}$), if for all $x_0 \in \mathbb{X}$ there exists an $i(x_0) \in \mathbb{Z}_+$ such that for all $\mathbf{w}$ and all $\mathbf{v}$ the corresponding state trajectory of (1) satisfies $x_k \in \mathcal{P}$ for all $k \in \mathbb{Z}_{\geqslant i(x_0)}$.

Definition 2.3. A real-valued scalar function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. It *belongs to class* $\mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and it is radially unbounded (i.e. $\varphi(s) \to \infty$ as $s \to \infty$). A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and $\lim_{k\to\infty} \beta(s, k) = 0$.

Next, we introduce a regional version of global ISpS [3–5] and global ISS [22,23,6], respectively, for the discrete-time nonlinear system (1). This is useful when dealing with constrained nonlinear systems, such as NMPC closed-loop systems, as it was observed in [16].

Definition 2.4 (*Regional ISpS (ISS)*). System (1) is said to be *ISpS* in $\mathbb{X} \subseteq \mathbb{R}^n$ if there exists a $\mathcal{KL}$-function β, a $\mathcal{K}$-function γ and a number $d \in \mathbb{R}_+$ such that, for each $x_0 \in \mathbb{X}$, all $\mathbf{w}$ and all $\mathbf{v}$, it holds that the corresponding state trajectory of (1) satisfies

$$\|x_k\| \leqslant \beta(\|x_0\|, k) + \gamma(\|\mathbf{v}_{[k-1]}\|) + d, \quad \forall k \in \mathbb{Z}_{\geqslant 1}. \tag{2}$$

If $0 \in \text{int}(\mathbb{X})$ and (2) holds for $d = 0$, system (1) is said to be *ISS* in $\mathbb{X}$.

In what follows we state a *discrete-time* version of the *continuous-time* ISpS sufficient conditions of Proposition 2.1 of [4]. This result will be used throughout the paper to prove ISpS and ISS for the particular case of min–max nonlinear MPC.

Theorem 2.5. *Let $d_1, d_2 \in \mathbb{R}_+$, let $a, b, c, \lambda \in \mathbb{R}_{>0}$ with $c \leqslant b$ and let*[1] $\alpha_1(s) := as^\lambda$, $\alpha_2(s) := bs^\lambda$, $\alpha_3(s) := cs^\lambda$ *and* $\sigma \in \mathscr{K}$. *Furthermore, let $\mathbb{X}$ be a RPI set for system* (1) *and let $V : \mathbb{X} \to \mathbb{R}_+$ be a function such that*

$$\alpha_1(\|x\|) \leqslant V(x) \leqslant \alpha_2(\|x\|) + d_1, \tag{3a}$$

$$V(G(x, w, v)) - V(x) \leqslant -\alpha_3(\|x\|) + \sigma(\|v\|) + d_2, \tag{3b}$$

for all $x \in \mathbb{X}$, $w \in \mathbb{W}$ and all $v \in \mathbb{V}$. Then it holds that:

(i) *System* (1) *is ISpS in $\mathbb{X}$ and the ISpS property of Definition* 2.4 *holds for*

$$\beta(s, k) := \alpha_1^{-1}(3\rho^k \alpha_2(s)), \quad \gamma(s) := \alpha_1^{-1}\left(\frac{3\sigma(s)}{1-\rho}\right),$$
$$d := \alpha_1^{-1}(3\xi), \tag{4}$$

where $\xi := d_1 + d_2/(1-\rho)$ and $\rho := 1 - c/b \in [0, 1)$.

(ii) *If $0 \in \mathrm{int}(\mathbb{X})$ and the inequalities* (3) *hold for $d_1 = d_2 = 0$, system* (1) *is ISS in $\mathbb{X}$ and the ISS property of Definition* 2.4 (*i.e. for $d = 0$*) *holds for*

$$\beta(s, k) := \alpha_1^{-1}(2\rho^k \alpha_2(s)), \quad \gamma(s) := \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right), \tag{5}$$

where $\rho := 1 - c/b \in [0, 1)$.

Proof. (i) From $V(x) \leqslant \alpha_2(\|x\|) + d_1$ for all $x \in \mathbb{X}$, we have that for any $x \in \mathbb{X}\backslash\{0\}$ it holds:

$$\begin{aligned} V(x) - \alpha_3(\|x\|) &\leqslant V(x) - \frac{\alpha_3(\|x\|)}{\alpha_2(\|x\|)}(V(x) - d_1) \\ &= \rho V(x) + (1-\rho)d_1, \end{aligned}$$

where $\rho := 1 - c/b \in [0, 1)$. In fact, the above inequality holds for all $x \in \mathbb{X}$, since $V(0) - \alpha_3(0) = V(0) = \rho V(0) + (1-\rho)V(0) \leqslant \rho V(0) + (1-\rho)d_1$. Then, inequality (3b) becomes

$$V(G(x, w, v)) \leqslant \rho V(x) + \sigma(\|v\|) + (1-\rho)d_1 + d_2, \tag{6}$$

for all $x \in \mathbb{X}$, $w \in \mathbb{W}$ and all $v \in \mathbb{V}$. Due to robust positive invariance of $\mathbb{X}$, inequality (6) yields repetitively

$$V(x_{k+1}) \leqslant \rho^{k+1} V(x_0) + \sum_{i=0}^{k} \rho^i [\sigma(\|v_{k-i}\|) + (1-\rho)d_1 + d_2],$$

for all $x_0 \in \mathbb{X}$, $\mathbf{w}_{[k]} = (w_0, w_1, \ldots, w_k) \in \mathbb{W}^{k+1}$, $\mathbf{v}_{[k]} = (v_0, v_1, \ldots, v_k) \in \mathbb{V}^{k+1}$, $k \in \mathbb{Z}_+$. Then, taking (3a) into account, using the property $\sigma(\|v_i\|) \leqslant \sigma(\|\mathbf{v}_{[k]}\|)$ for all $i \leqslant k$ and the identity $\sum_{i=0}^{k} \rho^i = (1-\rho^{k+1})/(1-\rho)$, it holds that:

$$\begin{aligned} V(x_{k+1}) &\leqslant \rho^{k+1}\alpha_2(\|x_0\|) + \rho^{k+1} d_1 \\ &\quad + \sum_{i=0}^{k} \rho^i [\sigma(\|v_{k-i}\|) + (1-\rho)d_1 + d_2] \\ &\leqslant \rho^{k+1}\alpha_2(\|x_0\|) + \rho^{k+1} d_1 \\ &\quad + [\sigma(\|\mathbf{v}_{[k]}\|) + (1-\rho)d_1 + d_2] \sum_{i=0}^{k} \rho^i \\ &= \rho^{k+1}\alpha_2(\|x_0\|) + \frac{1-\rho^{k+1}}{1-\rho}\sigma(\|\mathbf{v}_{[k]}\|) \\ &\quad + d_1 + \frac{1-\rho^{k+1}}{1-\rho} d_2 \\ &\leqslant \rho^{k+1}\alpha_2(\|x_0\|) + \frac{1}{1-\rho}\sigma(\|\mathbf{v}_{[k]}\|) \\ &\quad + d_1 + \frac{1}{1-\rho} d_2, \end{aligned}$$

for all $x_0 \in \mathbb{X}$, $\mathbf{w}_{[k]} \in \mathbb{W}^{k+1}$, $\mathbf{v}_{[k]} \in \mathbb{V}^{k+1}$, $k \in \mathbb{Z}_+$. Let $\xi := d_1 + d_2/(1-\rho)$. Taking (3a) into account and letting α_1^{-1} denote the inverse of α_1, we obtain:

$$\begin{aligned} \|x_{k+1}\| &\leqslant \alpha_1^{-1}(V(x_{k+1})) \\ &\leqslant \alpha_1^{-1}\left(\rho^{k+1}\alpha_2(\|x_0\|) + \xi + \frac{\sigma(\|\mathbf{v}_{[k]}\|)}{1-\rho}\right). \end{aligned} \tag{7}$$

Applying the inequality

$$\begin{aligned} \alpha_1^{-1}(z + y + s) &\leqslant \alpha_1^{-1}(3\max(z, y, s)) \\ &\leqslant \alpha_1^{-1}(3z) + \alpha_1^{-1}(3y) + \alpha_1^{-1}(3s), \end{aligned} \tag{8}$$

we obtain from (7)

$$\|x_{k+1}\| \leqslant \alpha_1^{-1}(3\rho^{k+1}\alpha_2(\|x_0\|)) + \alpha_1^{-1}\left(3\frac{\sigma(\|\mathbf{v}_{[k]}\|)}{1-\rho}\right) + \alpha_1^{-1}(3\xi),$$

for all $x_0 \in \mathbb{X}$, $\mathbf{w}_{[k]} \in \mathbb{W}^{k+1}$, $\mathbf{v}_{[k]} \in \mathbb{V}^{k+1}$, $k \in \mathbb{Z}_+$.

We distinguish between two cases: $\rho \neq 0$ and $\rho = 0$. First, suppose $\rho \in (0, 1)$ and let $\beta(s, k) := \alpha_1^{-1}(3\rho^k \alpha_2(s))$. For a fixed $k \in \mathbb{Z}_+$, we have that $\beta(\cdot, k) \in \mathscr{K}$ due to $\alpha_2 \in \mathscr{K}_\infty$, $\alpha_1^{-1} \in \mathscr{K}_\infty$ and $\rho \in (0, 1)$. For a fixed s, it follows that $\beta(s, \cdot)$ is non-increasing and $\lim_{k\to\infty} \beta(s, k) = 0$, due to $\rho \in (0, 1)$ and $\alpha_1^{-1} \in \mathscr{K}_\infty$. Thus, it follows that $\beta \in \mathscr{KL}$.

Now let $\gamma(s) := \alpha_1^{-1}(3\sigma(s)/(1-\rho))$. Since $1/(1-\rho) > 0$, it follows that $\gamma \in \mathscr{K}$ due to $\alpha_1^{-1} \in \mathscr{K}_\infty$ and $\sigma \in \mathscr{K}$.

Finally, let $d := \alpha_1^{-1}(3\xi)$. Since $\rho \in (0, 1)$ and $d_1, d_2 \geqslant 0$, we have that $\xi \geqslant 0$ and thus, $d \geqslant 0$.

Otherwise, if $\rho = 0$ we have from (7) that

$$\begin{aligned} \|x_k\| &\leqslant \alpha_1^{-1}(3\sigma(\|\mathbf{v}_{[k-1]}\|)) + \alpha_1^{-1}(3\xi) \\ &\leqslant \beta(\|x_0\|, k) + \alpha_1^{-1}(3\sigma(\|\mathbf{v}_{[k-1]}\|)) + \alpha_1^{-1}(3\xi) \end{aligned}$$

for any $\beta \in \mathscr{KL}$ and $k \in \mathbb{Z}_{\geqslant 1}$.

Hence, the perturbed system (1) is ISpS in $\mathbb{X}$ in the sense of Definition 2.4 and property (2) is satisfied with the functions given in (4).

[1] Note that $\alpha_1, \alpha_2, \alpha_3 \in \mathscr{K}_\infty$.

(ii) Following the proof of statement (i) it is straightforward to observe that when the sufficient conditions (3) are satisfied for $d_1 = d_2 = 0$, then ISS is achieved, since $d = \alpha_1^{-1}(3\xi) = \alpha_1^{-1}(0) = 0$. From (7) and $\alpha_1^{-1}(z + y) \leqslant \alpha_1^{-1}(2\max(z, y)) \leqslant \alpha_1^{-1}(2z) + \alpha_1^{-1}(2y)$, it can be easily shown that the ISS property of Definition 2.4 actually holds with the functions given in (5). □

Definition 2.6. A function $V(\cdot)$ that satisfies the hypothesis of Theorem 2.5 is called an *ISpS* (*ISS*) *Lyapunov function*.

Remark 2.7. The hypothesis of Theorem 2.5 part (i) does not require continuity of $G(\cdot,\cdot,\cdot)$ or $V(\cdot)$, nor that $G(0, 0, 0) = 0$ or $V(0) = 0$. The latter makes the ISpS framework suitable for analyzing stability of nonlinear systems in closed-loop with min–max MPC controllers, since, in general, the min–max MPC value function is not zero at zero (see Section 4 for details). The hypothesis of Theorem 2.5 part (ii), which deals with ISS, also does not require continuity of $G(\cdot,\cdot,\cdot)$ or $V(\cdot)$. However, it implies $G(0, w, 0)=0$ for all $w \in \mathbb{W}$ and $V(0)=0$, and continuity of $G(\cdot, w, \cdot)$ and $V(\cdot)$ at the point $x = 0$ only, for all $w \in \mathbb{W}$.

Note that, due to the use of $\mathscr{K}_\infty$-functions $\alpha_1, \alpha_2, \alpha_3$ of a special type (which is not restrictive for the commonly used cost functions in min–max MPC, as shown in Section 4), Theorem 2.5 provides *explicit* bounds on the evolution of the state.

3. Min–max nonlinear MPC: problem set-up

The results presented in this paper can be applied to both open-loop and feedback min–max MPC strategies. However, there seems to be a common agreement that open-loop min–max formulations are conservative and underestimate the set of feasible input trajectories. For this reason, although we present both problem formulations, the stability results are proven only for feedback min–max MPC set-ups. However, it is possible to prove, via a similar reasoning and using the *same* hypotheses, that all the results developed in this paper also hold for open-loop min–max MPC schemes.

Consider the discrete-time non-autonomous perturbed nonlinear system

$$x_{k+1} = g(x_k, u_k, w_k, v_k), \quad k \in \mathbb{Z}_+, \tag{9}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $w_k \in \mathbb{W} \subset \mathbb{R}^{d_w}$ and $v_k \in \mathbb{V} \subset \mathbb{R}^{d_v}$ are the state, the control action, unknown time-varying *parametric uncertainties* and other *disturbance inputs*, respectively. The mapping $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{d_w} \times \mathbb{R}^{d_v} \to \mathbb{R}^n$ is an arbitrary nonlinear, possibly discontinuous, function. Let $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ denote sets that contain the origin in their interior and represent state and input constraints for system (9). Furthermore, let $\mathbb{X}_T \subseteq \mathbb{X}$ with $0 \in \operatorname{int}(\mathbb{X}_T)$ denote a desired terminal set and let $F : \mathbb{R}^n \to \mathbb{R}_+$ with $F(0) = 0$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $L(0, 0) = 0$ be arbitrary functions. The objective is to regulate the system towards the origin while minimizing a performance index defined by the functions $F(\cdot)$, $L(\cdot,\cdot)$ and with the set $\mathbb{X}_T$ as terminal constraint.

For a fixed prediction horizon $N \in \mathbb{Z}_{\geqslant 1}$, *open-loop* min–max MPC evaluates a single sequence of controls, i.e. $\mathbf{u}_k := (u_{0|k}, \ldots, u_{N-1|k}) \in \mathbb{U}^N$. Let $\mathbf{x}_k(x_k, \mathbf{u}_k, \mathbf{w}_k, \mathbf{v}_k) := (x_{1|k}, \ldots, x_{N|k})$ denote the state sequence generated by system (9) from initial state $x_{0|k} := x_k$ and by applying the input sequence $\mathbf{u}_k$, where $\mathbf{w}_k := (w_{0|k}, \ldots, w_{N-1|k}) \in \mathbb{W}^N$ and $\mathbf{v}_k := (v_{0|k}, \ldots, v_{N-1|k}) \in \mathbb{V}^N$ are the corresponding disturbance sequences and $x_{i|k} := g(x_{i-1|k}, u_{i-1|k}, w_{i-1|k}, v_{i-1|k})$ for all $i = 1, \ldots, N$. The open-loop min–max MPC class of *admissible input sequences* defined for $\mathbb{X}_T$ and state $x_k \in \mathbb{X}$ is

$$\mathscr{U}_N(x_k) := \{\mathbf{u}_k \in \mathbb{U}^N \mid \mathbf{x}_k(x_k, \mathbf{u}_k, \mathbf{w}_k, \mathbf{v}_k) \in \mathbb{X}^N, \\ x_{N|k} \in \mathbb{X}_T,\ \forall \mathbf{w}_k \in \mathbb{W}^N, \forall \mathbf{v}_k \in \mathbb{V}^N\}.$$

Let the terminal set $\mathbb{X}_T \subseteq \mathbb{X}$ and $N \in \mathbb{Z}_{\geqslant 1}$ be given. At time $k \in \mathbb{Z}_+$ let $x_k \in \mathbb{X}$ be given. The *open-loop* min–max MPC approach minimizes the cost $J(x_k, \mathbf{u}_k) := \max_{\mathbf{w}_k \in \mathbb{W}^N, \mathbf{v}_k \in \mathbb{V}^N} [F(x_{N|k}) + \sum_{i=0}^{N-1} L(x_{i|k}, u_{i|k})]$, with prediction model (9), over all sequences $\mathbf{u}_k$ in $\mathscr{U}_N(x_k)$.

Feedback min–max MPC obtains a sequence of feedback control laws that minimizes a worst case cost function, while assuring robust constraint handling. In this paper we employ the *dynamic programming approach* to feedback min–max nonlinear MPC proposed in [10] for linear systems and in [18] for nonlinear systems.

In this approach, the feedback min–max optimal control input is obtained as follows:

$$V_i(x) := \min_{u \in \mathbb{U}} \Big\{ \max_{w \in \mathbb{W}, v \in \mathbb{V}} [L(x, u) + V_{i-1}(g(x, u, w, v))] \\ \text{such that } g(x, u, w, v) \in \mathbb{X}_f(i-1), \\ \forall w \in \mathbb{W},\ \forall v \in \mathbb{V} \Big\}, \tag{10}$$

where the set $\mathbb{X}_f(i)$ contains all the states $x_i \in \mathbb{X}$ which are such that (10) is feasible, $i = 1, \ldots, N$. The optimization problem is defined for $i = 1, \ldots, N$ with the boundary conditions

$$V_0(x_0) := F(x_0), \\ \mathbb{X}_f(0) := \mathbb{X}_T. \tag{11}$$

Taking into account the definition of the min–max problem (10), $\mathbb{X}_f(i)$ is now the set of all states that can be robustly controlled into the set $\mathbb{X}_T$ in $i \in \mathbb{Z}_{\geqslant 1}$ steps.

The control law is applied to system (9) in a receding horizon manner. At each sampling time the problem is solved for the current state x and the value function $V_N(x)$ is obtained. The *feedback* min–max MPC control law is defined as

$$\bar{u}(x) := u_N^*, \tag{12}$$

where u_N^* is the optimizer of problem (10) for $i = N$. For simplicity of exposition, in what follows we assume existence and uniqueness of u_N^*, and that the minimum and the maximum are well-defined in (10), for all $i = 1, \ldots, N$. Notice that it is possible to show that the results developed in this article also apply when the global optimum is not unique. Furthermore, following the reasoning employed in [19], ISpS results can also be obtained for the suboptimal case.

In the following sections the min–max MPC value function $V_N(x)$ will be used as a candidate ISpS Lyapunov function in order to establish ISpS of the nonlinear system (9) in closed-loop with the feedback min–max MPC control (12). To simplify the notation, for the reminder of the article we will use $V(x)$ to denote $V_N(x)$.

4. ISpS results for min–max nonlinear MPC

In this section we present sufficient conditions for ISpS of system (9) in closed-loop with the feedback min–max MPC control (12) and we derive explicit bounds on the evolution of the closed-loop system state. Let $h:\mathbb{R}^n\to\mathbb{R}^m$ denote an arbitrary nonlinear function with $h(0)=0$ and let $\mathbb{X}_{\mathbb{U}}:=\{x\in\mathbb{X}\mid h(x)\in\mathbb{U}\}$.

Assumption 4.1. There exist $a_L, a_F, b_F, \lambda\in\mathbb{R}_{>0}$ with $a_L\leqslant b_F$, $e_1, e_2\in\mathbb{R}_+$, a function $h:\mathbb{R}^n\to\mathbb{R}^m$ with $h(0)=0$ and a $\mathscr{K}$-function $\bar\sigma$ such that:

(i) $\mathbb{X}_T\subseteq\mathbb{X}_{\mathbb{U}}$ and $0\in\text{int}(\mathbb{X}_T)$;
(ii) $\mathbb{X}_T$ is a RPI set for system (9) in closed-loop with $u_k=h(x_k)$, $k\in\mathbb{Z}_+$;
(iii) $L(x,u)\geqslant a_L\|x\|^\lambda$ for all $x\in\mathbb{X}$ and all $u\in\mathbb{U}$;
(iv) $a_F\|x\|^\lambda\leqslant F(x)\leqslant b_F\|x\|^\lambda+e_1$ for all $x\in\mathbb{X}_T$;
(v) $F(g(x,h(x),w,v))-F(x)\leqslant -L(x,h(x))+\bar\sigma(\|v\|)+e_2$ for all $x\in\mathbb{X}_T$, $w\in\mathbb{W}$, and $v\in\mathbb{V}$.

Note that Assumption 4.1 implies that $F(\cdot)$ is a local[2] ISpS Lyapunov function. Then, from Theorem 2.5 it follows that system (9) in closed-loop with $u_k=h(x_k)$, $k\in\mathbb{Z}_+$ is ISpS in $\mathbb{X}_T$, as formally stated below.

Proposition 4.2. *Suppose that Assumption* 4.1 *holds. Then, system* (9) *in closed-loop with* $u_k=h(x_k)$, $k\in\mathbb{Z}_+$, *is ISpS in* $\mathbb{X}_T$. *Moreover, if Assumption* 4.1 *holds with* $e_1=e_2=0$, *system* (9) *in closed-loop with* $u_k=h(x_k)$, $k\in\mathbb{Z}_+$, *is ISS in* $\mathbb{X}_T$.

Assumption 4.1 can be regarded as a generalization of the usual sufficient conditions for nominal stability of MPC, which imply that $F(\cdot)$ is a local Lyapunov function, see, for example, the survey [17]. Techniques for computing a terminal cost and a function $h(\cdot)$ such that Assumption 4.1 is satisfied have been recently developed in [9] for relevant subclasses of system (9) (i.e. perturbed linear and piecewise affine systems). See also the illustrative nonlinear example in Section 6.

Theorem 4.3. *Suppose that* $F(\cdot)$, $L(\cdot,\cdot)$, $\mathbb{X}_T$ *and* $h(\cdot)$ *are such that Assumption* 4.1 *holds for system* (9). *Furthermore, suppose that there exists a number* $\theta\in\mathbb{R}_{\geqslant b_F}$ *such that* $V(x)\leqslant\theta\|x\|^\lambda$ *for all* $x\in\mathbb{X}_f(N)\backslash\mathbb{X}_T$. *Then, the perturbed nonlinear system* (9) *in closed-loop with the feedback min–max MPC control* (12) *is ISpS in* $\mathbb{X}_f(N)$. *Moreover, the property* (2) *holds with the following functions*:

$$\beta(s,k):=\left(\frac{3\theta}{a_L}\right)^{1/\lambda}\check\rho^k s,\quad \gamma(s):=\left(\frac{3\delta}{a_L(1-\rho)}\right)^{1/\lambda}s,$$
$$d:=\left(\frac{3\xi}{a_L}\right)^{1/\lambda},\tag{13}$$

where $\check\rho:=\rho^{1/\lambda}\in(0,1)$, $\rho:=1-a_L/\theta\in(0,1)$, $\delta>0$ *can be taken arbitrarily small*, $\xi:=d_1+d_2/(1-\rho)$, $d_1:=e_1+N[\max_{v\in\mathbb{V}}\bar\sigma(\|v\|)+e_2]$ *and* $d_2:=\max_{v\in\mathbb{V}}\bar\sigma(\|v\|)+e_2$.

Proof. The proof consists in showing that the min–max MPC value function $V(\cdot)$ is an ISpS Lyapunov function, i.e. it satisfies the hypothesis of Theorem 2.5. First, it is known (see [18,7]) that under Assumption 4.1(i), (ii) the set $\mathbb{X}_f(N)$ is a RPI set for system (9) in closed-loop with the feedback min–max MPC control (12).

Second, we will obtain lower and upper bounding functions on the min–max MPC value function that satisfy (3a). From Assumption 4.1(iii) it follows that $V(x)=V_N(x)\geqslant L(x,\bar u(x))\geqslant a_L\|x\|^\lambda$, for all $x\in\mathbb{X}_f(N)$, where $\bar u(x)$ is the feedback min–max MPC control law defined in (12).

Next, letting $x_0:=x\in\mathbb{X}_T$, by Assumption 4.1(ii) (i.e. due to robust positive invariance of $\mathbb{X}_T$) one can apply Assumption 4.1(v) repetitively for the sequence of predicted states. Summing up the resulting inequalities it follows that for any $\mathbf{w}_{[N-1]}\in\mathbb{W}^N$ and any $\mathbf{v}_{[N-1]}\in\mathbb{V}^N$

$$F(x_N)+\sum_{i=0}^{N-1}L(x_i,h(x_i))\leqslant F(x_0)+\sum_{i=0}^{N-1}\bar\sigma(\|v_i\|)+Ne_2,$$

where $x_i:=g(x_{i-1},h(x_{i-1}),w_{i-1},v_{i-1})$ for $i=1,\dots,N$. Then, by optimality and Assumption 4.1(iv) we have that for all $x\in\mathbb{X}_T$,

$$\begin{aligned}V(x)=V_N(x)&\leqslant\max_{w\in\mathbb{W},v\in\mathbb{V}}\left[F(x_N)+\sum_{i=0}^{N-1}L(x_i,h(x_i))\right]\\&\leqslant F(x)+N\left[\max_{v\in\mathbb{V}}\bar\sigma(\|v\|)+e_2\right]\leqslant b_F\|x\|^\lambda+d_1,\end{aligned}$$

where $d_1:=e_1+N[\max_{v\in\mathbb{V}}\bar\sigma(\|v\|)+e_2]>0$. As from the hypothesis of Theorem 4.3 we also have that $V(x)\leqslant\theta\|x\|^\lambda$ for all $x\in\mathbb{X}_f(N)\backslash\mathbb{X}_T$ (with $b_F\leqslant\theta$) it follows that $V(x)\leqslant\theta\|x\|^\lambda+d_1$ for all $x\in\mathbb{X}_f(N)$. Hence, $V(\cdot)$ satisfies condition (3a) for all $x\in\mathbb{X}_f(N)$ with $\alpha_1(s):=a_Ls^\lambda$, $\alpha_2(s):=\theta s^\lambda$ and $d_1=e_1+N[\max_{v\in\mathbb{V}}\bar\sigma(\|v\|)+e_2]>0$.

Next, we show that $V(\cdot)$ satisfies condition (3b). By Assumption 4.1(v) and optimality, for all $x\in\mathbb{X}_T=\mathbb{X}_f(0)$ we have that:

$$\begin{aligned}V_1(x)-V_0(x)&\leqslant\max_{w\in\mathbb{W},v\in\mathbb{V}}[L(x,h(x))+F(g(x,h(x),w,v))]\\&\quad-F(x)\\&\leqslant\max_{v\in\mathbb{V}}\bar\sigma(\|v\|)+e_2.\end{aligned}$$

[2] ISS Lyapunov function when $e_1=e_2=0$.

Then, it can be shown via induction that (see also [13]):

$$V_{i+1}(x) - V_i(x) \leqslant \max_{v\in\mathbb{V}} \bar{\sigma}(\|v\|) + e_2,$$
$$\forall x \in \mathbb{X}_f(i), \quad \forall i \in 0, \dots, N-1. \tag{14}$$

At time $k \in \mathbb{Z}_+$, for a given state $x_k \in \mathbb{X}$ and a fixed prediction horizon N the min–max MPC control law $\bar{u}(x_k)$ is calculated and then applied to system (9). The state evolves to $x_{k+1} = g(x_k, \bar{u}(x_k), w_k, v_k) \in \mathbb{X}_f(N)$. Then, by Assumption 4.1(v) and applying recursively (14) it follows that

$$\begin{aligned} V_N(x_{k+1}) - V_N(x_k) &= V_N(x_{k+1}) - \max_{w\in\mathbb{W}, v\in\mathbb{V}} [L(x_k, \bar{u}(x_k)) \\ &\quad + V_{N-1}(g(x_k, \bar{u}(x_k), w, v))] \\ &\leqslant V_N(x_{k+1}) - L(x_k, \bar{u}(x_k)) \\ &\quad - V_{N-1}(g(x_k, \bar{u}(x_k), w_k, v_k)) \\ &= V_N(x_{k+1}) - L(x_k, \bar{u}(x_k)) \\ &\quad - V_{N-1}(x_{k+1}) \\ &\leqslant -L(x_k, \bar{u}(x_k)) + \max_{v\in\mathbb{V}} \bar{\sigma}(\|v\|) + e_2 \\ &\leqslant -a_L\|x_k\|^\lambda + \max_{v\in\mathbb{V}} \bar{\sigma}(\|v\|) + e_2 \\ &= -a_L\|x_k\|^\lambda + d_2, \end{aligned} \tag{15}$$

for all $x_k \in \mathbb{X}_f(N)$, $w_k \in \mathbb{W}$, $v_k \in \mathbb{V}$ and all $k \in \mathbb{Z}_+$, where $d_2 := \max_{v\in\mathbb{V}} \bar{\sigma}(\|v\|) + e_2 > 0$. Hence, the feedback min–max nonlinear MPC value function $V(\cdot)$ satisfies (3b) with $\alpha_3(s) := a_L s^\lambda$, any $\sigma \in \mathscr{K}$ and $d_2 = \max_{v\in\mathbb{V}} \bar{\sigma}(\|v\|) + e_2 > 0$. The statements then follow from Theorem 2.5.

The functions $\beta(\cdot,\cdot)$, $\gamma(\cdot)$ and the constant d defined in (13) are obtained by letting $\sigma(s) := \delta s^\lambda$ for some (any) $\delta > 0$ and substituting the functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$, $\sigma(\cdot)$ and the constants d_1, d_2 obtained above in relation (4). □

5. Main result: ISS dual-mode min–max MPC

As shown in the previous section, the hypothesis of Theorem 4.3 is sufficient for ISpS, but not necessarily for ISS of system (9) in closed-loop with $\bar{u}(\cdot)$, even when $e_1 = e_2 = 0$. This is due to the min–max MPC value function $V(\cdot)$, which is only an ISpS Lyapunov function in general, and not an ISS Lyapunov function. Therefore, it is unclear whether the min–max MPC control law (12) results in an ISS closed-loop system.

In the case of persistent disturbances this is not necessarily a drawback, since ultimate boundedness in a RPI subset of $\mathbb{X}_f(N)$ is the most one can aim at, anyhow. It will be shown next that UB is indeed guaranteed under the hypothesis of Theorem 4.3. However, in the case when the disturbance input vanishes after a certain time it is desirable to have an ISS closed-loop system.

In this section we present sufficient conditions for ISS of system (9) in closed-loop with a dual-mode min–max MPC strategy. The following technical result will be employed to prove the main result for dual-mode min–max nonlinear MPC.

For any $\tau \in \mathbb{R}_{(0,a_L)}$ define

$$\mathbb{M}_\tau := \left\{ x \in \mathbb{X}_f(N) \,\middle|\, \|x\|^\lambda \leqslant \frac{d_2}{a_L - \tau} \right\}$$

and

$$\overline{\mathbb{M}}_\tau := \mathbb{X}_f(N) \setminus \mathbb{M}_\tau, \tag{16}$$

where $a_L \in \mathbb{R}_{>0}$ is from Assumption 4.1(iii) and $d_2 = \max_{v\in\mathbb{V}} \bar{\sigma}(\|v\|) + e_2 > 0$. Note that $0 \in \mathrm{int}(\mathbb{M}_\tau)$, as $d_2/(a_L - \tau) > 0$ and $0 \in \mathrm{int}(\mathbb{X}_T) \subseteq \mathrm{int}(\mathbb{X}_f(N))$.

Lemma 5.1. *Suppose that $F(\cdot)$, $L(\cdot,\cdot)$, $\mathbb{X}_T$ and $h(\cdot)$ are such that Assumption 4.1 holds for system (9). Let $\tau \in \mathbb{R}_{(0,a_L)}$ be such that $\overline{\mathbb{M}}_\tau \neq \emptyset$ and consider the closed-loop system (9)–(12). Then, for each $x_0 \in \overline{\mathbb{M}}_\tau$ there exists an $i(x_0) \in \mathbb{Z}_{\geqslant 1}$ such that for all disturbance realizations* **w** *and* **v**, *it holds that $x_{i(x_0)} \in \mathbb{M}_\tau$.*

Moreover, there exists a $\mathscr{KL}$-function $\bar{\beta}$ such that for all $x_0 \in \overline{\mathbb{M}}_\tau$ and all disturbance realizations **w** *and* **v**, *the corresponding trajectory of the closed-loop system (9)–(12) satisfies $\|x_k\| \leqslant \bar{\beta}(\|x_0\|, k)$ as long as $x_k \in \overline{\mathbb{M}}_\tau$ for all $k \in \mathbb{Z}_{[0,i)}$, $i \in \mathbb{Z}_{\geqslant 1}$.*

Proof. We prove the second statement of the lemma first. As shown in the proof of Theorem 4.3, the hypothesis implies that

$$a_L\|x\|^\lambda \leqslant V(x) \leqslant \theta\|x\|^\lambda + d_1, \quad \forall x \in \mathbb{X}_f(N).$$

Let $r > 0$ be such that $\mathscr{B}_r \subseteq \mathbb{M}_\tau$. For all state trajectories $\{x_k\}_{k\in\mathbb{Z}_{[0,i)}} \in \overline{\mathbb{M}}_\tau^i$ (and thus, $x_k \notin \mathbb{M}_\tau$ for all $k \in \mathbb{Z}_{[0,i)}$) we have that $\|x_k\| \geqslant r$ for all $k \in \mathbb{Z}_{[0,i)}$.

This yields:

$$\begin{aligned} V(x_k) &\leqslant \theta\|x_k\|^\lambda + d_1\left(\frac{\|x_k\|}{r}\right)^\lambda \\ &\leqslant \left(\theta + \frac{d_1}{r^\lambda}\right)\|x_k\|^\lambda, \quad \forall x_k \in \overline{\mathbb{M}}_\tau, \ \forall k \in \mathbb{Z}_{[0,i)}. \end{aligned}$$

The hypothesis also implies (see (15)) that

$$V(x_{k+1}) - V(x_k) \leqslant -a_L\|x_k\|^\lambda + d_2,$$
$$\forall x_k \in \mathbb{X}_f(N), \quad w_k \in \mathbb{W}, \quad v_k \in \mathbb{V}, \quad k \in \mathbb{Z}_+.$$

By the definitions in (16), for $x \in \overline{\mathbb{M}}_\tau$ it holds that $-a_L\|x\|^\lambda + d_2 \leqslant -\tau\|x\|^\lambda$, which yields:

$$V(x_{k+1}) - V(x_k) \leqslant -\tau\|x_k\|^\lambda,$$
$$\forall x_k \in \overline{\mathbb{M}}_\tau, \ w_k \in \mathbb{W}, \ v_k \in \mathbb{V}, \ k \in \mathbb{Z}_{[0,i)}. \tag{17}$$

Then, following the steps of the proof of Theorem 2.5, it is straightforward to show that the state trajectory satisfies for all $k \in \mathbb{Z}_{[0,i)}$,

$$\|x_k\| \leqslant \bar{\beta}(\|x_0\|, k),$$
$$\bar{\beta}(s,k) := \bar{\alpha}_1^{-1}(\bar{\rho}^k \bar{\alpha}_2(s)) = \left(\frac{\bar{b}}{a_L}\right)^{1/\lambda} s(\bar{\rho}^{1/\lambda})^k, \tag{18}$$

where $\bar{\alpha}_2(s) := \bar{b}s^\lambda$, $\bar{b} := \theta + d_1/r^\lambda$, $\bar{\alpha}_1(s) := a_L s^\lambda$ and $\bar{\rho} := 1 - \tau/\bar{b}$. Note that $\bar{\rho} \in (0,1)$ as $0 < \tau < a_L \leqslant b_F \leqslant \theta < \theta + d_1/r^\lambda = \bar{b}$.

Next, we prove that there exists an $i \in \mathbb{Z}_{\geqslant 1}$ such that $x_i \in \mathbb{M}_\tau$. Assume that there does not exist an $i \in \mathbb{Z}_{\geqslant 1}$ such that $x_i \in \mathbb{M}_\tau$. Then, for all $i \in \mathbb{Z}_+$ we have that

$$\|x_i\| \leqslant \bar{\beta}(\|x_0\|, i) = \left(\frac{\bar{b}}{a_L}\right)^{1/\lambda} \|x_0\|(\bar{\rho}^{1/\lambda})^i.$$

Since $\bar{\rho}^{1/\lambda} \in (0, 1)$, we have that $\lim_{i\to\infty}(\bar{\rho}^{1/\lambda})^i = 0$. Hence, there exists an $i \in \mathbb{Z}_{\geqslant 1}$ such that $x_i \in \mathscr{B}_r \subseteq \mathbb{M}_\tau$ and we reached a contradiction. Note that (18) is independent of w or v and thus, i can be taken to depend on x_0 only. □

Before stating the main result, we make use of Lemma 5.1 to prove that the ISpS sufficient conditions of Assumption 4.1 ensure ultimate boundedness of the min–max MPC closed-loop system. This property is achieved with respect to a RPI sublevel set of the min–max MPC value function induced by the set $\mathbb{M}_\tau$.

Theorem 5.2. *Suppose that the hypothesis of Lemma* 5.1 *holds and let*

$$\Upsilon := \max_{x\in\mathbb{M}_\tau} V(x) + d_2$$

and

$$\mathscr{V}_\Upsilon := \{x \in \mathbb{X}_f(N) \mid V(x) \leqslant \Upsilon\}.$$

Then, the closed-loop system (9)–(12) *is UB in the set* $\mathscr{V}_\Upsilon$ *for initial conditions in* $\mathbb{X}_f(N)$.

Proof. By definition of Υ, $x \in \mathbb{M}_\tau \subseteq \mathbb{X}_f(N)$ implies that

$$V(x) \leqslant \max_{x\in\mathbb{M}_\tau} V(x) \leqslant \max_{x\in\mathbb{M}_\tau} V(x) + d_2 = \Upsilon.$$

Therefore, $\mathbb{M}_\tau \subseteq \mathscr{V}_\Upsilon$. Suppose that $x_0 \in \mathbb{X}_f(N)\backslash\mathscr{V}_\Upsilon$ and thus, $x_0 \in \overline{\mathbb{M}}_\tau$. Then, by Lemma 5.1 it follows that there exists an $i(x_0) \in \mathbb{Z}_{\geqslant 1}$ such that $x_i(x_0) \in \mathbb{M}_\tau \subseteq \mathscr{V}_\Upsilon$.

Next, we prove that $\mathscr{V}_\Upsilon$ is a RPI set for the closed-loop system (9)–(12). As shown in the proof of Lemma 5.1 (see (17)), for any $x \in \mathscr{V}_\Upsilon\backslash\mathbb{M}_\tau$ it holds that

$$V(g(x, \bar{u}(x), w, v)) \leqslant V(x) - \tau\|x\|^\lambda \leqslant V(x) \leqslant \Upsilon,$$

for all $w \in \mathbb{W}$ and all $v \in \mathbb{V}$. Now let $x \in \mathbb{M}_\tau$. By inequality (15) it holds that

$$V(g(x, \bar{u}(x), w, v)) \leqslant V(x) - a_L\|x\|^\lambda + d_2 \leqslant V(x) + d_2 \leqslant \Upsilon.$$

Therefore, for any $x \in \mathscr{V}_\Upsilon$, it holds that $g(x, \bar{u}(x), w, v) \in \mathscr{V}_\Upsilon$ for all $w \in \mathbb{W}$ and all $v \in \mathbb{V}$, which implies that $\mathscr{V}_\Upsilon$ is a RPI set for the closed-loop system (9)–(12). Hence, the closed-loop system (9)–(12) is UB in $\mathscr{V}_\Upsilon$. □

In a worst case situation, i.e. when the disturbance input $v \in \mathbb{V}$ is too large and $\mathscr{V}_\Upsilon = \mathbb{X}_f(N)$ the result of Theorem 5.2 diminishes to ultimate boundedness of $\mathbb{X}_f(N)$ itself.

To state the main result, let the dual-mode feedback min–max MPC control law be defined as

$$\bar{u}^{\mathrm{DM}}(x) := \begin{cases} \bar{u}(x) & \text{if } x \in \mathbb{X}_f(N)\backslash\mathbb{X}_T, \\ h(x) & \text{if } x \in \mathbb{X}_T. \end{cases} \tag{19}$$

Theorem 5.3. *Suppose Assumption* 4.1 *holds with* $e_1 = e_2 = 0$ *for system* (9) *and there exists* $\tau \in \mathbb{R}_{(0,a_L)}$ *such that* $\mathbb{M}_\tau \subseteq \mathbb{X}_T$. *Then, the perturbed nonlinear system* (9) *in closed-loop with the dual-mode feedback min–max MPC control* $\bar{u}^{\mathrm{DM}}(\cdot)$ *is ISS in* $\mathbb{X}_f(N)$.

Proof. In order to prove ISS, we consider two situations: in Case 1 we assume that $x_0 \in \mathbb{X}_T$ and in Case 2 we assume that $x_0 \in \mathbb{X}_f(N)\backslash\mathbb{X}_T$. In Case 1, $F(\cdot)$ satisfies the hypothesis of Proposition 4.2 with $e_1 = e_2 = 0$ and hence, the closed-loop system (9)–(19) is ISS. Then, using the reasoning employed in the proof of Lemma 5.1, it can be shown that there exists a $\mathscr{KL}$-function $\tilde{\beta}(s, k) := \tilde{\alpha}_1^{-1}(2\tilde{\rho}^k\tilde{\alpha}_2(s))$, *with* $\tilde{\alpha}_1(s) := a_F s^\lambda$, $\tilde{\alpha}_2(s) := b_F s^\lambda$, $\tilde{\rho} := 1 - a_L/b_F$, and a $\mathscr{K}$-function γ such that for all $x_0 \in \mathbb{X}_T$ the state trajectory satisfies

$$\|x_k\| \leqslant \tilde{\beta}(\|x_0\|, k) + \gamma(\|\mathbf{v}_{[k-1]}\|), \quad \forall k \in \mathbb{Z}_{\geqslant 1}. \tag{20}$$

In Case 2, since $\mathbb{M}_\tau \subseteq \mathbb{X}_T$, by Lemma 5.1, for any $x_0 \in \mathbb{X}_f(N)$, $\mathbf{w}$ and any $\mathbf{v}$, there exists a $p \in \mathbb{Z}_{\geqslant 1}$ such that $x_k \notin \mathbb{X}_T$ for $k \in \mathbb{Z}_{[0,p)}$ and $x_p \in \mathbb{X}_T$. From Lemma 5.1 we also have that there exists a $\mathscr{KL}$-function $\bar{\beta}(s, k) = \bar{\alpha}_1^{-1}(\bar{\rho}^k\bar{\alpha}_2(s))$,with $\bar{\alpha}_1(s) = a_L s^\lambda$, $\bar{\alpha}_2(s) = \bar{b}s^\lambda$, $\bar{\rho} = 1 - \tau/\bar{b}$ such that the state trajectory satisfies

$$\|x_k\| \leqslant \bar{\beta}(\|x_0\|, k), \quad \forall k \in \mathbb{Z}_{\leqslant p} \text{ and } x_p \in \mathbb{X}_T.$$

Then, for all $p \in \mathbb{Z}_{\geqslant 1}$ and all $k \in \mathbb{Z}_{\geqslant p+1}$ it holds that

$$\begin{aligned}\|x_k\| &\leqslant \tilde{\beta}(\|x_p\|, k-p) + \gamma(\|\mathbf{v}_{[k-p,k-1]}\|) \\ &\leqslant \tilde{\beta}(\bar{\beta}(\|x_0\|, p), k-p) + \gamma(\|\mathbf{v}_{[k-p,k-1]}\|) \\ &\leqslant \hat{\beta}(\|x_0\|, k) + \gamma(\|\mathbf{v}_{[k-1]}\|),\end{aligned}$$

where $\mathbf{v}_{[k-p,k-1]}$ denotes the restriction of $\mathbf{v}$ to the interval $[k-p, k-1]$. In the above inequalities we used

$$\begin{aligned}&\tilde{\beta}(\bar{\beta}(s, p), k-p) \\ &\quad = \tilde{\alpha}_1^{-1}\left(2\tilde{\rho}^{k-p}\tilde{\alpha}_2\left(\left(\frac{\bar{b}}{a_L}\right)^{1/\lambda} s(\bar{\rho}^{1/\lambda})^p\right)\right) \\ &\quad \leqslant \left(\frac{2b_F\bar{b}}{a_L a_F}\right)^{1/\lambda} s(\hat{\rho}^{1/\lambda})^k := \hat{\beta}(s, k),\end{aligned}$$

and $\hat{\rho} := \max(\tilde{\rho}, \bar{\rho}) \in (0, 1)$. Hence, $\hat{\beta} \in \mathscr{KL}$.

Then, we have that

$$\|x_k\| \leqslant \beta(\|x_0\|, k) + \gamma(\|\mathbf{v}_{[k-1]}\|), \quad \forall k \in \mathbb{Z}_{\geqslant 1},$$

for all $x_0 \in \mathbb{X}_f(N)$, $\mathbf{w}$ and all $\mathbf{v}$, where $\beta(s, k) := \max(\tilde{\beta}(s, k), \bar{\beta}(s, k), \hat{\beta}(s, k))$.

Since $\tilde{\beta}, \bar{\beta}, \hat{\beta} \in \mathscr{KL}$ implies that $\beta \in \mathscr{KL}$, and we have $\gamma \in \mathscr{K}$, the statement then follows from Definition 2.4. □

The interpretation of the condition $\mathbb{M}_\tau \subseteq \mathbb{X}_T$ is that the min–max MPC controller steers the state of the system inside the terminal set $\mathbb{X}_T$ for all $\mathbf{w}$ and all $\mathbf{v}$. Then, ISS can be achieved by switching to the local feedback control law when the state enters the terminal set.

6. Illustrative example: a nonlinear double integrator

The following example will illustrate how one can verify the conditions for ISS of min–max nonlinear MPC presented in this article. For examples that illustrate the benefits of using a min–max MPC scenario compared to using a nominally stabilizing or inherently robust MPC approach we refer the interested reader to [10,20,15,24] and the references therein.

Consider a perturbed discrete-time nonlinear double integrator obtained from a continuous-time double integrator via a sample-and-hold device with a sampling period equal to one, as follows:

$$x_{k+1} = Ax_k + Bu_k + f(x_k) + v_k, \quad k \in \mathbb{Z}_+, \tag{21}$$

where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$, $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x) := 0.025 \begin{bmatrix} 1 \\ 1 \end{bmatrix} x^\top x$ is a nonlinear additive term and $v_k \in \mathbb{V} := \{v \in \mathbb{R}^2 \mid \|v\|_\infty \leqslant 0.03\}$ for all $k \in \mathbb{Z}_+$ is an additive disturbance input (we use $\|\cdot\|_\infty$ to denote the infinity norm). The state and the input are constrained at all times in the sets

$$\mathbb{X} := \{x \in \mathbb{R}^2 \mid \|x\|_\infty \leqslant 10\}$$

and

$$\mathbb{U} := \{u \in \mathbb{R} \mid |u| \leqslant 2\}.$$

The MPC cost function is defined using ∞-norms, i.e.

$$F(x) := \|Px\|_\infty, \quad L(x,u) := \|Qx\|_\infty + \|Rx\|_\infty,$$

where P is a full-column rank matrix (to be determined), $Q = 0.8I_2$ and $R = 0.1$. The stage cost satisfies Assumption 4.1(iii) for $\lambda = 1$ and any $a_L \in (0, 0.8)$.

We take the function $h(\cdot)$ as $h(x) := Kx$, where $K \in \mathbb{R}^{1\times 2}$ is the gain matrix. To compute the terminal cost matrix P and the gain matrix K such that Assumption 4.1(v) holds, we first calculate P and K for the linearization of system (21), i.e.:

$$x_{k+1} = Ax_k + Bu_k + v_k, \quad k \in \mathbb{Z}_+. \tag{22}$$

To accommodate for the nonlinear term $f(\cdot)$, we employ a "larger" stage cost weight matrix for the state, i.e. $\widetilde{Q} = 2.4I_2$, instead of $Q = 0.8I_2$, for which it holds that $\|\widetilde{Q}x\|_\infty \geqslant \|Qx\|_\infty$ for all $x \in \mathbb{R}^2$. The terminal cost $F(x) = \|Px\|_\infty$ and local control law $h(x) = Kx$ with the matrices

$$P = \begin{bmatrix} 12.1274 & 7.0267 \\ 0.4769 & 11.6072 \end{bmatrix}, \quad K = [-0.5885 \;\; -1.4169] \tag{23}$$

were computed (using a technique recently developed in [9]) such that the following inequality holds for the linear system (22), i.e.

$$\|P((A+BK)x+v)\|_\infty - \|Px\|_\infty \leqslant -\|\widetilde{Q}x\|_\infty - \|RKx\|_\infty + \bar{\sigma}(\|v\|), \tag{24}$$

for all $x \in \mathbb{R}^2$ and all $v \in \mathbb{R}^2$, where $\bar{\sigma}(s) := \|P\|_\infty s$.

The terminal cost satisfies Assumption 4.1(iv) for $\lambda = 1$, $b_F = \|P\|_\infty = 19.1541$, $a_F = 0.1$ and $e_1 = 0$. To obtain a suitable bound on $\|f(x)\|_\infty$ we employ the following tightened set of constraints for $h(\cdot)$ (see Fig. 1 for a plot of $\mathbb{X}_\mathbb{U}$):

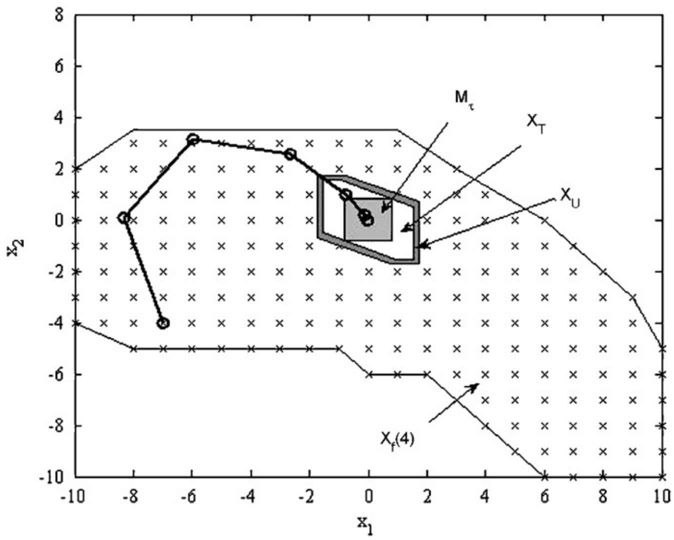

Fig. 1. State trajectory for the nonlinear system (21) in closed-loop with a dual-mode min–max MPC controller and an estimate of the set of feasible states $\mathbb{X}_f(4)$.

$$\mathbb{X}_\mathbb{U} := \{x \in \mathbb{X} \mid \|x\|_\infty \leqslant 1.72, \;\; |Kx| \leqslant 2\}.$$

The terminal set $\mathbb{X}_T$, also plotted in Fig. 1, is taken as the maximal RPI set contained in the set $\mathbb{X}_\mathbb{U}$ (and which is non-empty) for the linear system (22), in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$, and disturbances in the set $\{v \in \mathbb{R}^2 \mid \|v\|_\infty \leqslant 0.18\}$. One can easily check that $\max_{x \in \mathbb{X}_\mathbb{U}} \|f(x)\|_\infty < 0.15$ and thus, it follows that the terminal set $\mathbb{X}_T$ chosen as specified above is a RPI set for the *nonlinear system* (21) in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$, and all disturbances v in $\mathbb{V} = \{v \in \mathbb{R}^2 \mid \|v\|_\infty \leqslant 0.03\}$.

Using the fact that (notice that below, in some cases, $\|\cdot\|_\infty$ denotes the induced infinity matrix norm)

$$\|\widetilde{Q}x\|_\infty \geqslant 2.3515\|x\|_\infty, \quad \forall x \in \mathbb{R}^2,$$

$$\max_{x \in \mathbb{X}_T} \left\| P0.025 \begin{bmatrix} 1 \\ 1 \end{bmatrix} x^\top \right\|_\infty = 1.5515,$$

inequality (24) and the triangle inequality, for all $x \in \mathbb{X}_T$ and all $v \in \mathbb{R}^2$ we obtain

$$\begin{aligned}
&\|P((A+BK)x+v) + Pf(x)\|_\infty - \|Px\|_\infty \\
&\quad \leqslant \|P((A+BK)x+v)\|_\infty - \|Px\|_\infty + \|Pf(x)\|_\infty \\
&\quad \leqslant -\|\widetilde{Q}x\|_\infty - \|RKx\|_\infty + \bar{\sigma}(\|v\|_\infty) + \|Pf(x)\|_\infty \\
&\quad \leqslant -2.3515\|x\|_\infty - \|RKx\|_\infty + \bar{\sigma}(\|v\|_\infty) \\
&\qquad + \max_{x \in \mathbb{X}_T} \left(\left\| P0.025 \begin{bmatrix} 1 \\ 1 \end{bmatrix} x^\top \right\|_\infty \right) \|x\|_\infty \\
&\quad \leqslant -2.3515\|x\|_\infty - \|RKx\|_\infty + \bar{\sigma}(\|v\|_\infty) + 1.5515\|x\|_\infty \\
&\quad = -0.8\|x\|_\infty - \|RKx\|_\infty + \bar{\sigma}(\|v\|_\infty) \\
&\quad = -\|Q\|_\infty \|x\|_\infty - \|RKx\|_\infty + \bar{\sigma}(\|v\|_\infty) \\
&\quad \leqslant -\|Qx\|_\infty - \|RKx\|_\infty + \bar{\sigma}(\|v\|_\infty) \\
&\quad = -L(x, Kx) + \bar{\sigma}(\|v\|_\infty).
\end{aligned}$$

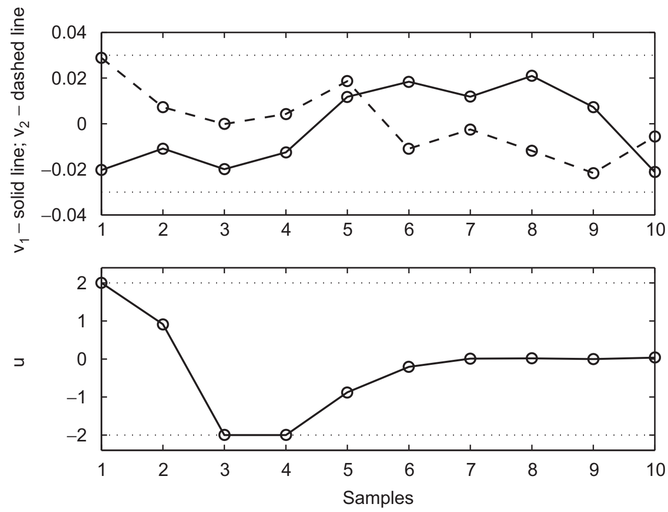

Fig. 2. Dual-mode min–max nonlinear MPC control input and disturbance input histories.

Hence, the terminal cost $F(x) = \|Px\|_\infty$ and the control law $h(x) = Kx$, with the matrices P and K given in (23), satisfy Assumption 4.1(v) for the nonlinear system (21) with $e_2 = 0$ and with $\bar{\sigma}(s) = \|P\|_\infty s$.

Consider now the set $\mathbb{M}_\tau$, which needs to be determined to establish ISS of the nonlinear system (21) in closed-loop with the dual-mode min–max MPC control law (19). We can choose $a_L = 0.79 < 0.8$, which ensures that $\|Qx\|_\infty \geqslant a_L\|x\|_\infty$ for all $x \in \mathbb{R}^2$. Since $d_2 = \max_{v\in\mathbb{V}} \bar{\sigma}(\|v\|_\infty) = 0.5746$, it follows that a necessary condition to be satisfied is $\tau \in (0, 0.79)$ (with the smallest set $\mathbb{M}_\tau$ obtained for $\lim_{\tau\to 0} d_2/(a_L - \tau) = 0.7273$). For $\tau = 0.0718$, which yields $d_2/(a_L - \tau) = 0.8001$, it holds that $\mathbb{M}_\tau \subset \mathbb{X}_T$, see Fig. 1 for an illustrative plot. Therefore, the closed-loop system (21)–(19) is ISS in $\mathbb{X}_f(N)$, as guaranteed by Theorem 5.3.

As the *feedback* min–max MPC optimization problem was computationally untractable for the nonlinear model (21), we have used an open-loop min–max MPC problem set-up, as the one described in Section 3, to calculate the control input. The developed theory applies also for the open-loop min–max MPC scheme, as pointed out in Section 3. Although the resulting open-loop min–max optimization problem still has a very high computational burden, we could obtain a solution using the *fmincon* Matlab solver. The closed-loop state trajectories for initial state $x_0 = [-7 \ \ -4]^\top$ and prediction horizon $N = 4$ are plotted in Fig. 1. The dual-mode min–max MPC control input and (randomly generated) disturbance input histories are plotted in Fig. 2. The min–max MPC controller manages to drive the state of the perturbed nonlinear system inside the terminal set, while satisfying constraints at all times.

7. Conclusions

In this article we have revisited the robust stability problem in min–max nonlinear MPC. The ISpS framework has been employed to study robust stability of perturbed nonlinear systems in closed-loop with min–max MPC controllers. New a priori conditions for ISpS were presented together with explicit bounds on the evolution of the closed-loop system state. Moreover, it was proven that these conditions also ensure ultimate boundedness. Novel conditions that guarantee ISS of min–max nonlinear MPC closed-loop systems were derived using a dual-mode approach. This result is useful as it provides a methodology for designing robustly asymptotically stable min–max MPC schemes without a priori assuming that the (additive) disturbance input converges to zero as the closed-loop system state converges to the origin.

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