STABILIZING RECEDING HORIZON CONTROL
OF PIECEWISE LINEAR SYSTEMS: AN LMI APPROACH[1]

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Abstract: Receding horizon control has recently been used for regulating discrete-time Piecewise Affine (PWA) systems. One of the obstructions for implementation consists in guaranteeing closed-loop stability *a priori*. This is an issue that has only been addressed marginally in the literature. In this paper we present an extension of the terminal cost method for guaranteeing stability in receding horizon control to the class of unconstrained Piecewise Linear (PWL) systems. A linear matrix inequalities set-up is developed to calculate the terminal weight matrix and the auxiliary feedback gains that ensure stability for quadratic cost based receding horizon control. It is shown that the PWL state-feedback control law employed in the stability proof globally asymptotically stabilizes the origin of the PWL system. The additional conditions needed to extend these results to constrained PWA systems are also pointed out. The implementation of the proposed method is illustrated by an example.

Keywords: Piecewise linear systems, Piecewise affine systems, Discrete-time systems, Receding horizon control, Stability, Linear matrix inequalities.

1. INTRODUCTION

Recently, research has focused on questions related to the optimal control and stabilization of hybrid systems in general and of Piecewise Affine (PWA) systems in particular, e.g. see (Rantzer and Johansson, 2000; Mignone *et al*., 2000; Bemporad *et al*., 2003; Borrelli *et al*., 2003) and the references therein. This interest in PWA systems is motivated by the fact that the PWA framework can model a broad class of hybrid systems (Sontag, 1996; Heemels *et al*., 2001). Also, PWA models can be obtained through appropriate conversion procedures (Bemporad, 2004) from the broad class of discrete-time hybrid automata that can be modeled in the language HYSDEL (Torrisi and Bemporad, 2004). Extension of methods for receding horizon control, also known as Model Predictive Control (MPC), to PWA systems led to successful implementations such as the ones reported in (Bemporad *et al*., 2000*a*; Bemporad *et al*., 2000*b*; De Schutter *et al*., 2002; Kerrigan and Mayne, 2002; Lazar and Heemels, 2003; Grieder *et al*., 2004). However, one of the serious drawbacks encountered in these implementations consists in guaranteeing closed-loop stability a priori. This aspect has only been addressed marginally in the previous work on receding horizon control of hybrid systems.

The first solution for guaranteeing stability of hybrid model based receding horizon control has been presented in (Bemporad and Morari, 1999) for Mixed Logical Dynamical (MLD) systems, which are equivalent to PWA systems under certain mild conditions (Heemels *et al*., 2001). This approach is based on enforcing a terminal state equality constraint. However, this method may require a long prediction horizon to guarantee feasibility for all initial states of interest, especially when constraints are present. As a result, a large sampling time is required for real-time implementation. Also, the system needs to have certain controllability properties, while stabilizability should be

[1]*This work has been financially sponsored by the Dutch Science Foundation (STW), Grant "Model Predictive Control for Hybrid Systems"* (DMR.5675).

sufficient in general. Most of the other MPC schemes mentioned above handle stability by keeping the state within a controllable (reachable) path (a sequence of controllable sets computed with respect to a desired target set) and by assuming positive invariance of the predefined target set. A notable exception is the paper (Bemporad *et al*., 2000*a*), where an extension of the results obtained for (linear) constrained LP-based receding horizon control (Bemporad *et al*., 2000*c*) has been pursued. Unfortunately, this MPC approach did not yield conclusive stabilization conditions, but only a heuristic criterion. Another option is to determine stability *a posteriori* by obtaining the explicit PWA solution of the MPC optimization problem and then analyzing the stability of the closed-loop system using piecewise quadratic Lyapunov functions (Grieder *et al*., 2004). Note that this method is not applicable for PWA systems if quadratic costs are used, since then the explicit solution of the MPC optimization problem is no longer piecewise affine.

In this paper we develop *a priori stabilization conditions* for quadratic cost based receding horizon control of unconstrained PWL systems. The proposed method is an extension of the terminal cost approach (Mayne *et al*., 2000) for guaranteeing stability in linear or nonlinear MPC. The procedure for deriving the stabilization conditions is based on Lyapunov arguments, which yield, after using non-trivial transformations, a set of Linear Matrix Inequalities (LMI). The feasibility of the resulting LMI implies that the value function of the MPC cost is a Lyapunov function of the controlled PWL system. The terminal weight on the state variables and the auxiliary feedback gains that ensure stability are obtained from the solution of the derived LMI. The stabilizing MPC algorithm leads to a Mixed Integer Quadratic Programming (MIQP) problem, which is standard in the context of hybrid MPC (Bemporad and Morari, 1999; Torrisi and Bemporad, 2004). Moreover, it is proved that the PWL state-feedback control law employed in the stability proof globally asymptotically stabilizes the origin of the PWL system. It is also pointed out that the stability problem solved in this paper represents a necessary step towards guaranteeing stability for receding horizon control of *constrained* PWA systems.

2. PROBLEM FORMULATION

Consider the time-invariant discrete-time PWL system described by equations of the form (Sontag, 1981):

$$x_{k+1} = A_j x_k + B_j u_k \quad \text{when} \quad x_k \in \Omega_j, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector and $u_k \in \mathbb{R}^m$ is the control input vector at the discrete-time instant $k \geq 0$. $A_j \in \mathbb{R}^{nxn}$, $B_j \in \mathbb{R}^{nxm}$, $j \in \mathcal{S}$, where $\mathcal{S} := \{1, 2, \ldots, s\}$ and s denotes the number of discrete modes. Here $\{\Omega_j \mid j \in \mathcal{S}\}$ is assumed to be a collection of polyhedral sub-sets of $\mathbb{R}^n$ with mutually disjointed interiors and $\bigcup_j \Omega_j = \mathbb{R}^n$.

The purpose is to regulate the state of system (1) to the origin. For a given $N \in \mathbb{N}$, let $\mathbf{x}_k(x_k, \mathbf{u}_k) = (x_k, \ldots, x_{k+N})$ denote a state sequence generated by system (1) with the measured state x_k as initial condition and with the input sequence $\mathbf{u}_k := (u_k, \ldots, u_{k+N-1})$.

Now consider the following problem.

Problem 1 *At time* $k \geq 0$ *let* x_k *be given and minimize the quadratic cost*

$$J(x_k, \mathbf{u}_k) := x_{k+N}^T P x_{k+N} + \sum_{i=0}^{N-1} x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i} \tag{2}$$

over all input sequences $\mathbf{u}_k \in (\mathbb{R}^m)^N$*, subject to*

$$x_{k+1+i} = A_j x_{k+i} + B_j u_{k+i} \quad \text{when} \quad x_{k+i} \in \Omega_j, \text{ for } i = 0, \ldots, N-1. \tag{3}$$

□

Here N denotes the prediction horizon and P, Q and R are positive definite and symmetric matrices. Let

$$V(x_k) := \min_{\mathbf{u}_k} J(x_k, \mathbf{u}_k) \tag{4}$$

denote the value function corresponding to cost (2), let

$$\mathbf{u}_k^* := (u_k^*,\ u_{k+1}^*, \ldots, u_{k+N-1}^*) \tag{5}$$

denote an optimal sequence of controls calculated for state $x_k \in \mathbb{R}^n$ by solving Problem 1 and let $\mathbf{x}_k^*(x_k, \mathbf{u}_k^*) = (x_{k+1}^*, \ldots, x_{k+N}^*)$ denote the corresponding optimal state trajectory. According to the receding horizon strategy, the MPC control is obtained as

$$u_k = \mathbf{u}_k^*(1) = u_k^*;\ \ k \in \mathbb{Z}_+. \tag{6}$$

However, the use of the receding horizon strategy does not necessarily guarantee that system (1) in closed-loop with the MPC control (6) is stable (Mayne *et al.*, 2000). Note that an optimal sequence of controls (5) may not be unique. However, this will not affect the stability analysis that follows.

The goal of this paper is to develop conditions with respect to the *ingredients* of Problem 1 such that system (1) in closed-loop with the MPC control (6) is asymptotically stable. Moreover, if possible, these conditions should be such that Problem 1 leads to an MIQP problem, as this is a standard tool in the context of hybrid MPC (Bemporad, 2004; Torrisi and Bemporad, 2004).

A way to ensure stability is to use the *terminal equality constraint* method in MPC (Keerthi and Gilbert, 1988), which is feasible for PWL systems. Although this method can be applied straightforwardly and is conceptually simple, it has the disadvantage that the system must be brought to the origin in finite time, over the prediction horizon. This requires that the PWL system is controllable, while stabilizability should be sufficient to develop a stabilizing MPC scheme. Also, the terminal equality constraint approach may require a long prediction horizon for ensuring feasibility of Problem 1 (e.g. see (Allgöwer *et al.*, 1999) for details).

In order to prove stability, we aim at using the value function (4) as a candidate Lyapunov function for the closed-loop system (1)-(6), i.e. we require that

$$V(x_{k+1}) - V(x_k) < 0;\ k \in \mathbb{Z}_+,\ \forall x_k \in \mathbb{R}^n \setminus \{0\}, \tag{7}$$

and we consider a local PWL controller of the form

$$u_k := K_j x_k \ \text{ when } \ x_k \in \Omega_j,\ K_j \in \mathbb{R}^{mxn}, j \in \mathcal{S}. \tag{8}$$

Note that this corresponds to one of the usual approaches for guaranteeing stability in MPC (Allgöwer *et al.*, 1999; Mayne *et al.*, 2000). A more precise problem formulation can now be given as follows: given Q and R, which are tuning factors of the MPC algorithm, determine P (and N) such that stability is ensured for the closed-loop system (1)-(6), i.e. such that (7) holds.

3. STABILIZING RECEDING HORIZON CONTROL OF PWL SYSTEMS

In this section we develop an LMI set-up for calculating the terminal weight matrix P such that (7) is satisfied for any N. Consider the nonlinear matrix inequality

$$P - (A_j + B_j K_j)^T P (A_j + B_j K_j) - Q - K_j^T R K_j > 0,\ j \in \mathcal{S} \tag{9}$$

in the unknowns (P, K_j), $j \in \mathcal{S}$, where the matrix P will be taken as the terminal weight employed in cost (2) and the feedbacks $\{K_j \mid j \in \mathcal{S}\}$ define the control law in (8).

Theorem 1 *Assume that* $\{(P, K_j) \mid j \in \mathcal{S}\}$ *with* $P > 0$ *satisfy (9). Then it holds that*

a) The MPC control (6) globally asymptotically stabilizes the PWL system (1);

b) The origin of the PWL system (1) in closed-loop with feedback (8) is globally asymptotically stable.

Proof: Consider (5) and the shifted sequence of controls

$$\mathbf{u}_{k+1} := (u^*_{k+1}, u^*_{k+2}, \ldots, u^*_{k+N-1}, u_{k+N}), \tag{10}$$

where the auxiliary control u_{k+N} is given by the PWL state-feedback (8).

a) In order to achieve stability we will prove that (7) is satisfied for all initial conditions $x_0 \in \mathbb{R}^n \setminus \{0\}$, which can be written as

$$\begin{aligned} V(x_{k+1}) - V(x_k) = J(x_{k+1}, \mathbf{u}^*_{k+1}) - J(x_k, \mathbf{u}^*_k) \le J(x_{k+1}, \mathbf{u}_{k+1}) - J(x_k, \mathbf{u}^*_k) = \\ = -x_k^{*T} Q x_k^* - u_k^{*T} R u_k^* + x_{k+N+1}^T P x_{k+N+1} - x_{k+N}^{*}{}^T (P - Q) x_{k+N}^* + u_{k+N}^T R u_{k+N} < 0, \;\; \forall x_k \in \mathbb{R}^n \setminus \{0\}. \end{aligned} \tag{11}$$

Here, $x_k^* = x_k \in \Omega_j$ is the measured state at the sampling instant k and $x_{k+1} = A_j x_k + B_j u_k^*$. Since the first two terms of the last inequality in (11) are always negative, it suffices to determine the matrix P and u_{k+N} given by (8) such that

$$x_{k+N+1}^T P x_{k+N+1} - x_{k+N}^{*}{}^T (P - Q) x_{k+N}^* + u_{k+N}^T R u_{k+N} < 0, \;\; \forall x_k \in \mathbb{R}^n \setminus \{0\}, \tag{12}$$

for condition (11) to hold. By substituting

$$x_{k+N+1} = A_j x_{k+N}^* + B_j u_{k+N} \;\; \text{when} \;\; x_{k+N}^* \in \Omega_j, j \in \mathcal{S}$$

and (8) in (12), i.e.

$$x_{k+N}^{*}{}^T \left(A_j^T P A_j - P + Q + K_j^T R K_j + K_j^T B_j^T P B_j K_j + K_j^T B_j^T P A_j + A_j^T P B_j K_j \right) x_{k+N}^* < 0, \;\; j \in \mathcal{S}, \tag{13}$$

we obtain the equivalent

$$P - (A_j + B_j K_j)^T P (A_j + B_j K_j) - Q - K_j^T R K_j > 0, \;\; j \in \mathcal{S}.$$

Since $\{(P, K_j) \mid j \in \mathcal{S}\}$ satisfy (9) for all $j \in \mathcal{S}$ it follows that (7) holds. This shows that the value function (4) is a Lyapunov function for the closed-loop system (1)-(6), thereby proving asymptotic stability;

b) If $\{(P, K_j) \mid j \in \mathcal{S}\}$ satisfy (9), then we have that

$$\begin{cases} P > 0 \\ (A_j + B_j K_j)^T P (A_j + B_j K_j) - P < 0 \end{cases}, \;\; j \in \mathcal{S}.$$

Therefore it directly follows that the function $\tilde{V}(x) := x^T P x$ is a common quadratic Lyapunov function for the matrices $A_j + B_j K_j, j \in \mathcal{S}$. Hence, the origin of the unconstrained PWL system (1) with feedback (8) is globally asymptotically stable. □

In order to implement the result of Theorem 1, it would be desirable that the nonlinear matrix inequality (9) is transformed into an LMI. A solution to transform the matrix inequality (9) without the terms $Q+K_j^T RK_j$ into an LMI has been presented in (Mignone *et al*., 2000), where state-feedback stabilization of PWA systems has been investigated. Note that the technique of (Mignone *et al*., 2000) no longer works for (9) due to the extra terms. In the sequel we transform (9) into an LMI using a new technique based on Schur complements (Boyd *et al*., 1994).

Consider the variables

$$Z := P^{-1} \text{ and } Y_j := K_j P^{-1},\ j \in \mathcal{S} \tag{14}$$

and the LMI

$$\Delta_j := \begin{pmatrix} Z & Z & Y_j^T & (A_jZ+B_jY_j)^T \\ Z & Q^{-1} & 0 & 0 \\ Y_j & 0 & R^{-1} & 0 \\ (A_jZ+B_jY_j) & 0 & 0 & Z \end{pmatrix} > 0. \tag{15}$$

Theorem 2 *Suppose that for* $j \in \mathcal{S}$ *the variables* (P, K_j) *and* (Z, Y_j) *are related according to (14). Then (9) and* $P > 0$ *are feasible if and only if (15) is feasible.*

Proof: We start by applying the Schur complement to (15) as follows:

$$\left(\begin{array}{c|ccc} Z & Z & Y_j^T & (A_jZ+B_jY_j)^T \\ \hline Z & Q^{-1} & 0 & 0 \\ Y_j & 0 & R^{-1} & 0 \\ (A_jZ+B_jY_j) & 0 & 0 & Z \end{array}\right) > 0,$$

which leads to

$$Z - \begin{pmatrix} Z & Y_j^T & (A_jZ+B_jY_j)^T \end{pmatrix} \begin{pmatrix} Q & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & Z^{-1} \end{pmatrix} \begin{pmatrix} Z \\ Y_j \\ (A_jZ+B_jY_j) \end{pmatrix} > 0$$

and

$$\begin{pmatrix} Q^{-1} & 0 & 0 \\ 0 & R^{-1} & 0 \\ 0 & 0 & Z \end{pmatrix} > 0.$$

Since $Q > 0$ and $R > 0$ it follows that

$$\begin{cases} Z > 0 \\ Z - ZQZ - Y_j^T RY_j - (A_jZ+B_jY_j)^T Z^{-1} (A_jZ+B_jY_j) > 0. \end{cases} \tag{16}$$

Hence, (15) is feasible if and only if (16) is feasible. Substituting the new variables defined in (14) in (16) gives

$$\begin{cases} P^{-1} > 0 \\ P^{-1} - (A_j P^{-1} + B_j K_j P^{-1})^T P (A_j P^{-1} + B_j K_j P^{-1}) - P^{-1} Q P^{-1} - (K_j P^{-1})^T R (K_j P^{-1}) > 0. \end{cases} \tag{17}$$

Then we pre-multiply and post-multiply both matrix inequalities in (17) by P, yielding the equivalent

$$\begin{cases} P > 0 \\ P - (A_j + B_j K_j)^T P (A_j + B_j K_j) - Q - K_j^{\,T} R K_j > 0. \end{cases} \tag{18}$$

Consequently, feasibility of (15) and feasibility of (18) are equivalent. As the second matrix inequality of (18) is (9) this completes the proof. □

If the LMI (15) is feasible then, by Theorem 2, the terminal weight matrix and the feedback gains are recovered as

$$P = Z^{-1} \text{ and } K_j = Y_j Z^{-1} \text{ for } j \in \mathcal{S}. \tag{19}$$

The main result of the paper can now be formulated as follows.

Theorem 3 *Assume that the LMI (15) is feasible, let* $\{(Z, Y_j) \mid j \in \mathcal{S}\}$ *be a solution and calculate* P *and* K_j *as in (19). Then it holds that*

a) The MPC control (6) globally asymptotically stabilizes the PWL system (1);

b) The origin of the PWL system (1) in closed-loop with feedback (8) is globally asymptotically stable. □

Remark 1 Theorem 3 provides *a priori* sufficient stabilization conditions for receding horizon control of unconstrained PWL systems. The nonlinear matrix inequality that was obtained from the Lyapunov requirement for stability (7) has been transformed into an LMI, which can be easily solved. If state and/or input constraints are imposed, or if the class of PWA systems is considered, a terminal state inequality constraint has to be added to Problem 1 to ensure stability. Then, Problem 1 corresponds to the terminal cost and constraint set method in MPC (Mayne *et al*., 2000). Moreover, one has to compute a positively invariant set for a PWA system in order to enable the result of Theorem 3 for application, which is a non-trivial problem. A possible solution for solving this problem is presented in (Lazar *et al*., 2004), where a stabilizing MPC set-up is developed for the class of constrained PWA systems. Hence, the stabilization conditions developed in this paper represent a necessary step towards guaranteeing stability for receding horizon control of *constrained* PWA systems. □

4. POSSIBILITIES TO REDUCE CONSERVATIVENESS

The result of Theorem 3 requires that a common terminal weight matrix P should satisfy the LMI (15) for all $j \in \mathcal{S}$ and this may result in some conservativeness. In this section we present a solution to reduce conservativeness that is based on using *piecewise terminal weights* in (2). Consider the following problem:

Problem 2 *At time* $k \geq 0$ *let* x_k *be given and minimize the quadratic cost*

$$J(x_k, \mathbf{u}_k) := x_{k+N}^T P_j x_{k+N} + \sum_{i=0}^{N-1} x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i} \quad \text{when} \quad x_{k+N} \in \Omega_j \tag{20}$$

over all input sequences $\mathbf{u}_k \in (\mathbb{R}^m)^N$*, subject to*

$$x_{k+1+i} = A_j x_{k+i} + B_j u_{k+i} \quad \text{when} \quad x_{k+i} \in \Omega_j, \text{ for } i = 0, \ldots, N-1.$$

□

Note that Problem 2 can still be transformed into an MIQP problem, e.g. see (Bemporad and Morari, 1999; Bemporad, 2004; Torrisi and Bemporad, 2004) for a systematic method. Let

$$V(x_k) := \min_{\mathbf{u}_k} J(x_k, \mathbf{u}_k) \tag{21}$$

denote now the value function corresponding to cost (20), let $\mathbf{u}_k^* := (u_k^*,\ u_{k+1}^*, \ldots, u_{k+N-1}^*)$ denote an optimal sequence of controls calculated for state $x_k \in \mathbb{R}^n$ by solving Problem 2 and let $\mathbf{x}_k^*(x_k, \mathbf{u}_k^*) = (x_{k+1}^*, \ldots, x_{k+N}^*)$ denote the corresponding optimal state trajectory. Then, according to the receding horizon strategy, the MPC control is obtained as in (6). Then Theorem 1 can be generalized as follows.

Consider the nonlinear matrix inequality

$$P_j - (A_j + B_j K_j)^T P_i (A_j + B_j K_j) - Q - K_j^T R K_j > 0,\ \ (i,j) \in \mathcal{S} \times \mathcal{S}, \tag{22}$$

in the unknowns $(P_j, K_j),\ j \in \mathcal{S}$ where the matrices $\{P_j \mid j \in \mathcal{S}\}$ will be taken as the terminal weights employed in cost (20) and the feedbacks $\{K_j \mid j \in \mathcal{S}\}$ define the control law in (8).

Theorem 4 *Assume that* $\{(P_j, K_j) \mid j \in \mathcal{S}\}$ with $P_j > 0$ for all $j \in \mathcal{S}$ satisfy *(22). Then it holds that*

a) The MPC control (6) that solves Problem 2 globally asymptotically stabilizes the PWL system (1);

b) The origin of the PWL system (1) in closed-loop with feedback (8) is globally asymptotically stable.

Proof: a) The Lyapunov requirement for stability (7) yields

$$x_{k+N+1}^T P_i x_{k+N+1} - {x_{k+N}^*}^T (P_j - Q) x_{k+N}^* + u_{k+N}^T R u_{k+N} < 0,\ \ \forall x_k \in \mathbb{R}^n \backslash \{0\}. \tag{23}$$

By substituting $x_{k+N+1} = A_j x_{k+N}^* + B_j u_{k+N}$ for $x_{k+N}^* \in \Omega_j,\ x_{k+N+1} \in \Omega_i,\ (i,j) \in \mathcal{S} \times \mathcal{S}$ and (8) in (23), i.e.

$${x_{k+N}^*}^T \left(A_j^T P_i A_j - P_j + Q + K_j^T R K_j + K_j^T B_j^T P_i B_j K_j + K_j^T B_j^T P_i A_j + A_j^T P_i B_j K_j \right) x_{k+N}^* < 0,\ (i,j) \in \mathcal{S} \times \mathcal{S}, \tag{24}$$

we obtain the equivalent

$$P_j - (A_j + B_j K_j)^T P_i (A_j + B_j K_j) - Q - K_j^T R K_j > 0,\ \ (i,j) \in \mathcal{S} \times \mathcal{S}.$$

Since $\{(P_j, K_j) \mid j \in \mathcal{S}\}$ satisfy (22) it follows that (7) holds, which shows that the value function (21) is a Lyapunov function for the closed-loop system (1)-(6), thereby proving asymptotic stability;

b) Since $\{(P_j, K_j) \mid j \in \mathcal{S}\}$ satisfy (22) we have that

$$\begin{cases} P_i > 0 \\ (A_j + B_j K_j)^T P_i (A_j + B_j K_j) - P_j < 0 \end{cases},\ \ (i,j) \in \mathcal{S} \times \mathcal{S}.$$

Therefore it directly follows that the function $\tilde{V}(x) := x^T P_j x$ when $x \in \Omega_j$ is a piecewise quadratic Lyapunov function for the matrices $A_j + B_j K_j, j \in \mathcal{S}$. Hence, the origin of the unconstrained PWL system (1) with feedback (8) is globally asymptotically stable. □

Remark 2 Theorem 4 takes into account all possible pairs of state space regions (i.e. all possible mode transitions from Ω_i to Ω_j) within a total of s regions (where s represents the number of elements of $\mathcal{S}$). Hence, this approach requires that s^2 linear matrix inequalities of the form (22) are feasible. A possible way to reduce the number of pairs employed is to consider only mode switches that can really occur in the system. This can be done using the tools developed in (Bemporad *et al.*, 2000*b*) to perform off-line a *one-step* rechability analysis for the considered PWL system. □

Next, we transform the nonlinear matrix inequality (22) into an LMI in a similar way as done for Problem 1. Consider the variables

$$Z_j := P_j^{-1} \text{ and } Y_j := K_j P_j^{-1}, \text{ for } j \in \mathcal{S} \tag{25}$$

and the LMI

$$\Delta_{ij} := \begin{pmatrix} Z_j & Z_j & Y_j^T & (A_j Z_j + B_j Y_j)^T \\ Z_j & Q^{-1} & 0 & 0 \\ Y_j & 0 & R^{-1} & 0 \\ (A_j Z_j + B_j Y_j) & 0 & 0 & Z_i \end{pmatrix} > 0. \tag{26}$$

Theorem 5 *Suppose that for* $j \in \mathcal{S}$ *the variables* (P_j, K_j) *and* (Z_j, Y_j) *are related according to (25). Then (22) and* $P_i > 0$ *are feasible if and only if (26) is feasible.*

Proof: We start by applying the Schur complement to (26) as follows:

$$\left(\begin{array}{c|ccc} Z_j & Z_j & Y_j^T & (A_j Z_j + B_j Y_j)^T \\ \hline Z_j & Q^{-1} & 0 & 0 \\ Y_j & 0 & R^{-1} & 0 \\ (A_j Z_j + B_j Y_j) & 0 & 0 & Z_i \end{array}\right) > 0,$$

which leads to

$$Z_j - \begin{pmatrix} Z_j & Y_j^T & (A_j Z_j + B_j Y_j)^T \end{pmatrix} \begin{pmatrix} Q & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & Z_i^{-1} \end{pmatrix} \begin{pmatrix} Z_j \\ Y_j \\ (A_j Z_j + B_j Y_j) \end{pmatrix} > 0$$

and

$$\begin{pmatrix} Q^{-1} & 0 & 0 \\ 0 & R^{-1} & 0 \\ 0 & 0 & Z_i \end{pmatrix} > 0.$$

Since $Q > 0$ and $R > 0$ it follows that

$$\begin{cases} Z_i > 0 \\ Z_j - Z_j Q Z_j - Y_j^T R Y_j - (A_j Z_j + B_j Y_j)^T Z_i^{-1} (A_j Z_j + B_j Y_j) > 0. \end{cases} \tag{27}$$

Hence, (26) is feasible if and only if (27) is feasible. Substituting the new variables defined in (25) in (27) gives

$$\begin{cases} P_i^{-1} > 0 \\ P_j^{-1} - (A_j P_j^{-1} + B_j K_j P_j^{-1})^T P_i \ (A_j P_j^{-1} + B_j K_j P_j^{-1}) - P_j^{-1} Q P_j^{-1} - (K_j P_j^{-1})^T R (K_j P_j^{-1}) > 0. \end{cases} \tag{28}$$

Then we pre-multiply and post-multiply the first matrix inequality in (28) by P_i and the second matrix inequality in (28) by P_j, yielding the equivalent

$$\begin{cases} P_i > 0 \\ P_j - (A_j + B_j K_j)^T P_i (A_j + B_j K_j) - Q - K_j^T R K_j > 0. \end{cases} \tag{29}$$

Consequently, feasibility of (26) and feasibility of (29) are equivalent. As the second matrix inequality of (29) is (22) this completes the proof. □

If the LMI (26) is feasible then, by Theorem 5, the terminal weight matrices and the feedback gains are recovered as

$$P_j = Z_j^{-1} \text{ and } K_j = Y_j Z_j^{-1} \text{ for } j \in \mathcal{S}. \tag{30}$$

Combining Theorem 4 and 5, the main result for Problem 2 can now be formulated as follows.

Theorem 6 *Assume that the LMI (26) is feasible, let* $\{(Z_j, Y_j) \mid j \in \mathcal{S}\}$ *be a solution and calculate* P_j *and* K_j *as in (30). Then it holds that*

a) The MPC control (6) that solves Problem 2 globally asymptotically stabilizes the PWL system (1);

b) The origin of the PWL system (1) in closed-loop with feedback (8) is globally asymptotically stable. □

Another possibility to reduce conservativeness is to employ the S-procedure (Boyd *et al.*, 1994) in (9) (or (22)), i.e. to require that

$$x^T((A_j + B_j K_j)^T P (A_j + B_j K_j) - P + Q + K_j^T R K_j) x < 0, \ j \in \mathcal{S} \tag{31}$$

only when $x \in \Omega_j$.

A sufficient condition for (31) to hold is to find a function $W_j(x) := x^T S_j x$ such that $W_j(x) \geq 0$ when $x \in \Omega_j$ such that the matrix inequality

$$x^T((A_j + B_j K_j)^T P (A_j + B_j K_j) - P + Q + K_j^T R K_j) x + x^T S_j x < 0, \quad j \in \mathcal{S} \tag{32}$$

is satisfied for all $x \in \mathbb{R}^n \setminus \{0\}$. Since $W_j(x)$ might be negative outside Ω_j, (32) is less conservative than (13).

In this case (9) can be replaced by

$$P - (A_j + B_j K_j)^T P (A_j + B_j K_j) - M_j - K_j^T R K_j > 0, \quad j \in \mathcal{S}, \tag{33}$$

where $M_j := (Q + S_j)$ becomes a new decision variable that needs to satisfy $M_j > 0$ and

$$x^T (M_j - Q) x > 0 \text{ for all } x \in \Omega_j.$$

If Ω_j is, for instance, of the form $\Omega_j = \{x \in \mathbb{R}^n \mid E_j x \geq 0\},\ j \in \mathcal{S},$ the approach of (Rantzer and Johansson, 2000) can be employed as an alternative to (33). Then S_j is taken of the form $E_j^T U_j E_j$ with U_j some matrix with nonnegative entries. □

5. EXAMPLE

Consider the following piecewise linear system with the partitioning corresponding to the four quadrants of the two dimensional $x_1 - x_2$ plane (Mignone *et al.*, 2000):

$$x_{k+1} = \begin{cases} A_1 x_k + B u_k, \quad E_1 x_k \geq 0; \quad A_1 = \begin{bmatrix} -0.04 & -0.461 \\ -0.139 & 0.341 \end{bmatrix}, E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ A_2 x_k + B u_k, \quad E_2 x_k \geq 0; \quad A_2 = \begin{bmatrix} 0.936 & 0.323 \\ 0.788 & -0.049 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ A_3 x_k + B u_k, \quad E_3 x_k \geq 0; \quad A_3 = \begin{bmatrix} -0.857 & 0.815 \\ 0.491 & 0.62 \end{bmatrix}, E_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ A_4 x_k + B u_k, \quad E_4 x_k \geq 0; \quad A_4 = \begin{bmatrix} -0.022 & 0.644 \\ 0.758 & 0.271 \end{bmatrix}, E_4 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases} \tag{34}$$

The LMI (15) has been solved using the LMI Control Toolbox (Gahinet *et al.*, 1995) for the tuning parameters $Q = I_2$ and $R = 0.1$, and for $j = 1,...,4$, yielding the following terminal weight matrix and feedback gains:

$$\begin{gathered} P = \begin{bmatrix} 2.8013 & 0.3772 \\ 0.3772 & 1.8104 \end{bmatrix}, \\ K_1 = [0.0566 \;\; 0.4011],\ K_2 = [-1.0097 \;\; -0.3093], \\ K_3 = [0.7625 \;\; \text{-}0.8677],\ K_4 = [-0.0812 \;\; -0.6595]. \end{gathered} \tag{35}$$

Problem 1 has been transformed in an equivalent MIQP form using the HYSDEL software package (Torrisi and Bemporad, 2004). A modified version of the solver developed in (Bemporad and Mignone, 2000*d*) has been used to solve Problem 1 at each sampling instant. The simulation results are plotted in Fig. 1 for system (34) with the initial state $x_0 = [4 \;\; \text{-}4]^T$ in closed-loop with the MPC control law (6) with the prediction horizon $N = 2$.

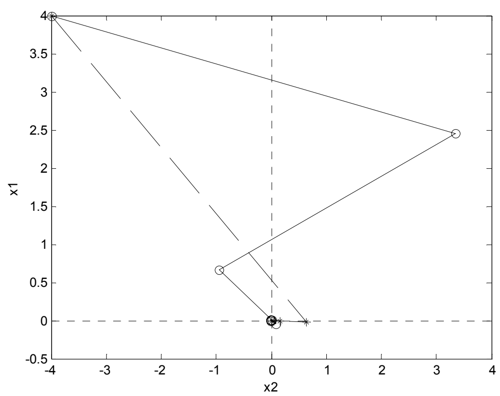

Fig. 1a Closed-loop state trajectory: MPC - solid/circle line; state-feedback - dashed/star line.

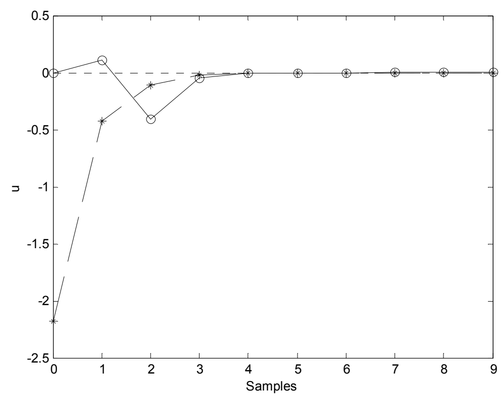

Fig. 1b Input history: MPC - solid/circle line; state-feedback - dashed/star line.

The MPC algorithm that solves Problem 1 with the terminal weight given in (35) (calculated as in (19)) successfully stabilizes the piecewise linear system (34). Note that the PWL state-feedback (8) also stabilizes system (34), but it yields larger control inputs in comparison with the unconstrained MPC controller.

6. CONCLUSIONS

In this paper we have derived *a priori stabilization conditions* for quadratic cost based receding horizon control of unconstrained piecewise linear systems using a terminal cost approach. An LMI set-up has been developed to calculate the terminal weight matrix (or matrices if multiple terminal weights are used) and auxiliary feedback gains such that the value function of the MPC cost is a Lyapunov function of the piecewise linear system in closed-loop with the predictive controller. As a benefit, the MPC optimization problem leads to an MIQP problem, which is standard in hybrid MPC. It has been shown that the state-feedback control law employed in the stability proof globally asymptotically stabilizes the origin of the PWL system. An example illustrated the results.

It has been pointed out that the stabilization conditions derived in this paper constitute a necessary step towards guaranteeing stability for receding horizon control of *constrained* PWA systems, which is part of future work.

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