Further Input-to-State Stability Subtleties for Discrete-time Systems

M. Lazar, W.P.M.H. Heemels, A.R. Teel

***Abstract*— This note considers input-to-state stability analysis of discrete-time systems using continuous Lyapunov functions. The main result reveals a relation between existence of a continuous Lyapunov function and inherent input-to-state stability on compact sets with respect to both inner and outer perturbations. If the Lyapunov function is $\mathscr{K}_\infty$-continuous, the result applies to unbounded sets as well.**

***Index Terms*— Discrete-time, Discontinuous systems, Input-to-state stability, Lyapunov methods.**

I. INTRODUCTION

This note considers the problem of inherent robustness analysis for general, stabilizing control laws, including those obtained *via* stabilizing model predictive control (MPC). A main motivation for this research is that nominally stabilizing (MPC) controllers might have no robustness properties with respect to disturbances. This aspect was for the first time shown in [1], where it was indicated that asymptotically stable MPC closed-loop systems may have zero robustness in the presence of arbitrary small perturbations. This undesired phenomenon was revealed in [1] by showing that an asymptotically stable MPC closed-loop system is not *robustly asymptotically stable* for arbitrary small perturbations. More recently, in [2] the same phenomenon was exposed for globally asymptotically stable (GAS) discrete-time systems in terms of a lack of input-to-state stability (ISS) [3] to *arbitrarily small* inputs. As the robust GAS property, as defined in [1], does not necessarily imply ISS, both observations are of independent interest. A further conclusion drawn in [2] is that GAS discrete-time systems which admit a discontinuous Lyapunov function are not necessarily ISS, not even locally. As such, this observation issued a warning for nominally stabilizing MPC schemes to potentially lack inherent ISS, as in the case of nonlinear or hybrid systems, the MPC candidate Lyapunov function is typically a discontinuous function.

To deal with the phenomenon of non-robustness, it would be useful to establish sufficient conditions under which nominally stable systems are inherently ISS. A conjecture that is frequently employed in the MPC literature is that the existence of a continuous Lyapunov function is sufficient for inherent ISS. The main contribution of this note is a formal statement and formal proof of this conjecture. To this end, we will exploit a property called $\mathscr{K}$-continuity [4], which generalizes Hölder continuity on compact sets, and a property called $\mathscr{K}_\infty$-continuity, which generalizes global Hölder continuity. In addition, we will provide a constructive proof of the fact that continuity on a compact set is equivalent with $\mathscr{K}$-continuity and we will prove that a stronger type of global uniform continuity is equivalent with $\mathscr{K}_\infty$-continuity. These results enable us to establish that every discrete-time system that admits a continuous Lyapunov function is inherently ISS on a robustly positively invariant compact set, with respect to both inner and outer perturbations. The inclusion of inner perturbations (e.g., measurement noise or estimation errors) is particularly relevant for MPC, as most of the ISS results in this framework are limited to outer perturbations (e.g., additive disturbances). A previous article that considered nominal robustness of MPC in terms of both inner and outer perturbations is [5], where it was established that existence of a continuous Lyapunov function is equivalent with robust GAS (RGAS) and semiglobal practical asymptotic stability (SPAS). Also, therein it was established that RGAS and SPAS are equivalent with attenuated ISS and integral ISS, respectively. As most robust stability results in nonlinear MPC make use of the ISS framework, see, e.g., [6]–[9], and integral ISS does not necessarily imply ISS [10], in this work we focus on establishing inherent ISS with respect to both inner and outer perturbations. In this context it is worth to mention the article [11], where a connection was established between ISS to outer perturbations and ISS to inner perturbations for general, constrained discrete-time systems.

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The remainder of the paper is organized as follows. Preliminaries are given in Section II, while a class $\mathscr{K}$ characterization of uniform continuity is provided in Section III. The main results on inherent ISS are presented in Section IV and conclusions are stated in Section V.

II. PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ we define $\Pi_{\geq c} := \{k \in \Pi \mid k \geq c\}$ and similarly $\Pi_{\leq c}$. For two sets $\mathscr{P}, \mathscr{S} \subseteq \mathbb{R}^n$, define $\mathscr{P}_{\mathscr{S}} := \mathscr{P} \cap \mathscr{S}$. For a sequence $\mathbf{w} := \{w(l)\}_{l\in\mathbb{Z}_+}$ with $w(l) \in \mathbb{R}^n$, $l \in \mathbb{Z}_+$, let $\|\mathbf{w}\| := \sup\{\|w(l)\| \mid l \in \mathbb{Z}_+\}$ and let $\mathbf{w}_{[k]} := \{w(l)\}_{l\in\mathbb{Z}_{[0,k]}}$. For a set $\mathscr{S} \subseteq \mathbb{R}^n$, we denote by $\operatorname{int}(\mathscr{S})$ the interior of $\mathscr{S}$. For two arbitrary sets $\mathscr{S} \subseteq \mathbb{R}^n$ and $\mathscr{P} \subseteq \mathbb{R}^n$, let $\mathscr{S} \oplus \mathscr{P} := \{x+y \mid x \in \mathscr{S}, y \in \mathscr{P}\}$ denote their Minkowski sum and let $\mathscr{S} \ominus \mathscr{P} := \{x \in \mathbb{R}^n \mid x + \mathscr{P} \subseteq \mathscr{S}\}$ denote their Pontryagin difference. Let $\|\cdot\|$ denote an arbitrary p-norm. For a matrix $Z \in \mathbb{R}^{m\times n}$ let $\|Z\| := \sup_{x\neq 0} \frac{\|Zx\|}{\|x\|}$ denote its corresponding induced matrix norm.

A real valued scalar function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(s) > 0$ for all $s \neq 0$ is called a positive function. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathscr{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathscr{K}_\infty$ if $\varphi \in \mathscr{K}$ and $\lim_{s\to\infty} \varphi(s) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathscr{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot,k) \in \mathscr{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s,\cdot)$ is decreasing and $\lim_{k\to\infty} \beta(s,k) = 0$.

III. UNIFORM CONTINUITY SUBTLETIES

This section presents certain properties of uniformly continuous functions (maps) that will be exploited to prove the main results.

Definition III.1 A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is called uniformly continuous on $\mathbb{X} \subseteq \mathbb{R}^n$ (or shortly, UC($\mathbb{X}$)) if there exists a positive function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $\varepsilon \in \mathbb{R}_{>0}$ and all $(x,y) \in \mathbb{X}^2 := \mathbb{X} \times \mathbb{X}$ with $\|x-y\| \leq \delta(\varepsilon)$ it holds that $\|f(x) - f(y)\| \leq \varepsilon$. If f is UC($\mathbb{R}^n$), then f is called globally uniformly continuous (GUC). If f is GUC and, moreover, the function δ can be chosen to satisfy $\lim_{\varepsilon\to\infty} \delta(\varepsilon) = \infty$, f is called unbounded GUC.

Definition III.2 A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is called Hölder continuous on $\mathbb{X} \subseteq \mathbb{R}^n$ (or shortly, HC($\mathbb{X}$)) if there exist $a \in \mathbb{R}_{>0}$, $\alpha \in \mathbb{R}_{>0}$ such that $\|f(x) - f(y)\| \leq a\|x-y\|^\alpha$ for all $(x,y) \in \mathbb{X}^2$. If f is HC($\mathbb{R}^n$), then f is called globally Hölder continuous (GHC). If $\alpha = 1$, then f is called Lipschitz continuous.

Fact III.3 A function $f : \mathbb{X} \to \mathbb{R}$ with $\mathbb{X} \subseteq \mathbb{R}^n$ that is UC($\mathbb{X}$) is continuous on $\mathbb{X}$. □

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Fact III.4 *Heine-Cantor Theorem.* Let $\mathbb{X} \subset \mathbb{R}^n$ be a compact set and let $f : \mathbb{X} \to \mathbb{R}$ denote a continuous function on $\mathbb{X}$. Then f is UC($\mathbb{X}$). □

Fact III.5 Let $A, B \subset \mathbb{R}^n$ be arbitrary compact sets and let $f : A \to \mathbb{R}$ denote a continuous function on A. Also, let $f(A) := \{f(x) \mid x \in A\}$. Then $f(A)$ and $A \oplus B$ are compact sets. □

Fact III.6 *Bolzano-Weierstrass Theorem.* Let $\mathbb{X} \subset \mathbb{R}^n$ be a compact set and let $f : \mathbb{X} \to \mathbb{R}$ denote a continuous function on $\mathbb{X}$. Then f attains its minimum and maximum on $\mathbb{X}$. □

Definition III.7 A map $f : \mathbb{X} \to \mathbb{R}^m$ is called $\mathscr{K}$-continuous on $\mathbb{X} \subset \mathbb{R}^n$ (or shortly, KC($\mathbb{X}$)) if there exists a function $\varphi \in \mathscr{K}$ such that $\|f(x) - f(y)\| \leq \varphi(\|x - y\|)$ for all $(x, y) \in \mathbb{X}^2$.

Next, we introduce a global version of the $\mathscr{K}$-continuity property.

Definition III.8 A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is called $\mathscr{K}_\infty$-continuous (or shortly, KIC) if there exists a function $\varphi \in \mathscr{K}_\infty$ such that $\|f(x) - f(y)\| \leq \varphi(\|x - y\|)$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Notice that the set of HC (GHC) functions is a subset of KC (KIC) functions. The following results will be instrumental in proving the main results of this note.

Lemma III.9 *Let $\mathbb{X} \subset \mathbb{R}^n$ be a compact set. A function $f : \mathbb{X} \to \mathbb{R}$ is UC($\mathbb{X}$)* if and only if *it is $\mathscr{K}$-continuous on $\mathbb{X}$.*

A proof of the above result can be found in [4], see Lemma A.2 of the $\mathscr{K}$-continuity Appendix. An alternative proof of Lemma III.9, which includes an explicit construction of the $\mathscr{K}$-function involved in Definition III.7, is given in Appendix A of this note.

Corollary III.10 *Let $\mathbb{X} \subset \mathbb{R}^n$ be a compact set. A function $f : \mathbb{X} \to \mathbb{R}$ is continuous on $\mathbb{X}$* if and only if *it is $\mathscr{K}$-continuous on $\mathbb{X}$.*

Proof: The claim follows from Lemma III.9 in combination with Fact III.3 and Fact III.4. ■

Lemma III.11 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is unbounded GUC* if and only if *it is $\mathscr{K}_\infty$-continuous.*

The proof of Lemma III.11, which includes an explicit construction of the $\mathscr{K}_\infty$-function involved in Definition III.8, is given in Appendix B of this note.

Corollary III.12 *Every $\mathscr{K}_\infty$-continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is globally uniformly continuous.*

Proof: The claim follows from Lemma III.11 and the fact that every unbounded GUC function is a GUC function. ■

The fact that not every globally uniformly continuous function is $\mathscr{K}_\infty$-continuous is consistent with the observation (see Example A.3 of [4]) that not every uniformly continuous function is $\mathscr{K}$-continuous, i.e., when the domain space is not restricted to compact subsets of $\mathbb{R}^n$.

Figure 1, subfigure a), summarizes the relations between continuity (C), uniform continuity (UC), $\mathscr{K}$-continuity (KC) and Hölder continuity (HC) on compact subsets of $\mathbb{R}^n$; subfigure b) of the same figure summarizes the relations between unbounded global uniform continuity (uGUC), global uniform continuity (GUC), $\mathscr{K}_\infty$-continuity (KIC) and global Hölder continuity (GHC).

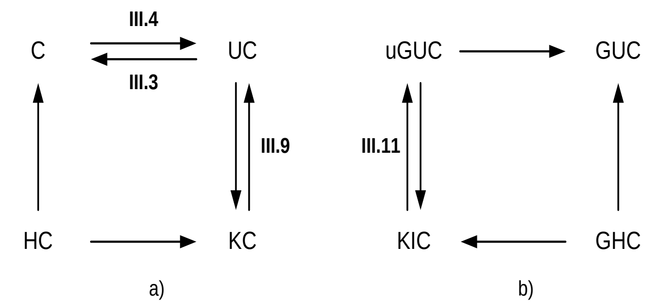

Fig. 1. Summary of uniform continuity results: a) compact subsets of $\mathbb{R}^n$; b) $\mathbb{R}^n$.

IV. INHERENT INPUT-TO-STATE STABILITY

Consider the discrete-time nominal system

$$x(k+1) = \Phi(x(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

and its perturbed counterpart

$$x(k+1) = \Psi(x(k), e(k), d(k)), \quad k \in \mathbb{Z}_+, \tag{2}$$

where $x : \mathbb{Z}_+ \to \mathbb{R}^{n_x}$ is the state trajectory, $e : \mathbb{Z}_+ \to \mathbb{R}^{n_e}$ is an unknown *inner* perturbation trajectory, $d : \mathbb{Z}_+ \to \mathbb{R}^{n_d}$ is an unknown *outer* perturbation trajectory and $\Phi : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$, $\Psi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_d} \to \mathbb{R}^{n_x}$ are nonlinear maps with $\Psi(x, 0, 0) := \Phi(x)$ for all $x \in \mathbb{R}^{n_x}$ and $\Phi(0) = 0$. The reason for distinguishing between inner and outer perturbations will become clear further on, within this section. For ease of notation we will use x, e and d, to also denote a vector in $\mathbb{R}^{n_x}$, $\mathbb{R}^{n_e}$ and $\mathbb{R}^{n_d}$, respectively. Let $\mathbb{X}$, $\mathbb{E}$ and $\mathbb{D}$ denote subsets of $\mathbb{R}^{n_x}$, $\mathbb{R}^{n_e}$ and $\mathbb{R}^{n_d}$ that contain the origin in their interior.

Definition IV.1 A set $\mathscr{P} \subseteq \mathbb{R}^{n_x}$ with $0 \in \text{int}(\mathscr{P})$ is called a *positively invariant (PI) set* for system (1) if for all $x \in \mathscr{P}$ it holds that $\Phi(x) \in \mathscr{P}$. A set $\mathscr{P} \subseteq \mathbb{R}^{n_x}$ with $0 \in \text{int}(\mathscr{P})$ is called a *robustly positively invariant (RPI) set* for system (2) and $(\mathbb{E}, \mathbb{D})$, or shortly, RPI($\mathbb{E}, \mathbb{D}$), if for all $x \in \mathscr{P}$ it holds that $\Psi(x, e, d) \in \mathscr{P}$ for all $(e, d) \in \mathbb{E} \times \mathbb{D}$.

Definition IV.2 We call system (1) asymptotically stable in $\mathbb{X}$, or shortly *AS($\mathbb{X}$)*, if there exists a $\mathscr{KL}$-function β such that for each $x(0) \in \mathbb{X}$ it holds that $\|x(k)\| \leq \beta(\|x(0)\|, k)$ for all $k \in \mathbb{Z}_+$. We call system (1) GAS if it is AS($\mathbb{R}^{n_x}$).

Definition IV.3 We call system (2) *input-to-state stable in $\mathbb{X}$ for inputs in $\mathbb{E}$ and $\mathbb{D}$*, or shortly *ISS($\mathbb{X}, \mathbb{E}, \mathbb{D}$)*, if there exist a $\mathscr{KL}$-function β and $\mathscr{K}$-functions γ_1, γ_2 such that, for each $x(0) \in \mathbb{X}$, all $\mathbf{e} = \{e(l)\}_{l \in \mathbb{Z}_+}$ with $e(l) \in \mathbb{E}$ for all $l \in \mathbb{Z}_+$ and all $\mathbf{d} = \{d(l)\}_{l \in \mathbb{Z}_+}$ with $d(l) \in \mathbb{D}$ for all $l \in \mathbb{Z}_+$, it holds that the corresponding state trajectory of (2) satisfies

$$\|x(k)\| \leq \beta(\|x(0)\|, k) + \gamma_1(\|\mathbf{e}_{[k-1]}\|) + \gamma_2(\|\mathbf{d}_{[k-1]}\|), \quad \forall k \in \mathbb{Z}_{\geq 1}.$$

The system (2) is *globally ISS* if it is ISS($\mathbb{R}^{n_x}$, $\mathbb{R}^{n_e}$, $\mathbb{R}^{n_d}$).

Throughout this article we will employ the following sufficient conditions for analyzing ISS.

Theorem IV.4 *Let $\alpha_1, \alpha_2, \alpha_3 \in \mathscr{K}_\infty$, $\sigma_1, \sigma_2 \in \mathscr{K}$, $\mathbb{X} \subseteq \mathbb{R}^{n_x}$ with $0 \in$ int($\mathbb{X}$). Let $V : \mathbb{X} \to \mathbb{R}_+$ be a function with $V(0) = 0$ and consider the following inequalities:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{3a}$$

$$V(\Psi(x, e, d)) - V(x) \leq -\alpha_3(\|x\|) + \sigma_1(\|e\|) + \sigma_2(\|d\|). \tag{3b}$$

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(i) *If* $\mathbb{X}$ *is a RPI(*$\mathbb{E}$,$\mathbb{D}$*) set for system* (2) *and inequalities* (3) *hold for all* $x \in \mathbb{X}$, $e \in \mathbb{E}$ *and all* $d \in \mathbb{D}$, *then system* (2) *is* ISS($\mathbb{X}$,$\mathbb{E}$,$\mathbb{D}$). *If inequalities* (3) *hold for all* $(x,e,d) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_d}$, *then system* (2) *is* globally ISS.

(ii) *If* $\mathbb{X}$ *is a PI set for system* (1) *and inequalities* (3) *hold for all* $x \in \mathbb{X}$ *(*$x \in \mathbb{R}^{n_x}$*)*, $e \in \mathbb{E} = \{0\}$ *and* $d \in \mathbb{D} = \{0\}$, *then system* (1) *is AS(*$\mathbb{X}$*) (GAS).*

Definition IV.5 A function $V : \mathbb{X} \to \mathbb{R}_+$ that satisfies the hypothesis of Theorem IV.4-(i) for some sets $\mathbb{E},\mathbb{D}$ is called an *ISS Lyapunov function on* $\mathbb{X}$ for system (2), or shortly, an ISS($\mathbb{X}$,$\mathbb{E}$,$\mathbb{D}$) Lyapunov function. An ISS($\mathbb{R}^{n_x}$, $\mathbb{R}^{n_e}$, $\mathbb{R}^{n_d}$) Lyapunov function is called a *global ISS Lyapunov function*.

Definition IV.6 A function $V : \mathbb{X} \to \mathbb{R}_+$ that satisfies the hypothesis of Theorem IV.4-(ii) is called a *Lyapunov function on* $\mathbb{X}$ for system (2). A Lyapunov function on $\mathbb{R}^{n_x}$ is called a *global Lyapunov function*.

The interested reader is referred to [3], [8] for a proof of Theorem IV.4. Notice that in contrast to the continuous-time case, in discrete-time the above sufficient conditions for ISS (GAS) implicitly require the continuity of the system dynamics and the (ISS) Lyapunov function *only at* $x = 0$, as indicated in [8], [12]. However, in what follows we will focus on *continuous* Lyapunov functions. This is a class of Lyapunov functions which is relevant to model predictive control in the case of linear or smooth, nonlinear models, see, e.g., [1], [13] and the references therein. The reader interested in ISS subtleties for discrete-time systems regarding *discontinuous* Lyapunov functions is referred to the previous, related work [2].

For the remainder of this note consider the case when $n_x = n_e = n_d = n$ for some $n \in \mathbb{Z}_{\geq 1}$. Further, suppose that the perturbed system (2) satisfies $\Psi(x,e,d) := \Phi(x+e)+d$ for all $(x,e,d) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, which exposes the difference between inner and outer perturbations. The next result relates existence of a continuous Lyapunov function to inherent ISS for system (1).

Theorem IV.7 *Let* $\mathbb{X}$, $\mathbb{E}$ *and* $\mathbb{D}$ *be compact subsets of* $\mathbb{R}^n$ *with the origin in their interior. Suppose that* $\mathbb{X}$ *is an RPI(*$\mathbb{E}$, $\mathbb{D}$*) set for system* (2). *Furthermore, suppose that system* (1) *admits a continuous Lyapunov function* V *on* $\mathbb{X}\oplus\mathbb{E}$. *Then,* V *is an ISS(*$\mathbb{X}$,$\mathbb{E}$,$\mathbb{D}$*) Lyapunov function for system* (2) *and hence system* (2) *is ISS(*$\mathbb{X}$,$\mathbb{E}$,$\mathbb{D}$*).*

Proof: The hypothesis implies that there exists a continuous function $V : \mathbb{X}\oplus\mathbb{E} \to \mathbb{R}_+$ that satisfies (3a) for all $x \in \mathbb{X}\oplus\mathbb{E}$. Thus, it satisfies (3a) for all $x \in \mathbb{X}$ as well. Next, we prove that V satisfies (3b) for all $(x,e,d) \in \mathbb{X}\times\mathbb{E}\times\mathbb{D}$. Let $\hat{x} := x+e$. As V is a Lyapunov function on $\mathbb{X}\oplus\mathbb{E}$ for system (1), by Definition IV.6 it follows that there exists a $\alpha_3 \in \mathscr{K}_\infty$ such that

$$V(\Phi(\hat{x})) - V(\hat{x}) + \alpha_3(\|\hat{x}\|) \leq 0, \quad \forall \hat{x} \in \mathbb{X}\oplus\mathbb{E}. \tag{4}$$

Since $\mathbb{X}$ is a RPI($\mathbb{E}$,$\mathbb{D}$) set for system (2), from Corollary III.10, Fact III.5, the reverse triangle inequality and using $\hat{x} = x+e$ we also have that there exist functions $\varphi_1, \varphi_2 \in \mathscr{K}$ such that

$$|V(\hat{x}) - V(x)| \leq \varphi_1(\|e\|), \tag{5a}$$
$$|V(\Phi(\hat{x})+d) - V(\Phi(\hat{x}))| \leq \varphi_1(\|d\|), \tag{5b}$$
$$|\alpha_3(\|\hat{x}\|) - \alpha_3(\|x\|)| \leq \varphi_2(\|e\|), \tag{5c}$$

for all $(x,e,d) \in \mathbb{X}\times\mathbb{E}\times\mathbb{D}$. Then, using the fact that $a-b \leq |a-b| = |b-a|$ for any $a,b \in \mathbb{R}$ and adding (5b) and (4) yield

$$V(\Phi(\hat{x})+d) - V(\hat{x}) + \alpha_3(\|\hat{x}\|) - \varphi_1(\|d\|) \leq 0$$

for all $\hat{x} \in \mathbb{X}\oplus\mathbb{E}$ and all $d \in \mathbb{D}$. Adding and subtracting $V(x)$ and $\alpha_3(\|x\|)$ in the above inequality and using (5a) and (5c), respectively, along with the fact that $a-b \leq |a-b| = |b-a|$ for any $a,b \in \mathbb{R}$, yield

$$V(\Phi(x+e)+d) - V(x) \leq -\alpha_3(\|x\|) + \sum_{i=1}^{2} \varphi_i(\|e\|) + \varphi_1(\|d\|),$$

for all $(x,e,d) \in \mathbb{X}\times\mathbb{E}\times\mathbb{D}$. By letting $\sigma_1(s) := \sum_{i=1}^{2} \varphi_i(s) \in \mathscr{K}$ and $\sigma_2(s) := \varphi_2(s) \in \mathscr{K}$ yields that V satisfies (3b) for all $(x,e,d) \in \mathbb{X}\times\mathbb{E}\times\mathbb{D}$. Hence, the claim follows from Theorem IV.4-(i). ■

A global correspondent of Theorem IV.7 is stated next.

Theorem IV.8 *Suppose that system* (1) *admits a global Lyapunov function* V *that satisfies* (3b) *for all* $x \in \mathbb{R}^n$, $e \in \mathbb{E} = \{0\}$ *and* $d \in \mathbb{D} = \{0\}$ *with a* $\alpha_3 \in \mathscr{K}_\infty$. *Moreover, suppose that* V *and* α_3 *are* $\mathscr{K}_\infty$*-continuous functions. Then,* V *is a global ISS Lyapunov function for system* (2) *and hence system* (2) *is globally ISS.*

Proof: The claim follows *via* the reasoning used in the proof of Theorem IV.7, in combination with Definition III.8. ■

Consider next the discrete-time nominal system with a control input

$$x(k+1) = \phi(x(k), u(x(k))), \quad k \in \mathbb{Z}_+, \tag{6}$$

and its perturbed counterpart

$$x(k+1) = \phi(x(k), u(x(k)+e(k))) + d(k), \quad k \in \mathbb{Z}_+, \tag{7}$$

where $u : \mathbb{R}^n \to \mathbb{R}^m$ is a state–feedback control law and $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear map with $\phi(0,0) = 0$. For ease of notation we will use u to also denote a vector in $\mathbb{R}^m$. Let $\mathbb{U}$ be a subset of $\mathbb{R}^m$ with $0 \in \text{int}(\mathbb{U})$.

Definition IV.9 Let $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ be compact sets. A map $\phi : \mathbb{X}\times\mathbb{U} \to \mathbb{R}^n$ is called uniformly $\mathscr{K}$-continuous on $\mathbb{X}$ (i.e., uniformly with respect to u) if there exists a function $\varphi \in \mathscr{K}$ such that $\|\phi(x,u) - \phi(y,u)\| \leq \varphi(\|x-y\|)$ for all $u \in \mathbb{U}$ and all $(x,y) \in \mathbb{X}^2$. A map $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is called uniformly $\mathscr{K}_\infty$-continuous if there exists a function $\varphi \in \mathscr{K}_\infty$ such that $\|\phi(x,u) - \phi(y,u)\| \leq \varphi(\|x-y\|)$ for all $u \in \mathbb{R}^m$ and all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Definition IV.10 Let $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$. Suppose that $u : \mathbb{X} \to \mathbb{U}$ is a known map with $u(0) = 0$. A set $\mathscr{P} \subseteq \mathbb{R}^n$ with $0 \in \text{int}(\mathscr{P})$ is called a PI set for system (6) if for all $x \in \mathscr{P}$ it holds that $\phi(x,u(x)) \in \mathscr{P}$. A set $\mathscr{P} \subseteq \mathbb{R}^n$ with $0 \in \text{int}(\mathscr{P})$ is called a RPI($\mathbb{E}$, $\mathbb{D}$) set for system (7) if for all $x \in \mathscr{P}$ it holds that $\phi(x,u(x+e))+d \in \mathscr{P}$ for all $(e,d) \in \mathbb{E}\times\mathbb{D}$.

Theorem IV.11 *Let* $\mathbb{X}$, $\mathbb{U}$, $\mathbb{E}$ *and* $\mathbb{D}$ *be compact sets with the origin in their interior and let* $u : \mathbb{X}\oplus\mathbb{E} \to \mathbb{U}$ *be a known map with* $u(0) = 0$. *Suppose that* $\mathbb{X}$ *is a RPI(*$\mathbb{E}$, $\mathbb{D}$*) set for system* (7). *Furthermore, suppose that system* (6) *admits a continuous Lyapunov function* V *on* $\mathbb{X}\oplus\mathbb{E}$ *and the map* ϕ *is uniformly* $\mathscr{K}$*-continuous on* $\mathbb{X}\oplus\mathbb{E}$. *Then,* V *is an ISS(*$\mathbb{X}$,$\mathbb{E}$,$\mathbb{D}$*) Lyapunov function for system* (7) *and hence system* (7) *is ISS(*$\mathbb{X}$,$\mathbb{E}$,$\mathbb{D}$*).*

Proof: Let $\hat{x} := x+e$. Observe that for any $x \in \mathbb{X}$ and $e \in \mathbb{E}$, $\hat{x} \in \mathbb{X}\oplus\mathbb{E}$ and thus, $\phi(\hat{x}, u(\hat{x})) \in \mathbb{X}\oplus\mathbb{E}$. The latter property follows since system (6) admits a Lyapunov function on $\mathbb{X}\oplus\mathbb{E}$, which by definition requires that $\mathbb{X}\oplus\mathbb{E}$ is a PI set for system (6). This further implies that there exists a continuous function $V : \mathbb{X}\oplus\mathbb{E} \to \mathbb{R}_+$ such that

$$V(\phi(\hat{x}, u(\hat{x}))) - V(\hat{x}) \leq -\alpha_3(\|\hat{x}\|), \quad \forall \hat{x} \in \mathbb{X}\oplus\mathbb{E}. \tag{8}$$

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From Corollary III.10 and Definition IV.9 it follows that there exist functions $\varphi_1, \varphi_2 \in \mathscr{K}$ such that

$$|V(\phi(\hat{x},u(\hat{x}))) - V(\phi(x,u(\hat{x})))| \leq \varphi_1(\|\phi(\hat{x},u(\hat{x})) - \phi(x,u(\hat{x}))\|) \leq \varphi_1(\varphi_2(\|e\|))$$

for all $(x,e) \in \mathbb{X} \times \mathbb{E}$. Adding and subtracting $V(\phi(x,u(\hat{x}))$ in (8) and using the above inequality along with the fact that $a-b \leq |a-b| = |b-a|$ for any $a,b \in \mathbb{R}$, yield

$$V(\phi(x,u(\hat{x}))) - V(\hat{x}) \leq -\alpha_3(\|\hat{x}\|) + \varphi_1(\varphi_2(\|e\|)).$$

Then, the claim follows *via* the reasoning used in the proof of Theorem IV.7, by considering a perturbed system (2) with $\Psi(x,e,d) := \phi(x,u(x+e)) + d$ for all $(x,e,d) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. ■

Theorem IV.12 *Let $u : \mathbb{R}^n \to \mathbb{R}^m$ be a known map with $u(0) = 0$. Suppose that system* (6) *admits a global Lyapunov function V that satisfies* (3b) *for all $x \in \mathbb{R}^n$, $e \in \mathbb{E} = \{0\}$ and $d \in \mathbb{D} = \{0\}$ with a $\alpha_3 \in \mathscr{K}_\infty$. Moreover, suppose that V and α_3 are $\mathscr{K}_\infty$-continuous functions and ϕ is a uniformly $\mathscr{K}_\infty$-continuous map. Then, V is a global ISS Lyapunov function for system* (7) *and hence system* (7) *is globally ISS.*

Proof: The claim follows *via* the reasoning used in the proof of Theorem IV.11 in combination with the reasoning employed in the proof of Theorem IV.8 and Definition IV.9. ■

Remark IV.13 The regional inherent ISS results of Theorem IV.7 and Theorem IV.11 are obtained using a similar reasoning as the one employed in [5] to establish regional inherent robustness (RGAS and SPAS) from a global continuous Lyapunov function. As it is also the case for the corresponding results in [5], the result of Theorem IV.7, which applies to systems without a control input, does not require continuity of the system dynamics and the result of Theorem IV.11, which applies to systems with a control input, does not require continuity of the state–feedback control law. The same holds for the global ISS results of Theorem IV.8 and Theorem IV.12, respectively, which do not a have a correspondent in [5]. Notice that if $\mathbb{E} = \{0\}$, then the continuity assumptions on ϕ can be removed in Theorem IV.11 and Theorem IV.12. □

Remark IV.14 The assumption that V is a Lyapunov function on $\mathbb{X} \oplus \mathbb{E}$, which is part of the hypothesis of Theorem IV.7 and Theorem IV.11, can be relaxed to the assumption that V is a Lyapunov function on $\mathbb{X}$. Then, following a similar reasoning, ISS($\mathbb{X} \ominus \mathbb{E}$,$\mathbb{E}$,$\mathbb{D}$) is obtained for the corresponding system. If additionally $\mathbb{X} \ominus \mathbb{E}$ is RPI($\mathbb{E}$,$\mathbb{D}$), it further holds that V is an ISS($\mathbb{X} \ominus \mathbb{E}$,$\mathbb{E}$,$\mathbb{D}$) Lyapunov function. Alternatively, under the assumption that V is a Lyapunov function on $\mathbb{X}$, if the system dynamics (in the autonomous case) and the state–feedback control law (in the non–autonomous case) is continuous, the results of [11] can be employed to translate the inner perturbation into an outer perturbation, at the price of a more conservative bound on the outer perturbation, i.e., the corresponding set $\mathbb{D}$. □

Remark IV.15 The robustness analysis framework provided in this note applies to a general class of Lyapunov function candidates, as indicated in what follows. Firstly, it is important to stress that existence of a *continuous* Lyapunov function is a necessary condition for guaranteeing inherent ISS of nominally stable discrete–time systems. *Example 2* in [2] demonstrates that GAS discrete–time systems that do not admit a continuous Lyapunov function can have zero robustness in terms of ISS. Secondly, the class of $\mathscr{K}$–continuous functions includes convex functions, such as positive definite quadratic or convex piecewise affine functions, and sub–linear functions, such as weighted norms or, in general, Minkowski functions of proper C–sets [14]. Lastly, the class of $\mathscr{K}_\infty$–continuous functions includes globally Lipschitz continuous functions, such as continuous piecewise affine functions, and symmetric sub–linear functions, such as weighted norms or, in general, Minkowski functions of symmetric proper C–sets. In this context, it is also worth to point out that sub–linear functions are less conservative than quadratic functions in terms of Lyapunov function candidates. For example, GAS linear polytopic difference inclusions (PDIs) always admit [15] a sub–linear (polyhedral) Lyapunov function. As such, the developed analysis framework can be used to establish inherent (global) ISS of linear PDIs that are GAS. Nevertheless, GAS linear PDIs do not necessarily admit a quadratic Lyapunov function. □

V. Conclusions

Input-to-state stability analysis of discrete-time systems using continuous Lyapunov functions was considered. The existence of a continuous Lyapunov function was related to inherent input-to-state stability on compact sets with respect to both inner and outer perturbations. For $\mathscr{K}_\infty$-continuous Lyapunov functions it was shown that this result applies to unbounded sets as well.

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References

[1] G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel, "Examples when nonlinear model predictive control is nonrobust," *Automatica*, vol. 40, no. 10, pp. 1729–1738, 2004.
[2] M. Lazar, W. P. M. H. Heemels, and A. R. Teel, "Lyapunov functions, stability and input-to-state stability subtleties for discrete-time discontinuous systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 10, pp. 2421–2425, 2009.
[3] Z.-P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, pp. 857–869, 2001.
[4] R. A. Freeman and P. V. Kokotović, *Robust nonlinear control design*. Birkäuser Boston, 2008, reprint of the 1996 edition.
[5] M. J. Messina, S. E. Tuna, and A. R. Teel, "Discrete-time certainty equivalence output feedback: allowing discontinuous control laws including those from model predictive control," *Automatica*, vol. 41, pp. 617–628, 2005.
[6] D. Limon, T. Alamo, F. Salas, and E. F. Camacho, "Input to state stability of min-max MPC controllers for nonlinear systems with bounded uncertainties," *Automatica*, vol. 42, pp. 797–803, 2006.
[7] L. Magni, D. M. Raimondo, and R. Scattolini, "Regional input-to-state stability for nonlinear model predictive control," *IEEE Transactions on Automatic Control*, vol. 51, no. 9, pp. 1548–1553, 2006.
[8] M. Lazar, D. Muñoz de la Peña, W. P. M. H. Heemels, and T. Alamo, "On input-to-state stability of min-max nonlinear model predictive control," *Systems & Control Letters*, vol. 57, pp. 39–48, 2008.
[9] M. Lazar and W. P. M. H. Heemels, "Predictive control of hybrid systems: Input-to-state stability results for sub-optimal solutions," *Automatica*, vol. 45, no. 1, pp. 180–185, 2009.
[10] A. Angeli, E. D. Sontag, and Y. Wang, "A characterization of integral input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 45, no. 6, pp. 1082–1097, 2000.

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[11] B. J. P. Roset, W. P. M. H. Heemels, M. Lazar, and H. Nijmeijer, "On robustness of constrained discrete-time systems to state measurement errors," *Automatica*, vol. 44, no. 4, pp. 1161–1165, 2008.
[12] M. Lazar, W. P. M. H. Heemels, S. Weiland, and A. Bemporad, "Stabilizing model predictive control of hybrid systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 11, pp. 1813–1818, 2006.
[13] G. Pannocchia, J. B. Rawlings, and S. J. Wright, "Conditions under which suboptimal nonlinear MPC is inherently robust," *Systems & Control Letters*, vol. 60, no. 9, pp. 747–755, 2011.
[14] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*. Cambridge University Press, 1993.
[15] A. P. Molchanov and E. S. Pyatnitskii, "Criteria of asymptotic stability of differential and difference inclusions encountered in control theory," *Systems & Control Letters*, vol. 13, pp. 59–64, 1989.

Appendix

A. Proof of Lemma III.9

Let us begin with the *only if* part. Let $M_x := \sup_{x\in\mathbb{X}} \|x\|$ and $M_f := \sup_{x\in\mathbb{X}} |f(x)|$, where the supremum is an attainable maximum by continuity of the norm and Fact III.6, and uniform continuity of f on $\mathbb{X}$, Fact III.3 and Fact III.6, respectively. As f is UC($\mathbb{X}$), without any loss of generality we can take $\delta : \mathbb{R}_{(0,2M_f]} \to \mathbb{R}_{>0}$ to be a positive, non-decreasing function. Let $\delta^* := \delta(2M_f) > 0$. Next, let

$$\bar{\delta}(\varepsilon) := \begin{cases} \delta(\varepsilon), & \varepsilon \in \mathbb{R}_{(0,2M_f]}, \\ \delta^* + \varepsilon - 2M_f, & \varepsilon \in \mathbb{R}_{(2M_f,2(M_x+M_f)-\delta^*]}. \end{cases}$$

Notice that $\delta^* \le 2M_x$, as one can always pick δ such that $\delta(2M_f) \le 2M_x$. Furthermore, if $\delta^* = 2M_x$, then $\bar{\delta}(\varepsilon) = \delta(\varepsilon)$ for all $\varepsilon \in \mathbb{R}_{(0,2(M_x+M_f)-\delta^*]}$ and $\bar{\delta}(2(M_x+M_f)-\delta^*) = 2M_x$. Observe that the function $\bar{\delta} : \mathbb{R}_{(0,2(M_x+M_f)-\delta^*]} \to \mathbb{R}_{(0,2M_x]}$ is non-decreasing and it extends the domain of δ when $\delta^* < 2M_x$. Next, we prove that there exists a continuous, strictly increasing function $\rho : \mathbb{R}_{[0,2(M_x+M_f)-\delta^*]} \to \mathbb{R}_+$ with $\rho(0) = 0$ and such that $\rho(\varepsilon) \le \bar{\delta}(\varepsilon)$ for all $\varepsilon \in \mathbb{R}_{(0,2(M_x+M_f)-\delta^*]}$. Define

$$s_k := \inf\{\varepsilon \in \mathbb{R}_{(0,2(M_x+M_f)-\delta^*]} \mid \bar{\delta}(\varepsilon) \ge 2M_x 0.5^k\}, \quad \forall k \in \mathbb{Z}_+. \quad (9)$$

Then, define

$$\rho(\varepsilon) := M_x \left(0.5^k + \frac{0.5^k}{s_k - s_{k+1}}(\varepsilon - s_{k+1})\right), \quad (10)$$

for all $\varepsilon \in \mathbb{R}_{[s_{k+1},s_k)}$ and all $k \in \mathbb{Z}_+$, $\rho(0) := 0$. Observe that $\lim_{k\to\infty} s_k = 0$, which implies that ρ is continuous at zero. As $\lim_{\varepsilon\downarrow s_k} \rho(\varepsilon) = \lim_{\varepsilon\uparrow s_k} \rho(\varepsilon) = M_x 0.5^{k-1}$ for all $k \in \mathbb{Z}_{\ge 1}$, $\rho(s_0) = 2M_x$ and $s_0 = 2(M_x+M_f)-\delta^*$, it follows that ρ is continuous on $\mathbb{R}_{[0,2(M_x+M_f)-\delta^*]}$. Next, observing that $\rho(s_k) = M_x 0.5^{k-1} = 2M_x 0.5^k = 2\rho(s_{k+1})$ for all $k \in \mathbb{Z}_+$ yields that ρ is strictly increasing. Hence, the constructed function $\rho : \mathbb{R}_{[0,2(M_x+M_f)-\delta^*]} \to \mathbb{R}_+$ satisfies the desired properties and, from (9) and (10) it follows that $\rho(\varepsilon) \le \bar{\delta}(\varepsilon)$ for all $\varepsilon \in \mathbb{R}_{(0,2(M_x+M_f)-\delta^*]}$. As $\rho(2(M_x+M_f)-\delta^*) = 2M_x$, it holds that the function $\rho^{-1} : \mathbb{R}_{[0,2M_x]} \to \mathbb{R}_{[0,2(M_x+M_f)-\delta^*]}$ is continuous, strictly increasing and $\rho^{-1}(0) = 0$. As $\|x-y\| \le 2M_x$ for any $(x,y) \in \mathbb{X}^2$, we can define $q := \|x-y\|$ and $w := \rho^{-1}(q)$. Since f is UC($\mathbb{X}$) it follows that for all $(x,y) \in \mathbb{X}^2$, $\|x-y\| = q = \rho(w) \le \delta(w)$, which implies that

$$|f(x) - f(y)| \le w = \rho^{-1}(\|x-y\|).$$

It is then straightforward to construct a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\varphi(s) = \rho^{-1}(s)$ for all $s \in \mathbb{R}_{[0,2M_x]}$ and $\varphi \in \mathscr{K}$, which completes the *only if* part of the proof.

The *if* part of the proof proceeds as follows. Let $\varepsilon > 0$ and take $\delta(\varepsilon) := \varphi^{-1}(\min\{\varepsilon, 2M_f\})$. Then, by Definition III.7, for all $(x,y) \in \mathbb{X}^2$ with $\|x-y\| \le \delta(\varepsilon)$ it holds that

$$|f(x) - f(y)| \le \varphi(\|x-y\|) \le \varphi(\delta(\varepsilon)) = \min\{\varepsilon, 2M_f\} \le \varepsilon.$$

B. Proof of Lemma III.11

The claim for the *only if* part follows *mutatis mutandis* by applying the reasoning used in the proof of Lemma III.9. The difference is that δ can be chosen so as to satisfy $\lim_{\varepsilon\to\infty} \delta(\varepsilon) = \infty$ and, as such, it suffices to construct a $\mathscr{K}_\infty$ function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho(\varepsilon) \le \delta(\varepsilon)$ for all $\varepsilon \ge 0$. It is straightforward to verify that the function

$$\rho(\varepsilon) := 0.5^k + \frac{0.5^k}{s_k - s_{k+1}}(\varepsilon - s_{k+1}), \quad \forall \varepsilon \in \mathbb{R}_{[s_{k+1},s_k)}, \quad \forall k \in \mathbb{Z}$$

and $\rho(0) := 0$, where

$$s_k := \inf\{\varepsilon \in \mathbb{R}_{>0} \mid \delta(\varepsilon) \ge 0.5^k\}, \quad \forall k \in \mathbb{Z}$$

satisfies the desired properties. Similarly, for the *if* part it suffices to observe that $\varphi^{-1} \in \mathscr{K}_\infty$.

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