

# Global input-to-state stability and stabilization of discrete-time piecewise affine systems

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## Abstract

This article presents conditions for global input-to-state stability (ISS) and stabilization of discrete-time, possibly discontinuous, piecewise affine (PWA) systems. Piecewise quadratic, possibly discontinuous candidate ISS Lyapunov functions are employed for both analysis and synthesis purposes. This enables us to obtain sufficient conditions based on linear matrix inequalities, which can be solved efficiently. One of the advantages of using the ISS framework is that additive disturbance inputs are explicitly taken into account in the analysis and synthesis procedures. Furthermore, the results apply to PWA systems in their full generality, i.e. non-zero affine terms are allowed in the regions in the partition whose closure contains the origin.

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## 1. Introduction

Discrete-time piecewise affine (PWA) systems form a powerful modeling class for the approximation of hybrid and nonlinear dynamics [1,2]. They also arise from the interconnection of discrete-time linear systems and automata [3]. The modeling capability of discrete-time PWA systems has already been shown in several applications, including switched power converters [4], direct torque control of three-phase induction motors [5], applications in automotive systems [6] and systems biology [7,8]. Therefore, there is an increasing interest in developing efficient tools for stability analysis and stabilizing controller synthesis for discrete-time PWA systems, as it is illustrated by several articles on this topic, see, for example, [9–12], to mention just a few. These works employ the Lyapunov stability framework and typically consider piecewise quadratic (PWQ) candidate Lyapunov functions.

Recently, in [13]<sup>1</sup> the authors demonstrated that exponentially stable discrete-time PWA systems may have no robustness. More precisely, they showed that the exponential stability property cannot prevent that *arbitrarily small* additive disturbances keep the state trajectory far from the origin. Mathematically speaking, this means that the system is not input-to-state stable (ISS) [14] with respect to arbitrarily small disturbances. The non-robustness property is

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<sup>1</sup> For completeness, an example from [13] is presented in Section 2.

related to the absence of a continuous Lyapunov function. As asymptotically stable closed-loop systems are always affected by perturbations in practice, apparently it is crucial that disturbances are taken into account when analyzing the stability of PWA systems. This strongly motivates the study of robust stability.

Only few results on robust stability are available in the literature for discrete-time hybrid systems. Robust stability in terms of  $l_2$ -gain analysis of discrete-time PWA systems was investigated in [10], based on linear matrix inequalities (LMI), while in [15], it was observed that if a robustly positively invariant set can be calculated for an asymptotically stable PWA system, then *local* ultimate boundedness is ensured. It is also worth mentioning that the robust stability results for discrete-time linear parameter varying systems presented in [16,17] could be used in combination with the asymptotic stability results of [12] to derive robust stability results for discrete-time *switched linear systems*. For local input-to-state stabilization of constrained discrete-time PWA systems using model predictive control we refer the interested reader to [18,19]. In case the constraints are absent, the results in [18,19] can result in global ISS results. For ISS results for *continuous-time* switched systems and hybrid systems we refer the reader to the recent works [20–22] based on *smooth* Lyapunov functions and to [23,24] based on multiple (ISS) Lyapunov functions. However, a global robust stability analysis methodology for *discrete-time PWA systems* that can be used for both analysis and synthesis purposes seems to be missing from the literature.

This motivates the current paper that considers discrete-time PWA systems subject to unbounded additive disturbance inputs and uses the ISS framework [14,25] to obtain global robust stability results. For simplicity and clarity of exposition, only PWQ candidate ISS Lyapunov functions are considered, but the results can be extended *mutatis mutandis* to piecewise polynomial or piecewise affine candidate functions. The paper consists of two parts: the first part deals with ISS analysis, while the second part provides techniques for the synthesis of input-to-state stabilizing controllers. In both sections the sufficient conditions for ISS are expressed in terms of LMIs, which can be solved efficiently [26].

One of the advantages of using the ISS framework for studying robust stability of discrete-time PWA systems is that additive disturbance inputs are explicitly taken into account in the analysis and synthesis procedures. Also, the ISS framework enables us to obtain robust stability results for PWA systems in their full generality, i.e. non-zero affine terms are allowed in the regions in the state-space partition whose closure contains the origin. This situation is often excluded in other works. Notice that in this paper we develop a new LMI technique for dealing with non-zero affine terms, which does not rely on a system transformation and the  $S$ -procedure, as, for instance, in [10]. The technique presented here leads to LMI-based sufficient conditions for both ISS analysis and synthesis of ISS controllers.

### 1.1. Notation and basic definitions

Let  $\mathbb{R}$ ,  $\mathbb{R}_+$ ,  $\mathbb{Z}$  and  $\mathbb{Z}_+$  denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. We use the notation  $\mathbb{Z}_{\geq c_1}$  and  $\mathbb{Z}_{(c_1, c_2]}$  to denote the sets  $\{k \in \mathbb{Z}_+ \mid k \geq c_1\}$  and  $\{k \in \mathbb{Z}_+ \mid c_1 < k \leq c_2\}$ , respectively, for some  $c_1, c_2 \in \mathbb{Z}_+$ . We denote by  $\|\cdot\|$  the Euclidean norm. For a sequence  $\{z_p\}_{p \in \mathbb{Z}_+}$  with  $z_p \in \mathbb{R}^l$  let  $\|\{z_p\}_{p \in \mathbb{Z}_+}\| := \sup\{\|z_p\| \mid p \in \mathbb{Z}_+\}$ . For a sequence  $\{z_p\}_{p \in \mathbb{Z}_+}$  with  $z_p \in \mathbb{R}^l$ ,  $z_{[k]}$  denotes the truncation of  $\{z_p\}_{p \in \mathbb{Z}_+}$  at time  $k \in \mathbb{Z}_+$ , i.e.  $z_{[k]} = \{z_p\}_{p \in \mathbb{Z}_{[0, k]}}$ .

For a matrix  $Z \in \mathbb{R}^{m \times n}$  let  $\|Z\| := \sup_{x \neq 0} \frac{\|Zx\|}{\|x\|}$  denote its induced Euclidean norm. For a positive definite matrix  $Z \in \mathbb{R}^{n \times n}$ ,  $\lambda_{\min}(Z)$  and  $\lambda_{\max}(Z)$  denote the smallest and the largest eigenvalue of  $Z$ , respectively.

For a set  $\mathcal{P} \subseteq \mathbb{R}^n$ , we denote by  $\partial\mathcal{P}$  the boundary, by  $\text{int}(\mathcal{P})$  the interior and by  $\text{cl}(\mathcal{P})$  the closure of  $\mathcal{P}$ . A polyhedron (or a polyhedral set) is a set obtained as the intersection of a finite number of open and/or closed half-spaces.

A function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}$  if it is continuous, strictly increasing and  $\varphi(0) = 0$ . A function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}_\infty$  if  $\varphi \in \mathcal{K}$  and it is unbounded (i.e.  $\varphi(s) \rightarrow \infty$  as  $s \rightarrow \infty$ ). A function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{KL}$  if for each fixed  $k \in \mathbb{R}_+$ ,  $\beta(\cdot, k) \in \mathcal{K}$  and for each fixed  $s \in \mathbb{R}_+$ ,  $\beta(s, \cdot)$  is non-increasing and  $\lim_{k \rightarrow \infty} \beta(s, k) = 0$ .

### 1.2. Preliminary results on input-to-state stability

Consider the discrete-time nonlinear system described by

$$x_{k+1} = G(x_k, v_k), \quad k \in \mathbb{Z}_+, \quad (1)$$

where  $x_k \in \mathbb{R}^n$  is the state,  $v_k \in \mathbb{R}^{d_v}$  is an unknown disturbance input and  $G : \mathbb{R}^n \times \mathbb{R}^{d_v} \rightarrow \mathbb{R}^n$  is an arbitrary nonlinear function. Next, we define the notions of input-to-state practical stability (ISpS) [27,28] and input-to-state stability [14,25] for the discrete-time perturbed nonlinear system (1).

**Definition 1.1.** The system (1) is said to be globally ISpS if there exist a  $\mathcal{KL}$ -function  $\beta(\cdot, \cdot)$ , a  $\mathcal{K}$ -function  $\gamma(\cdot)$  and a non-negative constant  $d$  such that, for each  $x_0 \in \mathbb{R}^n$  and all  $\{v_p\}_{p \in \mathbb{Z}_+}$  with  $v_p \in \mathbb{R}^{d_v}$  for all  $p \in \mathbb{Z}_+$ , it holds that the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|) + d, \quad \forall k \in \mathbb{Z}_{\geq 1}. \tag{2}$$

If the above condition holds for  $d = 0$ , the system (1) is said to be globally ISS.

Notice that the global ISS property implies that the origin is an equilibrium in (1) for zero disturbance input, meaning that  $G(0, 0) = 0$ .

In what follows we state a discrete-time version of the continuous-time ISpS sufficient conditions of Proposition 2.1 of [28], and a version of the discrete-time ISS result of [25]. These results will be used throughout the paper to establish ISpS and ISS for the particular case of PWA systems. A proof is provided for completeness.

**Theorem 1.2.** Let  $d_1, d_2 \in \mathbb{R}_+$ , let  $a, b, c, \lambda \in \mathbb{R}_{>0}$  with  $c \leq b$  and let  $\alpha_1(s) := as^\lambda$ ,  $\alpha_2(s) := bs^\lambda$ ,  $\alpha_3(s) := cs^\lambda$  and  $\sigma \in \mathcal{K}$ . Furthermore, let  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  be a function such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) + d_1 \tag{3a}$$

$$V(G(x, v)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|) + d_2 \tag{3b}$$

for all  $x \in \mathbb{R}^n$  and all  $v \in \mathbb{R}^{d_v}$ . Then it holds that:

(i) The system (1) is globally ISpS. Moreover, the ISpS property (2) of Definition 1.1 holds for

$$\beta(s, k) := \alpha_1^{-1}(3\rho^k \alpha_2(s)), \quad \gamma(s) := \alpha_1^{-1}\left(\frac{3\sigma(s)}{1-\rho}\right), \quad d := \alpha_1^{-1}(3\xi), \tag{4}$$

where  $\xi := d_1 + \frac{d_2}{1-\rho}$  and  $\rho := 1 - \frac{c}{b} \in [0, 1)$ .

(ii) If inequalities (3) hold for  $d_1 = d_2 = 0$ , the system (1) is globally ISS. Moreover, the ISS property (2) of Definition 1.1 (i.e. when  $d = 0$ ) holds for

$$\beta(s, k) := \alpha_1^{-1}(2\rho^k \alpha_2(s)), \quad \gamma(s) := \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right), \tag{5}$$

where  $\rho := 1 - \frac{c}{b} \in [0, 1)$ .

**Proof.** (i) From  $V(x) \leq \alpha_2(\|x\|) + d_1$  for all  $x \in \mathbb{R}^n$ , we have that for any  $x \in \mathbb{R}^n \setminus \{0\}$  it holds:

$$V(x) - \alpha_3(\|x\|) \leq V(x) - \frac{\alpha_3(\|x\|)}{\alpha_2(\|x\|)}(V(x) - d_1) = \rho V(x) + (1 - \rho)d_1,$$

where  $\rho := 1 - \frac{c}{b} \in [0, 1)$ . In fact, the previous inequality holds for all  $x \in \mathbb{R}^n$ , since  $V(0) - \alpha_3(0) = V(0) = \rho V(0) + (1 - \rho)V(0) \leq \rho V(0) + (1 - \rho)d_1$ . Then, inequality (3b) becomes

$$V(G(x, v)) \leq \rho V(x) + \sigma(\|v\|) + (1 - \rho)d_1 + d_2, \tag{6}$$

for all  $x \in \mathbb{R}^n$  and all  $v \in \mathbb{R}^{d_v}$ . Applying inequality (6) repetitively yields:

$$V(x_{k+1}) \leq \rho^{k+1} V(x_0) + \sum_{i=0}^k \rho^i [\sigma(\|v_{k-i}\|) + (1 - \rho)d_1 + d_2]$$

for all  $x_0 \in \mathbb{R}^n$ ,  $v_{[k]} = \{v_0, v_1, \dots, v_k\} \in \{\mathbb{R}^{d_v}\}^{k+1}$ ,  $k \in \mathbb{Z}_+$ . Here,  $v_{[k]}$  is the truncation of some corresponding disturbance sequence. Then, taking (3a) into account and using the property  $\sigma(\|v_i\|) \leq \sigma(\|v_{[k]}\|)$  for all  $i \leq k$  and

the identity  $\sum_{i=0}^k \rho^i = \frac{1-\rho^{k+1}}{1-\rho}$ , the following inequalities hold:

$$\begin{aligned} V(x_{k+1}) &\leq \rho^{k+1} \alpha_2(\|x_0\|) + \rho^{k+1} d_1 + \sum_{i=0}^k \rho^i [\sigma(\|v_{k-i}\|) + (1-\rho)d_1 + d_2] \\ &\leq \rho^{k+1} \alpha_2(\|x_0\|) + \rho^{k+1} d_1 + [\sigma(\|v_{[k]}\|) + (1-\rho)d_1 + d_2] \sum_{i=0}^k \rho^i \\ &= \rho^{k+1} \alpha_2(\|x_0\|) + \frac{1-\rho^{k+1}}{1-\rho} \sigma(\|v_{[k]}\|) + d_1 + \frac{1-\rho^{k+1}}{1-\rho} d_2 \\ &\leq \rho^{k+1} \alpha_2(\|x_0\|) + \frac{1}{1-\rho} \sigma(\|v_{[k]}\|) + d_1 + \frac{1}{1-\rho} d_2, \end{aligned}$$

for all  $x_0 \in \mathbb{R}^n$ ,  $v_{[k]} \in \{\mathbb{R}^{d_v}\}^{k+1}$ ,  $k \in \mathbb{Z}_+$ . Let  $\xi := d_1 + \frac{d_2}{1-\rho}$ . Taking (3a) into account and letting  $\alpha_1^{-1}$  denote the inverse of  $\alpha_1$ , we obtain:

$$\|x_{k+1}\| \leq \alpha_1^{-1}(V(x_{k+1})) \leq \alpha_1^{-1} \left( \rho^{k+1} \alpha_2(\|x_0\|) + \xi + \frac{\sigma(\|v_{[k]}\|)}{1-\rho} \right). \quad (7)$$

Applying the following inequality (notice that  $\alpha_1^{-1}$  is a  $\mathcal{K}_\infty$  function as well),

$$\alpha_1^{-1}(z + y + s) \leq \alpha_1^{-1}(3 \max(z, y, s)) \leq \alpha_1^{-1}(3z) + \alpha_1^{-1}(3y) + \alpha_1^{-1}(3s), \quad (8)$$

we obtain from (7)

$$\|x_{k+1}\| \leq \alpha_1^{-1}(3\rho^{k+1} \alpha_2(\|x_0\|)) + \alpha_1^{-1} \left( 3 \frac{\sigma(\|v_{[k]}\|)}{1-\rho} \right) + \alpha_1^{-1}(3\xi),$$

for all  $x_0 \in \mathbb{R}^n$ ,  $v_{[k]} \in \{\mathbb{R}^{d_v}\}^{k+1}$ ,  $k \in \mathbb{Z}_+$ .

We distinguish between two cases:  $\rho \neq 0$  and  $\rho = 0$ . First, suppose that  $\rho \in (0, 1)$  and let  $\beta(s, k) := \alpha_1^{-1}(3\rho^k \alpha_2(s))$ . For a fixed  $k \in \mathbb{Z}_+$ , we have that  $\beta(\cdot, k) \in \mathcal{K}$  due to  $\alpha_2 \in \mathcal{K}_\infty$ ,  $\alpha_1^{-1} \in \mathcal{K}_\infty$  and  $\rho \in (0, 1)$ . For a fixed  $s$ , it follows that  $\beta(s, \cdot)$  is non-increasing and  $\lim_{k \rightarrow \infty} \beta(s, k) = 0$ , due to  $\rho \in (0, 1)$  and  $\alpha_1^{-1} \in \mathcal{K}_\infty$ . Thus, it follows that  $\beta \in \mathcal{KL}$ .

Now let  $\gamma(s) := \alpha_1^{-1} \left( \frac{3\sigma(s)}{1-\rho} \right)$ . Since  $\frac{1}{1-\rho} > 0$ , it follows that  $\gamma \in \mathcal{K}$  due to  $\alpha_1^{-1} \in \mathcal{K}_\infty$  and  $\sigma \in \mathcal{K}$ .

Finally, let  $d := \alpha_1^{-1}(3\xi)$ . Since  $\rho \in (0, 1)$  and  $d_1, d_2 \geq 0$ , we have that  $\xi \geq 0$  and thus,  $d \geq 0$ .

Otherwise, if  $\rho = 0$  we have from (7) that

$$\begin{aligned} \|x_k\| &\leq \alpha_1^{-1}(3\sigma(\|v_{[k-1]}\|)) + \alpha_1^{-1}(3\xi) \\ &\leq \beta(\|x_0\|, k) + \alpha_1^{-1}(3\sigma(\|v_{[k-1]}\|)) + \alpha_1^{-1}(3\xi) \end{aligned}$$

for any  $\beta \in \mathcal{KL}$  and  $k \in \mathbb{Z}_{\geq 1}$ .

Hence, the perturbed system (1) is globally ISpS and property (2) is satisfied with the functions given in (4).

(ii) Following the proof of statement (i), it is straightforward to observe that when the sufficient conditions (3) are satisfied for  $d_1 = d_2 = 0$ , then global ISS is achieved, since  $d = \alpha_1^{-1}(3\xi) = \alpha_1^{-1}(0) = 0$ . From (7) and (8), it can be easily shown that global ISS actually holds with the functions given in (5).  $\square$

**Definition 1.3.** A function  $V(\cdot)$  that satisfies the hypothesis of Theorem 1.2 is called an *ISpS (ISS) Lyapunov function*. A function  $V(\cdot)$  that satisfies the hypothesis of Theorem 1.2, part (ii), only for  $v = 0$  is called a *Lyapunov function*.

**Remark 1.4.** While ISS results commonly require continuity of the system dynamics  $G(\cdot, \cdot)$  and smoothness of the ISS Lyapunov function  $V(\cdot)$ , the hypothesis of Theorem 1.2 allows for both  $G(\cdot, \cdot)$  and  $V(\cdot)$  to be discontinuous. If inequality (3a) holds for  $d_1 = 0$ , then the hypothesis of Theorem 1.2 *only* implies continuity at the point  $x = 0$ , and *not* necessarily on a neighborhood of  $x = 0$ . For the continuous-time case, results that replace smoothness of  $V(\cdot)$  by piecewise smoothness have been recently presented in [24].

## 2. A motivating example and problem statement

In the remainder of this article we focus on perturbed discrete-time, possibly discontinuous, PWA systems of the form

$$x_{k+1} = G(x_k, v_k) := A_j x_k + f_j + D_j v_k \quad \text{if } x_k \in \Omega_j, \tag{9}$$

where  $A_j \in \mathbb{R}^{n \times n}$ ,  $f_j \in \mathbb{R}^n$ ,  $D_j \in \mathbb{R}^{n \times d_v}$  for all  $j \in \mathcal{S}$  and  $\mathcal{S} := \{1, 2, \dots, s\}$  is a finite set of indices. The collection  $\{\Omega_j \mid j \in \mathcal{S}\}$  consists of (not necessarily closed) polyhedra that define a partition of  $\mathbb{R}^n$ , meaning that  $\cup_{j \in \mathcal{S}} \Omega_j = \mathbb{R}^n$ ,  $\Omega_i \cap \Omega_j = \emptyset$  for  $i \neq j$  and  $\text{int}(\Omega_j) \neq \emptyset$  for all  $j \in \mathcal{S}$ . Let  $\mathcal{S}_0 := \{j \in \mathcal{S} \mid 0 \in \text{cl}(\Omega_j)\}$ ,  $\mathcal{S}_1 := \{j \in \mathcal{S} \mid 0 \notin \text{cl}(\Omega_j)\}$  and let  $\mathcal{S}_{\text{aff}} := \{j \in \mathcal{S} \mid f_j \neq 0\}$ ,  $\mathcal{S}_{\text{lin}} := \{j \in \mathcal{S} \mid f_j = 0\}$ , so that  $\mathcal{S}_0 \cup \mathcal{S}_1 = \mathcal{S}$  and  $\mathcal{S}_{\text{aff}} \cup \mathcal{S}_{\text{lin}} = \mathcal{S}$ .

To illustrate that globally exponentially stable PWA systems can be non-robust to arbitrarily small perturbations, we briefly recall an example from [13]. Consider the perturbed PWA system given by (9) with  $v_k \in \mathcal{B}_\mu = \{v \in \mathbb{R} \mid |v| \leq \mu\}$  for some small positive parameter  $\mu > 0$ ,  $j \in \mathcal{S} := \{1, 2\}$ ,  $k \in \mathbb{Z}_+$ , and where  $A_1 = A_2 = 0$ ,  $f_1 = 0$ ,  $f_2 = 1$ ,  $D_1 = D_2 = 1$  and the partition is given by  $\Omega_1 = \{x \in \mathbb{R} \mid x \leq 1\}$ ,  $\Omega_2 = \{x \in \mathbb{R} \mid x > 1\}$ .

Suppose that  $v_k = 0$  for all  $k \in \mathbb{Z}_+$ . Then, one can easily observe that any solution  $x_k$  at time  $k \in \mathbb{Z}_+$  of the unperturbed system corresponding to (9) starting from an initial condition  $x_0 \in \mathbb{R}$  satisfies  $|x_k| \leq |x_0|$  (even  $|x_k| < |x_0|$  when  $x_0 \neq 0$ ) and converges exponentially to the origin. Moreover, any trajectory  $x_k$  reaches the origin in 2 discrete time steps or less. Furthermore, it can be proven that  $V(x) := \sum_{i=0}^{\infty} x_i^2$  is a Lyapunov function, where  $x_i$  denotes the solution of the unperturbed system corresponding to (9) at time  $i \in \mathbb{Z}_+$ , obtained from an initial condition  $x_0 := x \in \mathbb{R}$ . More specifically, since  $V(x) = \sum_{i=0}^{\infty} x_i^2 = x_0^2 + x_1^2$  for any  $x_0 = x \in \mathbb{R}$ , it holds that  $V(G(x, 0)) - V(x) \leq -\alpha_3(|x|)$  for all  $x \in \mathbb{R}$ , where  $\alpha_3(s) := s^2$ . An explicit expression for  $V(\cdot)$  is:

$$V(x) = \sum_{i=0}^{\infty} x_i^2 = x_0^2 + x_1^2 = \begin{cases} x^2 + 1 & \text{if } x > 1 \\ x^2 & \text{if } x \leq 1, \end{cases}$$

which shows that  $V(\cdot)$  is discontinuous at  $x = 1$ .

Now consider the case where  $v_k = \mu > 0$  for all  $k \in \mathbb{Z}_+$  in (9). Then, the origin of the perturbed system (9) is not ISS, as  $x_k = 1 + \mu$  is an equilibrium of (9) to which all trajectories with initial conditions  $x_0 \in \mathbb{R}_{>1}$  converge. Hence, no matter how small  $\mu > 0$  is taken, the PWA system (9) is not ISS in  $\mathbb{R}$  for disturbances in  $\mathcal{B}_\mu$ .

The above example shows that globally exponentially stable PWA systems that admit a discontinuous Lyapunov function, which could be found via the tools presented in [9–12], do not necessarily have any robustness when affected by arbitrarily small perturbations. Therefore, the aim of this paper is to derive sufficient conditions for global ISpS and global ISS, respectively, of perturbed PWA systems of the form (9).

Although discontinuous Lyapunov functions may have the disadvantage that they do not transfer exponential stability for the nominal system to ISS for the perturbed system, as demonstrated by the above example (and hence, are not necessarily ISS Lyapunov functions), they still can be used, if properly constructed, to establish ISpS or ISS of perturbed PWA systems (see Remark 1.4). As discontinuous (ISpS or ISS) Lyapunov functions are common in hybrid systems theory via the multiple Lyapunov approach [29] and in particular PWQ functions have computational advantages as they can often be found via linear matrix inequalities, we also adopt PWQ, possibly discontinuous, candidate ISpS (ISS) Lyapunov functions of the form

$$V : \mathbb{R}^n \rightarrow \mathbb{R}_+, \quad V(x) = x^\top P_j x \quad \text{if } x \in \Omega_j, \tag{10}$$

where  $P_j$ ,  $j \in \mathcal{S}$ , are positive definite and symmetric matrices. Observe that  $V(\cdot)$  satisfies condition (3a) with

$$\alpha_1(\|x\|) := \min_{j \in \mathcal{S}} \lambda_{\min}(P_j) \|x\|^2, \quad \alpha_2(\|x\|) := \max_{j \in \mathcal{S}} \lambda_{\max}(P_j) \|x\|^2 \tag{11}$$

and  $d_1 = 0$ .

## 3. Input-to-state stability analysis tools for PWA systems

In this section we present LMI-based sufficient conditions for global ISpS (ISS) of the PWA system (9). Let  $Q$  be a known positive definite and symmetric matrix and let  $\gamma_1, \gamma_2$  be known positive numbers with  $\gamma_1 \gamma_2 > 1$ .

For any  $(j, i) \in \mathcal{S} \times \mathcal{S}$  consider now the LMI

$$\Delta_{ji} := \begin{pmatrix} \Xi_{ji} & -A_j^\top P_i & -A_j^\top P_i \\ -P_i A_j & \gamma_1 P_i & -P_i \\ -P_i A_j & -P_i & \gamma_2 P_i \end{pmatrix} > 0, \quad (12)$$

where

$$\Xi_{ji} := P_j - A_j^\top P_i A_j - E_j^\top U_{ji} E_j - Q - M_{ji}.$$

The matrix  $E_j$ ,  $j \in \mathcal{S}$ , defines the cone  $\mathcal{C}_j := \{x \in \mathbb{R}^n \mid E_j x \geq 0\}$  that is chosen such that  $\Omega_j \subseteq \mathcal{C}_j$ . The role of these matrices is to introduce an  $S$ -procedure relaxation [30]. The unknown variables in (12) are the matrices  $P_j$ ,  $j \in \mathcal{S}$ , which are required to be positive definite and symmetric, the matrices  $U_{ji}$ ,  $(j, i) \in \mathcal{S} \times \mathcal{S}$ , which need to have non-negative elements, and the matrices  $M_{ji}$ ,  $(j, i) \in \mathcal{S}_{\text{aff}} \times \mathcal{S}$ , which are required to be positive definite and symmetric. For all  $(j, i) \in \mathcal{S}_{\text{lin}} \times \mathcal{S}$  we take  $M_{ji} = 0$ . For any  $(j, i) \in \mathcal{S}_{\text{aff}} \times \mathcal{S}$ , define

$$\mathcal{E}_{ji} := \{x \in \mathbb{R}^n \mid x^\top M_{ji} x < (1 + \gamma_1) f_j^\top P_i f_j\}.$$

**Theorem 3.1.** *Let system (9), the matrix  $Q > 0$  and the numbers  $\gamma_1, \gamma_2 > 0$  with  $\gamma_1 \gamma_2 > 1$  be given. Suppose that the LMIs*

$$\Delta_{ji} > 0, \quad (j, i) \in \mathcal{S} \times \mathcal{S} \quad (13)$$

are feasible. Then, it holds that:

- (i) The system (9) is globally ISpS;
- (ii) If  $^2 (\cup_{i \in \mathcal{S}} \mathcal{E}_{ji}) \cap \Omega_j = \emptyset$  for all  $j \in \mathcal{S}_{\text{aff}}$ , then system (9) is globally ISS;
- (iii) If system (9) is piecewise linear (PWL), i.e.  $\mathcal{S}_{\text{lin}} = \mathcal{S}$ , then system (9) is globally ISS.

**Proof.** The proof consists in showing that  $V(\cdot)$ , as defined in (10), is an ISpS (ISS) Lyapunov function.

(i) By the hypothesis  $\Delta_{ji} > 0$  for all  $(j, i) \in \mathcal{S} \times \mathcal{S}$  it follows that:

$$\begin{pmatrix} x^\top & f_j^\top & (D_j v)^\top \end{pmatrix} \Delta_{ji} \begin{pmatrix} x \\ f_j \\ D_j v \end{pmatrix} \geq 0, \quad \forall x \in \Omega_j, \forall (j, i) \in \mathcal{S} \times \mathcal{S}, \forall v \in \mathbb{R}^{d_v}.$$

The above inequality yields for all  $x \in \Omega_j$ :

$$\begin{aligned} & (A_j x + f_j + D_j v)^\top P_i (A_j x + f_j + D_j v) - x^\top P_j x \\ & \leq -x^\top Q x + (1 + \gamma_2) (D_j v)^\top P_i (D_j v) - x^\top E_j^\top U_{ji} E_j x + (1 + \gamma_1) f_j^\top P_i f_j - x^\top M_{ji} x \\ & \leq -\lambda_{\min}(Q) \|x\|^2 + (1 + \gamma_2) \max_{i \in \mathcal{S}} \lambda_{\max}(P_i) \max_{j \in \mathcal{S}} \|D_j\|^2 \|v\|^2 + (1 + \gamma_1) \max_{i \in \mathcal{S}} \lambda_{\max}(P_i) \max_{j \in \mathcal{S}} \|f_j\|^2. \end{aligned} \quad (14)$$

Hence,

$$V(A_j x + f_j + D_j v) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|) + d_2$$

for all  $x \in \Omega_j$ ,  $(j, i) \in \mathcal{S} \times \mathcal{S}$  and all  $v \in \mathbb{R}^{d_v}$ , where

$$\begin{aligned} \alpha_3(\|x\|) & := \lambda_{\min}(Q) \|x\|^2, \\ \sigma(\|v\|) & := (1 + \gamma_2) \max_{i \in \mathcal{S}} \lambda_{\max}(P_i) \max_{j \in \mathcal{S}} \|D_j\|^2 \|v\|^2, \\ d_2 & := (1 + \gamma_1) \max_{i \in \mathcal{S}} \lambda_{\max}(P_i) \max_{j \in \mathcal{S}} \|f_j\|^2. \end{aligned}$$

Notice that  $\Delta_{ji} > 0$  for all  $(j, i) \in \mathcal{S} \times \mathcal{S}$  also implies that  $P_i > 0$  for all  $i \in \mathcal{S}$  and thus, (11) holds.

<sup>2</sup> Note that  $(\cup_{i \in \mathcal{S}} \mathcal{E}_{ji}) \cap \Omega_j = \emptyset$  for all  $j \in \mathcal{S}_{\text{aff}}$  implies  $\mathcal{S}_0 \subseteq \mathcal{S}_{\text{lin}}$ .

From (12) we have that for all  $(j, i) \in \mathcal{S} \times \mathcal{S}$ ,

$$\Delta_{ji} > 0 \Rightarrow \Xi_{ji} > 0 \Rightarrow x^\top (P_j - Q)x \geq 0 \quad \text{for all } x \in \Omega_j.$$

Then, it follows that for all  $j \in \mathcal{S}$  and all  $x \in \Omega_j$ :

$$\lambda_{\min}(Q)\|x\|^2 \leq x^\top Qx \leq x^\top P_j x \leq \max_{j \in \mathcal{S}} \lambda_{\max}(P_j)\|x\|^2,$$

which yields  $\lambda_{\min}(Q) =: c \leq b := \max_{j \in \mathcal{S}} \lambda_{\max}(P_j)$ . Hence, the function  $V(\cdot)$  defined in (10) satisfies the hypothesis of Theorem 1.2 with  $d_1 = 0$  and  $d_2 = (1 + \gamma_1) \max_{i \in \mathcal{S}} \lambda_{\max}(P_i) \max_{j \in \mathcal{S}} \|f_j\|^2$ . Then, the statement follows from Theorem 1.2.

(ii) To establish global ISS, we need to prove that in the above setting, we can take  $d_2 = 0$  under the additional hypothesis. Consider the first inequality in (14). For  $j \in \mathcal{S}_{\text{lin}}$ , if  $x \in \Omega_j$  we obtain  $d_2 = 0$  due to  $f_j = 0$  and  $M_{ji} = 0$ . For any  $j \in \mathcal{S}_{\text{aff}}$ , it holds that  $x \in \Omega_j$  implies  $x \notin \cup_{i \in \mathcal{S}} \mathcal{E}_{ji}$ . This yields for any  $j \in \mathcal{S}_{\text{aff}}$  that

$$x \in \Omega_j \Rightarrow (1 + \gamma_1)f_j^\top P_i f_j - x^\top M_{ji}x \leq 0$$

and thus, from the first inequality in (14) it follows that the function  $V(\cdot)$  defined in (10) satisfies the hypothesis of Theorem 1.2 for  $d_1 = d_2 = 0$ . Hence, the statement follows from Theorem 1.2.

(iii) This is a special case of part (ii).  $\square$

The matrix  $Q$  gives the gain of the  $\mathcal{K}$ -function  $\alpha_3(\cdot)$  and is related to the decrease of the state norm, and hence, to the transient behavior (see the proof of Theorem 1.2 and the role of  $\rho$ ). If ISpS (ISS) is the only goal,  $Q$  can be chosen “less” positive definite to reduce conservatism of (13). The numbers  $\gamma_1, \gamma_2$  and the matrices  $\{P_j \mid j \in \mathcal{S}\}$  yield the constant  $d_2 = (1 + \gamma_1) \max_{i \in \mathcal{S}} \lambda_{\max}(P_i) \max_{j \in \mathcal{S}} \|f_j\|^2$  and the gain of the  $\mathcal{K}$ -function  $\sigma(\|v\|) = (1 + \gamma_2) \max_{i \in \mathcal{S}} \lambda_{\max}(P_i) \max_{j \in \mathcal{S}} \|D_j\|^2 \|v\|^2$ , which influence the functions  $\beta(\cdot, \cdot), \gamma(\cdot)$  and the constant  $d$  defined in (4).

**Remark 3.2.** A necessary condition for feasibility of the LMI (13) is  $\gamma_1 \gamma_2 > 1$ . As it would be desirable to obtain a constant  $d_2$  and gain of the function  $\sigma(\cdot)$  as small as possible, one has to make a trade-off in choosing  $\gamma_1$  and  $\gamma_2$ . One could add a cost criterion to (13) and specify  $\gamma_1, \gamma_2$  as unknown variables in the resulting optimization problem, which might solve the trade-off. However, if  $\gamma_1$  and  $\gamma_2$  are not fixed, (13) is a bilinear matrix inequality (due to  $\gamma_1 P_i, \gamma_2 P_i$ ). Since the unknowns  $\gamma_1, \gamma_2$  are scalars, this problem can still be solved efficiently via semi-definite programming solvers (software), e.g. [31,32], by setting lower and upper bounds for  $\gamma_1, \gamma_2$  and doing bisections.

**Remark 3.3.** The LMI conditions (13) are independent of the matrices  $D_j$  and  $f_j, j \in \mathcal{S}$ . This indicates that feasibility of the LMIs (13) implies that all systems (9) with matrices  $A_j, j \in \mathcal{S}$  as used in (13) are ISpS. Only the particular value of the constant  $d_2 = (1 + \gamma_1) \max_{i \in \mathcal{S}} \lambda_{\max}(P_i) \max_{j \in \mathcal{S}} \|f_j\|^2$  and the gain of the  $\mathcal{K}$ -function  $\sigma(\|v\|) = (1 + \gamma_2) \max_{i \in \mathcal{S}} \lambda_{\max}(P_i) \max_{j \in \mathcal{S}} \|D_j\|^2 \|v\|^2$  depend on the particular values of  $f_j$  and  $D_j, j \in \mathcal{S}$ . If certain additional conditions are satisfied by  $f_j$  and  $D_j, j \in \mathcal{S}$  as formulated in statements (ii) or (iii) of Theorem 3.1, then one even obtains ISS. Hence, in summary, the LMI conditions (13) guarantee ISpS for a whole class of perturbed PWA systems with the same system matrices  $A_j, j \in \mathcal{S}$  and, when additional properties in terms of the matrices  $D_j$  or  $f_j, j \in \mathcal{S}$  are fulfilled, then certain gains  $\sigma(\cdot)$  or constants  $d_2$  are obtained or even ISS can be derived.

**Remark 3.4.** Since we take  $M_{ji} = 0$  for all  $(j, i) \in \mathcal{S}_{\text{lin}} \times \mathcal{S}$ , the LMI-based sufficient conditions for ISS (12) recover the LMI-based sufficient conditions for asymptotic stability presented in [10] (for PWL systems) in the absence of disturbances (i.e. when  $v_k = 0$  for all  $k \in \mathbb{Z}_+$ ).

**Remark 3.5.** If the disturbance inputs are bounded, which is a reasonable assumption in practice, ISpS implies, by definition, ultimate boundedness (see also [18]). This means that Theorem 3.1 part (i) implies ultimate boundedness, a result that was also obtained in [15] via a different route. However, the result of Theorem 3.1 part (i) applies to a more general class of PWA systems than the one considered in [15], since we allow for non-zero affine terms in regions  $\Omega_j$  with  $0 \in \text{cl}(\Omega_j)$ .

#### 4. Synthesis of input-to-state stabilizing controllers for PWA systems

In this section we address the problem of synthesizing input-to-state stabilizing controllers for perturbed discrete-time non-autonomous PWA systems:

$$x_{k+1} = g(x_k, u_k, v_k) := A_j x_k + B_j u_k + f_j + D_j v_k \quad \text{if } x_k \in \Omega_j, \quad (15)$$

where  $u_k \in \mathbb{R}^m$  is the input and  $B_j \in \mathbb{R}^{n \times m}$  for all  $j \in \mathcal{S}$ . The notation in (15) is similar with the one used in Section 3 for system (9).

In this section we take the control input as a PWL state-feedback control law of the form:

$$u_k = h(x_k) := K_j x_k \quad \text{if } x_k \in \Omega_j, \quad (16)$$

where  $K_j \in \mathbb{R}^{m \times n}$  for all  $j \in \mathcal{S}$ . The aim is to calculate the feedback gains  $\{K_j \mid j \in \mathcal{S}\}$  such that the PWA closed-loop system (15) and (16) is globally ISpS and ISS, respectively. For this purpose we make use again of PWQ candidate ISpS (ISS) Lyapunov functions of the form (10).

For any  $(j, i) \in \mathcal{S} \times \mathcal{S}$ , consider now the following LMI:

$$\Delta_{ji} := \begin{pmatrix} \Delta_{ji}^{11} & \Delta_{ji}^{12} \\ \Delta_{ji}^{21} & \Delta_{ji}^{22} \end{pmatrix} > 0, \quad (17)$$

where

$$\Delta_{ji}^{11} := \begin{pmatrix} Z_j & -(A_j Z_j + B_j Y_j)^\top & -(A_j Z_j + B_j Y_j)^\top \\ -(A_j Z_j + B_j Y_j) & \gamma_1 Z_i & -Z_i \\ -(A_j Z_j + B_j Y_j) & -Z_i & \gamma_2 Z_i \end{pmatrix}$$

and, for  $j \in \mathcal{S}_{\text{aff}}$

$$\Delta_{ji}^{22} := \text{diag} \left( \left[ \begin{pmatrix} Z_i & 0 & 0 \\ 0 & Z_i & 0 \\ 0 & 0 & Z_i \end{pmatrix}, \begin{pmatrix} Q^{-1} & 0 & 0 \\ 0 & Q^{-1} & 0 \\ 0 & 0 & Q^{-1} \end{pmatrix}, \begin{pmatrix} N_{ji} & 0 & 0 \\ 0 & N_{ji} & 0 \\ 0 & 0 & N_{ji} \end{pmatrix} \right] \right)$$

$$\Delta_{ji}^{12} = \Delta_{ji}^{21\top} := \begin{pmatrix} (A_j Z_j + B_j Y_j)^\top & 0 & 0 & Z_j & 0 & 0 & Z_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

while for  $j \in \mathcal{S}_{\text{lin}}$ ,

$$\Delta_{ji}^{22} := \text{diag} \left( \left[ \begin{pmatrix} Z_i & 0 & 0 \\ 0 & Z_i & 0 \\ 0 & 0 & Z_i \end{pmatrix}, \begin{pmatrix} Q^{-1} & 0 & 0 \\ 0 & Q^{-1} & 0 \\ 0 & 0 & Q^{-1} \end{pmatrix} \right] \right)$$

$$\Delta_{ji}^{12} = \Delta_{ji}^{21\top} := \begin{pmatrix} (A_j Z_j + B_j Y_j)^\top & 0 & 0 & Z_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the above definitions,  $\text{diag}([L_1, \dots, L_n])$  denotes a diagonal matrix of appropriate dimensions with the matrices  $L_1, \dots, L_n$  on the main diagonal, and 0 denotes everywhere a matrix of appropriate dimensions with all the elements zero.

The unknown variables in (17) are the matrices  $Z_j \in \mathbb{R}^{n \times n}$ ,  $j \in \mathcal{S}$ , which are required to be positive definite and symmetric, the matrices  $Y_j \in \mathbb{R}^{m \times n}$ ,  $j \in \mathcal{S}$ , and the matrices  $N_{ji}$ ,  $(j, i) \in \mathcal{S}_{\text{aff}} \times \mathcal{S}$ , which are required to be positive definite and symmetric. The matrix  $Q$  is a known positive definite and symmetric matrix and the numbers  $\gamma_1, \gamma_2 > 0$  with  $\gamma_1 \gamma_2 > 1$  play the same role as discussed in Section 3.

For any  $(j, i) \in \mathcal{S}_{\text{aff}} \times \mathcal{S}$ , define

$$\mathcal{E}_{ji} := \{x \in \mathbb{R}^n \mid x^\top N_{ji}^{-1} x < (1 + \gamma_1) f_j^\top P_i f_j\}.$$



**Theorem 4.1.** Let system (15), the matrix  $Q > 0$  and the numbers  $\gamma_1, \gamma_2 > 0$  with  $\gamma_1\gamma_2 > 1$  be given. Suppose that the LMIs

$$\Delta_{ji} > 0, \quad (j, i) \in \mathcal{S} \times \mathcal{S} \tag{18}$$

are feasible and let  $\{Z_j, Y_j \mid j \in \mathcal{S}\}$  and  $\{N_{ji} \mid (j, i) \in \mathcal{S}_{\text{aff}} \times \mathcal{S}\}$  be a solution. For all  $j \in \mathcal{S}$  let  $P_j := Z_j^{-1}$  and let  $K_j := Y_j Z_j^{-1}$ . For all  $(j, i) \in \mathcal{S}_{\text{lin}} \times \mathcal{S}$  take  $M_{ji} = 0$ . For all  $(j, i) \in \mathcal{S}_{\text{aff}} \times \mathcal{S}$  take  $M_{ji} = N_{ji}^{-1}$ . Then, it holds that:

- (i) The closed-loop system (15) and (16) is globally ISpS;
- (ii) If  $(\cup_{i \in \mathcal{S}} \mathcal{E}_{ji}) \cap \Omega_j = \emptyset$  for all  $j \in \mathcal{S}_{\text{aff}}$ , then the closed-loop system (15) and (16) is globally ISS;
- (iii) If system (15) is PWL, i.e.  $\mathcal{S}_{\text{lin}} = \mathcal{S}$ , then the closed-loop system (15) and (16) is globally ISS.

**Proof.** By applying the Schur complement [26] to (18), for any  $(j, i) \in \mathcal{S} \times \mathcal{S}$  we obtain

$$\Delta_{ji}^{11} - \Delta_{ji}^{21\top} \Delta_{ji}^{22^{-1}} \Delta_{ji}^{21} > 0,$$

which yields the equivalent matrix inequality:

$$\Phi_{ji} := \begin{pmatrix} \Gamma_{ji} & -(A_j Z_j + B_j Y_j)^\top & -(A_j Z_j + B_j Y_j)^\top \\ -(A_j Z_j + B_j Y_j) & \gamma_1 Z_i & -Z_i \\ -(A_j Z_j + B_j Y_j) & -Z_i & \gamma_2 Z_i \end{pmatrix} > 0 \tag{19}$$

where

$$\Gamma_{ji} := Z_j - (A_j Z_j + B_j Y_j)^\top Z_i^{-1} (A_j Z_j + B_j Y_j) - Z_j Q Z_j - Z_j N_{ji}^{-1} Z_j.$$

By pre- and post-multiplying (19) with  $\begin{pmatrix} Z_j^{-1} & 0 & 0 \\ 0 & Z_i^{-1} & 0 \\ 0 & 0 & Z_i^{-1} \end{pmatrix}$  and by replacing  $Z_j^{-1}$  by  $P_j$ ,  $Y_j Z_j^{-1}$  by  $K_j$  and  $N_{ji}^{-1}$  by  $M_{ji}$  turns inequality (19) into the equivalent matrix inequality:

$$\begin{pmatrix} \Xi_{ji} & -(A_j + B_j K_j)^\top P_i & -(A_j + B_j K_j)^\top P_i \\ -P_i (A_j + B_j K_j) & \gamma_1 P_i & -P_i \\ -P_i (A_j + B_j K_j) & -P_i & \gamma_2 P_i \end{pmatrix} > 0,$$

for all  $(j, i) \in \mathcal{S} \times \mathcal{S}$ , where

$$\Xi_{ji} := P_j - (A_j + B_j K_j)^\top P_i (A_j + B_j K_j) - Q - M_{ji}.$$

Then, it follows that the LMI (13) is feasible for the closed-loop system (15) and (16) for all  $(j, i) \in \mathcal{S} \times \mathcal{S}$ . The proof now follows from Theorem 3.1.  $\square$

Note that as in the case of ISS analysis, it can be proven that if the hypothesis of Theorem 4.1 part (i) is satisfied, then the closed-loop system (15) and (16) is ultimately bounded in the presence of bounded disturbances.

**Remark 4.2.** Instead of PWL state feedbacks in (16) we could also have used PWA state feedbacks, i.e.  $u_k = K_j x_k + k_j$  when  $x_k \in \Omega_j$ . Note that this replaces the affine terms  $f_j$  in the closed-loop system by  $f_j + B_j k_j$ . In view of Remark 3.3 feasibility of the LMIs (18) does not depend on  $D_j$  or  $f_j + B_j k_j$ ,  $j \in \mathcal{S}$ . However, the affine terms  $k_j$  in the state feedback could be used subsequently, to minimize the constant  $d_2 = (1 + \gamma_1) \max_{i \in \mathcal{S}} \lambda_{\max}(P_i) \max_{j \in \mathcal{S}} \|f_j + B_j k_j\|^2$  (reducing the size of the ultimate bound) or make sure that the conditions (ii) or (iii) in Theorem 4.1 are satisfied and obtain ISS (instead of ISpS).

### 5. Illustrative examples

In this section we illustrate the theoretical results presented in Sections 3 and 4 by means of two examples.

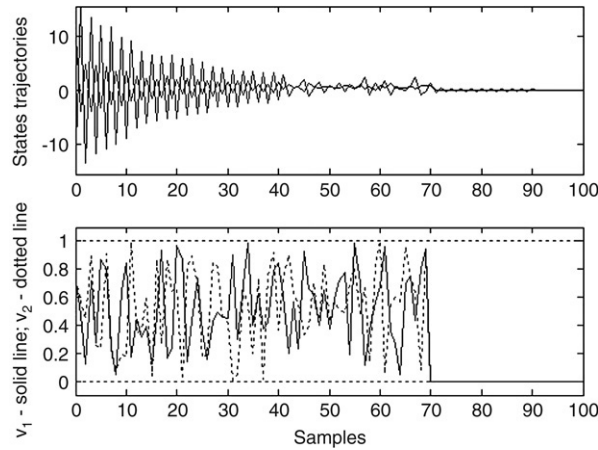


Fig. 1. States trajectories and disturbances histories for system (20).

5.1. Example 1

In this example we illustrate the  $S$ -procedure relaxation and the result of Theorem 3.1 part (iii). Consider the following perturbed PWL system:

$$x_{k+1} = \begin{cases} A_1 x_k + v_k & \text{if } E_1 x_k > 0 \\ A_2 x_k + v_k & \text{if } E_2 x_k \geq 0 \\ A_3 x_k + v_k & \text{if } E_3 x_k > 0 \\ A_4 x_k + v_k & \text{if } E_4 x_k \geq 0 \ \& \ x_k \neq 0, \end{cases} \tag{20}$$

where all inequalities hold componentwise,  $A_1 = \begin{bmatrix} 0.5 & 0.61 \\ 0.9 & 1.345 \end{bmatrix}$ ,  $A_2 = \begin{bmatrix} -0.92 & 0.644 \\ 0.758 & -0.71 \end{bmatrix}$ ,  $A_3 = A_1$  and  $A_4 = A_2$ . The state-space partition of system (20) is given by the matrices  $E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$  and  $E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ .

Searching for a common quadratic or a PWQ *without* the  $S$ -relaxation ISS Lyapunov function did not succeed for system (20). However, by solving the LMI (12) for  $Q = 10^{-4}I_2$ ,  $\gamma_1 = 100$  and  $\gamma_2 = 11$  we obtained the following PWQ with an  $S$ -relaxation ISS Lyapunov function  $V(x) = x^T P_j x$  if  $x \in \Omega_j$ ,  $j = 1, 2, 3, 4$ :

$$\begin{aligned} P_1 &= \begin{bmatrix} 0.1845 & 0.0494 \\ 0.0494 & 0.0335 \end{bmatrix}, & P_2 &= \begin{bmatrix} 0.0851 & -0.0110 \\ -0.0110 & 0.0336 \end{bmatrix}, \\ P_3 &= P_1, & P_4 &= P_2, \\ U_{11} &= \begin{bmatrix} 0.0119 & 0.0519 \\ 0.0519 & 0.0223 \end{bmatrix}, & U_{12} &= \begin{bmatrix} 0.0120 & 0.0540 \\ 0.0540 & 0.0053 \end{bmatrix}, \\ U_{21} &= \begin{bmatrix} 0.0035 & 0.0048 \\ 0.0048 & 0.0041 \end{bmatrix}, & U_{22} &= 10^{-3} \begin{bmatrix} 0.1185 & 0.2265 \\ 0.2265 & 0.3749 \end{bmatrix}. \end{aligned}$$

States trajectories for system (20) with initial state  $x_0 = [-10 \ 10]^T$  are plotted in Fig. 1 together with the additive disturbance inputs history. The disturbance inputs were randomly generated in the interval  $[0, 1]$  until the sampling time 70 and then set equal to zero. The gain of the function

$$\sigma(\|v\|) = (1 + \gamma_2) \max_{j=1,2,3,4} \lambda_{\max}(P_j) \|v\|^2$$

corresponding to  $\gamma_2 = 11$  is 2.3911. This yields an ISS gain equal to 15.4243 for system (20) via the relation  $\gamma(s) := \alpha_1^{-1} \left( \frac{2\sigma(s)}{1-\rho} \right)$  established in (5). As guaranteed by Theorem 3.1, system (20) is globally ISS, which ensures asymptotic stability when the disturbance inputs converge to zero (convergence of the state to zero can be observed in Fig. 1).

5.2. Example 2

In this example we illustrate the result of Theorem 4.1 part (ii). Let

$$A(T_s) := \begin{pmatrix} 1 & T_s & \frac{T_s^2}{2!} & \frac{T_s^3}{3!} \\ 0 & 1 & T_s & \frac{T_s^2}{2!} \\ 0 & 0 & 1 & T_s \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B(T_s) := \begin{pmatrix} \frac{T_s^4}{4!} \\ \frac{T_s^3}{3!} \\ \frac{T_s^2}{2!} \\ \frac{T_s}{1!} \end{pmatrix}$$

denote the dynamics corresponding to a discrete-time quadruple integrator, i.e.  $x_{k+1} = A(T_s)x_k + B(T_s)u_k$ , obtained from a continuous-time quadruple integrator via a sampled-and-hold device with sampling period  $T_s > 0$ . Let  $x_i$ ,  $i = 1, 2, 3, 4$ , denote the  $i$ th component of the state vector. Let  $\Omega_1 := \{x \in \mathbb{R}^4 \mid x_4 \geq 2\}$ ,  $\Omega_4 := \{x \in \mathbb{R}^4 \mid x_4 \leq -2\}$ ,  $\Omega_2 := \{x \in \mathbb{R}^4 \mid 2 > x_4 \geq 0\}$  and let  $\Omega_3 := \{x \in \mathbb{R}^4 \mid -2 < x_4 < 0\}$ . Consider now the following perturbed PWA system:

$$x_{k+1} = \begin{cases} A_1x_k + B_1u_k + f_1 + D_1v_k & \text{if } x_k \in \Omega_1 \\ A_2x_k + B_2u_k + f_2 + D_2v_k & \text{if } x_k \in \Omega_2 \\ A_3x_k + B_3u_k + f_3 + D_3v_k & \text{if } x_k \in \Omega_3 \\ A_4x_k + B_4u_k + f_4 + D_4v_k & \text{if } x_k \in \Omega_4, \end{cases} \tag{21}$$

where

$$\begin{aligned} A_1 = A_4 = A(1.2), \quad B_1 = B_4 = B(1.2), \\ A_2 = A(0.9), \quad B_2 = B(0.9), \quad A_3 = A(0.8), \quad B_3 = B(0.8), \\ f_2 = f_3 = 0, \quad f_1 = f_4 = [0.1 \ 0.1 \ 0.1 \ 0.1]^\top, \quad D_1 = D_2 = D_3 = D_4 = [1 \ 1 \ 1 \ 1]^\top. \end{aligned}$$

The LMIs (18) were solved<sup>3</sup> for  $Q = 0.01I_4$ ,  $\gamma_1 = 2$  and  $\gamma_2 = 4$ , yielding the following weights of the PWQ ISS Lyapunov function  $V(x) = x^\top P_j x$  if  $x \in \Omega_j$ ,  $j = 1, 2, 3, 4$ , feedbacks  $\{K_j \mid j = 1, 2, 3, 4\}$  and matrix  $M$ :

$$\begin{aligned} P_1 = P_4 &= \begin{bmatrix} 0.3866 & 0.7019 & 0.5532 & 0.1903 \\ 0.7019 & 1.5632 & 1.3131 & 0.4688 \\ 0.5532 & 1.3131 & 1.2255 & 0.4552 \\ 0.1903 & 0.4688 & 0.4552 & 0.1955 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} 0.3574 & 0.6052 & 0.4420 & 0.1407 \\ 0.6052 & 1.2725 & 0.9894 & 0.3278 \\ 0.4420 & 0.9894 & 0.8812 & 0.3046 \\ 0.1407 & 0.3278 & 0.3046 & 0.1328 \end{bmatrix}, \\ P_3 &= \begin{bmatrix} 0.3779 & 0.6410 & 0.4597 & 0.1453 \\ 0.6410 & 1.3414 & 1.0298 & 0.3390 \\ 0.4597 & 1.0298 & 0.9007 & 0.3118 \\ 0.1453 & 0.3390 & 0.3118 & 0.1334 \end{bmatrix}, \\ K_1 = K_4 &= [-0.3393 \quad -1.1789 \quad -1.8520 \quad -1.7028], \\ K_2 &= [-0.5584 \quad -1.7607 \quad -2.4729 \quad -2.0012], \\ K_3 &= [-0.6814 \quad -2.0895 \quad -2.8249 \quad -2.1705], \end{aligned}$$

<sup>3</sup> For simplicity we used a common matrix  $N$  for all possible mode transitions that can occur when the state is in mode 1 (region  $\Omega_1$ ) or mode 4 (region  $\Omega_4$ ), i.e.  $N = N_{11} = N_{12} = N_{13} = N_{14} = N_{44} = N_{41} = N_{42} = N_{43}$ , which yields  $M = N^{-1}$ .

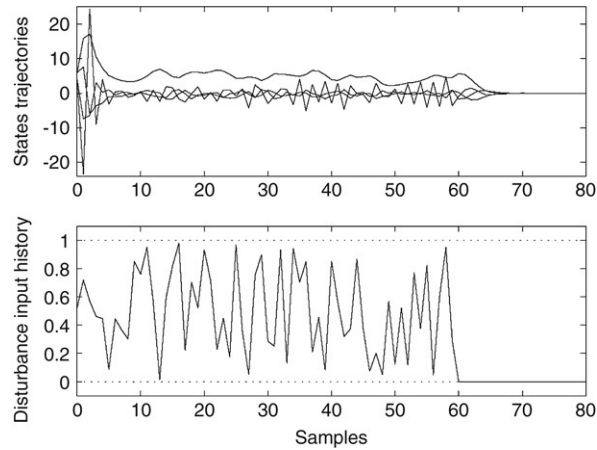


Fig. 2. States trajectories and disturbance histories for the closed-loop system (21)–(16).

$$M = \begin{bmatrix} 0.0156 & 0.0075 & 0.0023 & 0.0005 \\ 0.0075 & 0.0212 & 0.0082 & 0.0016 \\ 0.0023 & 0.0082 & 0.0146 & 0.0044 \\ 0.0005 & 0.0016 & 0.0044 & 0.0081 \end{bmatrix}.$$

One can easily establish that the hypothesis of [Theorem 4.1](#) part (ii) is satisfied, i.e.  $\mathcal{E}_i \cap \Omega_1 = \emptyset$  and  $\mathcal{E}_{4i} \cap \Omega_4 = \emptyset$  for all  $i = 1, 2, 3, 4$ , by observing that

$$\begin{aligned} \min_{x \in \Omega_1} x^\top M x &= \min_{x \in \Omega_4} x^\top M x \\ &= 0.4340 > 0.3221 = \max_{i=1,2,3,4} (1 + \gamma_1) f_1^\top P_i f_1 = \max_{i=1,2,3,4} (1 + \gamma_1) f_4^\top P_i f_4. \end{aligned}$$

Hence, system (21) in closed-loop with (16) is globally ISS. The gain of the function

$$\sigma(\|v\|) = (1 + \gamma_2) \max_{j=1,2,3,4} \lambda_{\max}(P_j) \max_{j=1,2,3,4} \|D_j\|^2 \|v\|^2,$$

corresponding to  $\gamma_2 = 4$  is 15.8772. This yields an ISS gain equal to 42.52 for system (21)–(16) via the relation  $\gamma(s) = \alpha_1^{-1} \left( \frac{2\sigma(s)}{1-\rho} \right) = 42.52s$  established in (5). The closed-loop state trajectories obtained for initial state  $x_0 = [6 \ 6 \ 4 \ 4]^\top$  are plotted in [Fig. 2](#) together with the additive disturbance input history. The disturbance input was randomly generated in the interval  $[0, 1]$  until the sampling time 60 and then set equal to zero. As guaranteed by [Theorem 4.1](#), the closed-loop system (21)–(16) is globally ISS, which ensures asymptotic stability when the disturbance inputs converges to zero (convergence of the state to zero can be observed in [Fig. 2](#)).

## 6. Concluding remarks

In this paper we presented LMI-based sufficient conditions for global input-to-state stability and stabilization of discrete-time perturbed, possibly discontinuous, PWA systems. These results are important as nominally exponentially stable discrete-time PWA systems can be non-robust to *arbitrarily small* additive disturbances [13]. This indicates that special precautions with respect to robustness must be taken when designing stabilizing controllers for PWA systems that will be implemented in practice. The developed methodology has a wide applicability, including the class of PWA systems and any hybrid system that can be transformed into an equivalent PWA form [2], e.g. mixed logical dynamical systems or linear complementarity systems.

State and input constraints have not been considered in order to obtain global ISS results. However, the usual LMI techniques [26] for specifying state and/or input constraints can be added to the sufficient conditions presented in this paper, resulting in local ISS results for constrained PWA systems.

For simplicity and clarity of exposition we employed PWQ (with an  $S$ -procedure relaxation for analysis) candidate ISS Lyapunov functions of the form (10). However, the results can be extended to piecewise polynomial or piecewise

affine candidate ISS Lyapunov functions. Furthermore, an extension to PWA systems affected by both parametric uncertainties and additive disturbances is possible.

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## References

- [1] E.D. Sontag, Nonlinear regulation: The piecewise linear approach, *IEEE Transactions on Automatic Control* 26 (2) (1981) 346–357.
- [2] W.P.M.H. Heemels, B. De Schutter, A. Bemporad, Equivalence of hybrid dynamical models, *Automatica* 37 (2001) 1085–1091.
- [3] E.D. Sontag, Interconnected automata and linear systems: A theoretical framework in discrete-time, in: R. Alur, T.A. Henzinger, E.D. Sontag (Eds.), *Hybrid Systems III: Verification and Control*, Springer, New York, 1996, pp. 436–448.
- [4] D.M.W. Leenaerts, Further extensions to Chua’s explicit piecewise linear function descriptions, *International Journal of Circuit Theory and Applications* 24 (1996) 621–633.
- [5] T. Geyer, G. Papafotiou, M. Morari, Model predictive control in power electronics: A hybrid systems approach, in: *IEEE Conference on Decision and Control*, Seville, Spain, 2005.
- [6] A. Bemporad, P. Borodani, M. Mannelli, Hybrid control of an automotive robotized gearbox for reduction of consumptions and emissions, in: *Hybrid Systems: Computation and Control*, in: *Lecture Notes in Computer Science*, vol. 2623, Springer Verlag, 2003, pp. 81–96.
- [7] S. Drulhe, G. Ferrari-Trecate, H. de Jong, A. Viari, Reconstruction of switching thresholds in piecewise-affine models of genetic regulatory networks, in: *Hybrid Systems: Computation and Control*, in: *Lecture Notes in Computer Science*, vol. 3927, Springer Verlag, 2006, pp. 184–199.
- [8] B. Yordanov, C. Belta, G. Batt, Model checking discrete time piecewise affine systems: application to gene networks, in: *European Control Conference*, Kos, Greece, 2007.
- [9] D. Mignone, G. Ferrari-Trecate, M. Morari, Stability and stabilization of piecewise affine and hybrid systems: An LMI approach, in: *39th IEEE Conference on Decision and Control*, 2000, pp. 504–509.
- [10] G. Ferrari-Trecate, F.A. Cuzzola, D. Mignone, M. Morari, Analysis of discrete-time piecewise affine and hybrid systems, *Automatica* 38 (12) (2002) 2139–2146.
- [11] G. Feng, Stability analysis of piecewise discrete-time linear systems, *IEEE Transactions on Automatic Control* 47 (7) (2002) 1108–1112.
- [12] J. Daafouz, P. Riedinger, C. Lung, Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach, *IEEE Transactions on Automatic Control* 47 (2002) 1883–1887.
- [13] M. Lazar, W.P.M.H. Heemels, A.R. Teel, Subtleties in robust stability of discrete-time piecewise affine systems, in: *American Control Conference*, New York City, USA, 2007, pp. 3464–3469.
- [14] E.D. Sontag, Further facts about input to state stabilization, *IEEE Transactions on Automatic Control* AC–35 (1990) 473–476.
- [15] P. Grieder, Efficient computation of feedback controllers for constrained systems, Ph.D. Thesis, Swiss Federal Institute of Technology (ETH) Zürich, Switzerland, 2004.
- [16] G. Millerioux, L. Rosier, G. Bloch, J. Daafouz, Bounded state reconstruction error for LPV systems with estimated parameters, *IEEE Transactions on Automatic Control* 49 (2004) 1385–1389.
- [17] J. Daafouz, G. Millerioux, L. Rosier, Observer design with guaranteed bound for LPV systems, in: *16th IFAC World Congress*, Prague, 2005.
- [18] M. Lazar, Model predictive control of hybrid systems: Stability and robustness, Ph.D. Thesis, Eindhoven University of Technology, The Netherlands, 2006.
- [19] M. Lazar, W.P.M.H. Heemels, A. Bemporad, S. Weiland, Discrete-time non-smooth nonlinear MPC: Stability and robustness, in: *Assessment and Future Directions of Nonlinear Model Predictive Control*, in: *Lecture Notes in Control and Information Sciences*, vol. 358, Springer, 2007, pp. 93–103.
- [20] J.P. Hespanha, D. Liberzon, A.R. Teel, On input-to-state stability of impulsive systems, in: *Proceedings of the 44th IEEE Conference on Decision and Control*, Seville, Spain, 2005, pp. 3992–3997.
- [21] C. Cai, A. Teel, Results on input-to-state stability for hybrid systems, in: *Proceedings of the 44th IEEE Conference on Decision and Control*, Seville, Spain, 2005, pp. 5403–5408.
- [22] D. Liberzon, D. Nesić, Stability analysis of hybrid systems via small-gain theorems, in: *Hybrid Systems: Computation and Control*, in: *Lecture Notes in Computer Science*, vol. 3927, Springer Verlag, Santa Barbara, CA, 2006, pp. 421–435.
- [23] L. Vu, D. Chatterjee, D. Liberzon, ISS of switched systems and applications to switching adaptive control, in: *44th IEEE Conference on Decision and Control*, Seville, Spain, 2005, pp. 120–125.
- [24] W.P.M.H. Heemels, S. Weiland, A. Juloski, Input-to-state stability of discontinuous dynamical systems with an observer-based control application, in: *Hybrid Systems: Computation and Control*, in: *Lecture Notes in Computer Science*, vol. 4416, Springer Verlag, Pisa, Italy, 2007, pp. 259–272.
- [25] Z.-P. Jiang, Y. Wang, Input-to-state stability for discrete-time nonlinear systems, *Automatica* 37 (2001) 857–869.
- [26] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in Control Theory*, in: *Studies in Applied Mathematics*, vol. 15, SIAM, 1994.

- [27] Z.-P. Jiang, Quelques résultats de stabilisation robuste. Application á la commande, Ph.D. Thesis, École des Mines de Paris, France, 1993 (in French).
- [28] Z.-P. Jiang, I. Mareels, Y. Wang, A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems, *Automatica* 32 (8) (1996) 1211–1215.
- [29] M.S. Branicky, V.S. Borkar, S.K. Mitter, A unified framework for hybrid control: Model and optimal control theory, *IEEE Transactions on Automatic Control* 43 (1) (1998) 31–45.
- [30] M. Johansson, A. Rantzer, Computation of piecewise quadratic Lyapunov functions for hybrid systems, *IEEE Transactions on Automatic Control* 43 (4) (1998) 555–559.
- [31] J.F. Sturm, SeDuMi: Matlab toolbox for solving optimization problems over symmetric cones, toolbox available online at: <http://sedumi.mcmaster.ca/>, 2001.
- [32] J. Löfberg, YALMIP: Matlab toolbox for rapid prototyping of optimization problems, web: <http://control.ee.ethz.ch/joloef/yalmip.msql>, 2002.