INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2008; **18**:890–904
Published online 19 July 2007 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/rnc.1249

Input-to-state stabilizing sub-optimal NMPC with an application to DC–DC converters

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SUMMARY

This article focuses on the synthesis of computationally friendly sub-optimal nonlinear model predictive control (NMPC) algorithms with guaranteed robust stability. To analyse the robustness of the MPC closed-loop system, we employ the input-to-state stability (ISS) framework. To design ISS sub-optimal NMPC schemes, a new Lyapunov-based method is proposed. ISS is ensured *via* a set of constraints, which can be specified as a finite number of linear inequalities for input affine nonlinear systems. Furthermore, the method allows for online optimization over the ISS gain of the resulting closed-loop system. The potential of the developed theory for the control of fast nonlinear systems, with sampling periods below 1 ms, is illustrated by applying it to control a Buck-Boost DC–DC converter. Copyright © 2007 John Wiley & Sons, Ltd.

Received 15 November 2006; Revised 30 April 2007; Accepted 29 May 2007

KEY WORDS: sub-optimality; nonlinear model predictive control; input-to-state stability; DC–DC converters

1. INTRODUCTION

One of the essential properties of nonlinear model predictive control (NMPC) is the stability of the closed-loop system. Perhaps the most embraced stabilization method is the so-called terminal cost and constraint set approach, see, for example, the survey [1]. This method uses the value function of the MPC cost as a candidate Lyapunov function for the closed-loop system

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Contract/grant sponsor: Dutch Science Foundation
Contract/grant sponsor: European Community through the Network of Excellence HYCON; contract/grant numbers: DMR. 5675, FP6-IST-511368

and achieves stability *via* a particular terminal cost and an additional constraint on the terminal state, i.e. the predicted state at the end of the prediction horizon. Its advantage consists in the fact that initial feasibility of the NMPC optimization problem implies recursive feasibility, and the finite horizon MPC cost can be a good approximation of the infinite horizon MPC cost by suitable selection of the terminal cost. However, these properties are only guaranteed under the standing assumption that the global optimum of the MPC optimization problem is attained at each sampling instant. Clearly, when dealing with nonlinear prediction models and hard constraints, it is difficult, if not impossible, to guarantee that this assumption holds, as numerical solvers usually provide (in the limited computational time available) a feasible, sub-optimal input sequence. Such a sub-optimal input sequence needs to have certain properties to still guarantee stability of the MPC closed-loop system. Therefore, in practice, especially for fast nonlinear systems, there is a need for sub-optimal NMPC algorithms based on simpler optimization problems, which can be solved faster, and that still have an *a priori* stability guarantee.

An important result regarding sub-optimal NMPC was presented in [2], where it is shown that feasibility of the NMPC optimization problem rather than optimality is sufficient for stability. To be precise, in [2], stability is achieved without requiring optimality *via* an additional constraint that forces the MPC value function to decrease at each sampling instant. However, when nonlinear prediction models are used, this constraint becomes highly nonlinear and difficult to implement from a computational point of view, as the MPC value function depends on the whole sequence of predicted future inputs. Regarding the MPC algorithms of [2], two issues remain to be investigated: how to guarantee *robust stability* for the closed-loop system and how to decrease the computational burden, so that implementation becomes possible for fast systems.

This paper proposes new solutions for designing input-to-state stabilizing (ISS) [3, 4] and computationally friendly sub-optimal MPC algorithms. We achieve this goal *via* new, simpler stabilizing constraints, which can be implemented as a finite number of linear inequalities for input affine nonlinear systems. The method uses an infinity norm-based artificial Lyapunov function, which can be computed off-line. The resulting ISS constraints only depend on the measured state and the first element of the sub-optimal sequence of the predicted future inputs, which results in a considerable simplification with respect to [2].

The proposed ISS NMPC scheme belongs to the category of inherently robust MPC, as opposed to min–max MPC [1]. By this we mean that knowledge about disturbances is not taken into account when computing the control law. However, in the case of disturbances that take values in a bounded, polyhedral set, we show how the developed MPC scheme can be modified to incorporate feedback to disturbances. This is achieved *via* additional constraints that allow for online optimization of the ISS gain [4] of the MPC closed-loop system.

To illustrate the potential for applications to fast nonlinear systems, with sampling periods below 1 ms, the developed theory is applied to control a Buck-Boost DC–DC converter. This type of DC–DC converter is currently used in a wide variety of relevant processes, including electric and hybrid vehicles, solar plants, DC motor drives, switched-mode DC power supplies and many more [5]. Existing control techniques for DC–DC converters mainly rely on PID controllers, which cannot always cope with the desired control objectives: a very fast start-up response with no overshoot and good robust performance in steady state, while satisfying constraints on the inductor current and the duty cycle. With the algorithms developed in this article, we manage to obtain good start-up behaviour and performance in the presence of

DOI: 10.1002/rnc

significant load and additive disturbances. Moreover, a preliminary estimate shows that the required NMPC calculations can safely be performed within the very short sampling time of 0.65 ms. This indicates that new application domains for fast systems are opened up for NMPC, next to the traditional (slower) process control applications.

1.1. Notation and basic definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. We use the notations $\mathbb{Z}_{\geqslant c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ to denote the sets $\{k \in \mathbb{Z}_+ | k \geqslant c_1\}$ and $\{k \in \mathbb{Z}_+ | c_1 < k \leqslant c_2\}$, respectively, for some $c_1, c_2 \in \mathbb{Z}_+$.

We denote the Hölder p-norm of a vector $x \in \mathbb{R}^n$ as $\|x\|_p$. Let $[x]_i$, $i = 1, \ldots, n$ denote the ith component of a vector $x \in \mathbb{R}^n$ and let $|\cdot|$ denote the absolute value. In the remainder of this article, we use $\|\cdot\|$ to denote the ∞-norm $\|\cdot\|_\infty$, for shortness. For a sequence $(z_0, z_1, \ldots) =: \{z_j\}_{j \in \mathbb{Z}_+}$ with $z_j \in \mathbb{R}^l$, let $\|\{z_j\}_{j \in \mathbb{Z}_+}\| := \sup\{\|z_j\| \,|\, j \in \mathbb{Z}_+\}$. Furthermore, $z_{[k]}$ denotes the truncation of $\{z_j\}_{j \in \mathbb{Z}_+}$ at time $k \in \mathbb{Z}_+$, i.e. $z_{[k]} = \{z_j\}_{j \in \mathbb{Z}_{[0,k]}}$. For an arbitrary sequence $\mathbf{u} := (u_0, u_1, \ldots) = \{u_j\}_{j \in \mathbb{Z}_+}$, we use $\mathbf{u}(j)$ to denote u_j. For a matrix $Z \in \mathbb{R}^{m \times n}$, let $\|Z\| := \sup_{x \neq 0} \|Zx\| / \|x\|$ denote its corresponding induced matrix norm. It is well known that $\|Z\|_\infty = \max_{1 \leqslant i \leqslant m} \sum_{j=1}^n |Z^{\{ij\}}|$, where $Z^{\{ij\}}$ is the ijth entry of Z.

For a set $\mathscr{S} \subseteq \mathbb{R}^n$, we denote by $\partial\mathscr{S}$ the boundary, by $\text{int}(\mathscr{S})$ the interior and by $\text{cl}(\mathscr{S})$ the closure of $\mathscr{S}$. A polyhedron (or a polyhedral set) in $\mathbb{R}^n$ is a set obtained as the intersection of a finite number of open and/or closed half-spaces. Given $(n+1)$ *affinely independent* points $(\theta_0, \ldots, \theta_n)$ of $\mathbb{R}^n$, i.e. $(1\ \theta_0^\top)^\top, \ldots, (1\ \theta_n^\top)^\top$ are linearly independent in $\mathbb{R}^{n+1}$, we define a simplex S as

$$S := \text{Co}(\theta_0, \ldots, \theta_n) := \left\{ x \in \mathbb{R}^n \,\middle|\, x = \sum_{l=0}^{n} \mu_l \theta_l, \sum_{l=0}^{n} \mu_l = 1, \mu_l \geqslant 0 \text{ for } l = 0, 1, \ldots, n \right\}$$

where $\text{Co}(\cdot)$ denotes the convex hull.

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathscr{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathscr{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathscr{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and $\lim_{k \to \infty} \beta(s, k) = 0$.

2. INPUT-TO-STATE STABILITY PRELIMINARIES

Consider the discrete-time perturbed nonlinear system described by

$$x_{k+1} \in G(x_k, w_k), \quad k \in \mathbb{Z}_+ \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $w_k \in \mathbb{R}^l$ is an unknown disturbance input and $G : \mathbb{R}^n \times \mathbb{R}^l \hookrightarrow \mathbb{R}^n$ is an arbitrary nonlinear, possibly discontinuous, set-valued function. For simplicity of notation, we assume that the origin is an equilibrium in (1) for zero disturbance, i.e. $G(0, 0) = 0$. Consider the case when w_k takes values at all times $k \in \mathbb{Z}_+$ in a bounded set $\mathbb{W} \subset \mathbb{R}^l$.

Definition 2.1 (Robust positive invariance (RPI))
We call a set $\mathscr{P} \subseteq \mathbb{R}^n$ *robustly positively invariant* for system (1) with respect to $\mathbb{W}$ if for all $x \in \mathscr{P}$ it holds that $G(x, w) \subseteq \mathscr{P}$ for all disturbances $w \in \mathbb{W}$.

DOI: 10.1002/rnc

Definition 2.2 (Regional ISS (Magni et al. [6], Lazar [7], Limon et al. [8]))
Let $\mathbb{X}$ with $0 \in \text{int}(\mathbb{X})$ and $\mathbb{W}$ be subsets of $\mathbb{R}^n$ and $\mathbb{R}^l$, respectively. We call system (1) *ISS in* $\mathbb{X}$ *for disturbances in* $\mathbb{W}$ if there exist a $\mathscr{KL}$-function $\beta(\cdot,\cdot)$ and a $\mathscr{K}$-function $\gamma(\cdot)$ such that, for each $x_0 \in \mathbb{X}$ and all $\{w_j\}_{j\in\mathbb{Z}_+}$ with $w_j \in \mathbb{W}$ for all $j \in \mathbb{Z}_+$, it holds that the corresponding state trajectories $\{x_k\}_{k\in\mathbb{Z}_+}$ satisfy $\|x_k\| \leqslant \beta(\|x_0\|, k) + \gamma(\|w_{[k-1]}\|)$, $\forall k \in \mathbb{Z}_{\geqslant 1}$.

We call the $\mathscr{K}$-function $\gamma(\cdot)$ an ISS gain of system (1). The following conditions for regional ISS will be used throughout the article to establish ISS of MPC closed-loop systems.

Theorem 2.3
Let $\mathbb{W}$ be a subset of $\mathbb{R}^l$ and let $\mathbb{X}$ be a RPI set for (1) with respect to $\mathbb{W}$ with $0 \in \text{int}(\mathbb{X})$. Furthermore, let $\alpha_1(s) := as^\delta$, $\alpha_2(s) := bs^\delta$, $\alpha_3(s) := cs^\delta$ for some $a, b, c, \delta > 0$, $\sigma \in \mathscr{K}$ and let $V : \mathbb{R}^n \to \mathbb{R}_+$ with $V(0) = 0$ be a function such that

$$\alpha_1(\|x\|) \leqslant V(x) \leqslant \alpha_2(\|x\|) \tag{2a}$$

$$V(x^+) - V(x) \leqslant -\alpha_3(\|x\|) + \sigma(\|w\|) \tag{2b}$$

for all $x \in \mathbb{X}$, $w \in \mathbb{W}$ and all $x^+ \in G(x, w)$. Then system (1) is ISS in $\mathbb{X}$ for disturbances in $\mathbb{W}$.

The proof of Theorem 2.3 is similar in nature to the proof given in [6–8] by replacing the difference *equation* by a difference *inclusion* (1) and is omitted here for brevity. We call a function $V(\cdot)$ that satisfies the hypothesis of Theorem 2.3 an *ISS Lyapunov function*.

3. SUB-OPTIMAL NONLINEAR MODEL PREDICTIVE CONTROL

We consider nominal and perturbed discrete-time nonlinear systems of the form:

$$x_{k+1} = f(x_k) + g(x_k)u_k, \quad k \in \mathbb{Z}_+ \tag{3a}$$

$$\tilde{x}_{k+1} = f(\tilde{x}_k) + g(\tilde{x}_k)u_k + w_k, \quad k \in \mathbb{Z}_+ \tag{3b}$$

where $x_k, \tilde{x}_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $w_k \in \mathbb{W} \subset \mathbb{R}^n$ are the state, the input and the additive disturbance, respectively, at discrete time $k \in \mathbb{Z}_+$, and $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are nonlinear functions with $f(0) = 0$. We will consider the case when sub-optimal NMPC is used to generate the input u_k in (3). We assume that the state and the input vectors are constrained for both systems (3a) and (3b), in a compact subset $\mathbb{X}$ of $\mathbb{R}^n$ and a compact subset $\mathbb{U}$ of $\mathbb{R}^m$, respectively, which contain the origin in their interior. For a fixed $N \in \mathbb{Z}_{\geqslant 1}$, let $\mathbf{x}_k(x_k, \mathbf{u}_k) := (x_{1|k}, \ldots, x_{N|k})$ denote the state sequence generated by the nominal system (3a) from initial state $x_{0|k} := x_k$ and by applying an input sequence $\mathbf{u}_k := (u_{0|k}, \ldots, u_{N-1|k})$. Let $F : \mathbb{R}^n \to \mathbb{R}_+$ with $F(0) = 0$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $L(0, 0) = 0$ be arbitrary mappings. At time $k \in \mathbb{Z}_+$ let $x_k \in \mathbb{X}$ be given. The basic NMPC scenario consists in minimizing at each sampling instant $k \in \mathbb{Z}_+$ a finite horizon cost function of the form

$$J(x_k, \mathbf{u}_k) := F(x_{N|k}) + \sum_{i=0}^{N-1} L(x_{i|k}, u_{i|k}) \tag{4}$$

DOI: 10.1002/rnc

with prediction model (3a) and initial state $x_{0|k} := x_k$, over all input sequences $\mathbf{u}_k$, subject to state and input constraints. In the MPC literature [1], $F(\cdot)$ is called the terminal cost, $L(\cdot,\cdot)$ is called the stage cost and N is called the prediction horizon. Let $\mathbb{X}_f(N) \subseteq \mathbb{X}$ denote the set of *feasible states* with respect to the above optimization problem. Then, $V_{\text{MPC}} : \mathbb{X}_f(N) \to \mathbb{R}_+$, $V_{\text{MPC}}(x_k) := \inf_{\mathbf{u}_k} J(x_k, \mathbf{u}_k)$ is the MPC value function corresponding to the cost (4). If there exists an optimal sequence of controls $\mathbf{u}_k^* := (u^*_{0|k}, u^*_{1|k}, \ldots, u^*_{N-1|k})$ that minimizes (4), the infimum above is a minimum and $V_{\text{MPC}}(x_k) = J(x_k, \mathbf{u}_k^*)$. Then, an *optimal* MPC control law is defined as $u^{\text{MPC}}(x_k) := u^*_{0|k}$, $k \in \mathbb{Z}_+$. Sufficient conditions for the existence of an optimal sequence for NMPC can be found in [9]. Stability of the resulting MPC closed-loop system is usually guaranteed by adding a particular constraint on the terminal state $x_{N|k}$, see, for example, the survey [1].

As mentioned in the Introduction, in practice, the available solvers provide only a feasible, sub-optimal sequence of inputs $\bar{\mathbf{u}}_k := (\bar{u}_{0|k}, \bar{u}_{1|k}, \ldots, \bar{u}_{N-1|k})$ and the control applied to the plant, i.e. $\bar{u}_{0|k}$, is a *sub-optimal* MPC control. The resulting value function is then $\bar{V}(x_k) := J(x_k, \bar{\mathbf{u}}_k)$. The stability of the resulting MPC closed-loop system may be unclear now, or may even be lost. Our goal is to develop a sub-optimal NMPC algorithm that still guarantees robust stability *a priori*.

3.1. Sub-optimal NMPC algorithm and input-to-state stability aspects

In this article we consider ∞-norm-based MPC costs, i.e.

$$F(x) := \|Px\| \quad \text{and} \quad L(x,u) := \|Qx\| + \|R_u u\| \tag{5}$$

where $P \in \mathbb{R}^{p\times n}$, $Q \in \mathbb{R}^{q\times n}$ and $R_u \in \mathbb{R}^{r_u\times m}$ are assumed to be known matrices that have full-column rank. In what follows we will employ an ∞-norm-based ISS Lyapunov function of the form $V(x) := \|P_V x\|$, where $P_V \in \mathbb{R}^{p_v\times n}$ is a full-column rank matrix. Let $Q_V \in \mathbb{R}^{q_v\times n}$ be a known matrix with full-column rank.

Algorithm 3.1 (ISS sub-optimal NMPC)

Step 1: At time $k \in \mathbb{Z}_+$ measure the state x_k, let $x_{0|k} := x_k$ and find a control sequence $\mathbf{u}_k = (u_{0|k}, \ldots, u_{N-1|k})$ that satisfies (optionally, also minimizes the cost (4)–(5)):

$$x_{i+1|k} = f(x_{i|k}) + g(x_{i|k})u_{i|k}, \quad i = 0, \ldots, N-1 \tag{6a}$$

$$\|P_V(f(x_{0|k}) + g(x_{0|k})u_{0|k})\| - \|P_V x_{0|k}\| \leqslant -\|Q_V x_{0|k}\| \tag{6b}$$

$$x_{i|k} \in \mathbb{X}, \quad i = 1, \ldots, N \tag{6c}$$

$$u_{i|k} \in \mathbb{U}, \quad i = 0, \ldots, N-1 \tag{6d}$$

Step 2: Let $\Pi(x_k) := \{\mathbf{u} \in \{\mathbb{R}^m\}^N | \mathbf{u} \text{ satisfies (6)}\}$ and let $\pi(x_k) := \{\mathbf{u}(0) \in \mathbb{R}^m | \mathbf{u} \in \Pi(x_k)\}$. Select a feasible sequence of inputs $\bar{\mathbf{u}}_k := (\bar{u}_{0|k}, \bar{u}_{1|k}, \ldots, \bar{u}_{N-1|k}) \in \Pi(x_k)$ and apply the input $\bar{u}_{0|k} \in \pi(x_k)$ to the perturbed system (3b), increment k by one and go to Step 1.

Let $\widetilde{\mathbb{X}}_f(N) \subseteq \mathbb{X}$ with $0 \in \text{int}(\widetilde{\mathbb{X}}_f(N))$ be a set of states x for which the optimization problem in Step 1 of Algorithm 3.1 is recursively feasible for system $x_{k+1} \in h(x_k, \pi(x_k), w_k)$, $k \in \mathbb{Z}_+$, for any $w_k \in \mathbb{W}$. Here, $h(x, \pi(x), w) := \{f(x) + g(x)u + w | x \in \widetilde{\mathbb{X}}_f(N),\ u \in \pi(x)\}$ denotes the set-valued

DOI: 10.1002/rnc

map corresponding to the perturbed system (3b) in closed-loop with the set-valued sub-optimal MPC control law $\pi(\cdot)$. Notice that $\widetilde{\mathbb{X}}_{\mathrm{f}}(N)$ is an RPI set for $x_{k+1} \in h(x_k, \pi(x_k), w_k)$, $w_k \in \mathbb{W}$, $k \in \mathbb{Z}_+$.

Theorem 3.2
The closed-loop system $x_{k+1} \in h(x_k, \pi(x_k), w_k)$ is ISS in $\widetilde{\mathbb{X}}_{\mathrm{f}}(N)$ for disturbances in $\mathbb{W}$.

Proof
The proof consists in showing that $V(x) = ||P_V x||$ is an ISS Lyapunov function for the system $x_{k+1} \in h(x_k, \pi(x_k), w_k)$. Since P_V has full-column rank, there exist $c_2 \geqslant c_1 > 0$ such that $c_1||x|| \leqslant ||P_V x|| \leqslant c_2||x||$ for all x. For example, $c_2 = ||P_V||$ and $c_1 = \lambda_v/\sqrt{p_v}$, where $\lambda_v > 0$ is the smallest singular value of P_V. Hence, $V(\cdot)$ satisfies condition (2a) for $\alpha_1(||x||) := c_1||x||$ and $\alpha_2(||x||) := c_2||x||$. Next, we show that $V(\cdot)$ satisfies condition (2b). From constraint (6b) and using the triangle inequality, we have that for any $x_{k+1} = h(x_k, \bar{u}_k, w_k)$ with $x_k \in \widetilde{\mathbb{X}}_{\mathrm{f}}(N)$, $w_k \in \mathbb{W}$ and $\bar{u}_k \in \pi(x_k)$:

$$
\begin{aligned}
V(x_{k+1}) - V(x_k) &= V(f(x_k) + g(x_k)\bar{u}_k + w_k) - V(x_k) = ||P_V(f(x_k) + g(x_k)\bar{u}_k + w_k)|| - ||P_V x_k|| \\
&\leqslant ||P_V(f(x_k) + g(x_k)\bar{u}_k)|| + ||P_V w_k|| - ||P_V x_k|| \leqslant -||Q_V x_k|| + ||P_V w_k|| \\
&\leqslant -\alpha_3(||x_k||) + \sigma(||w_k||)
\end{aligned}
$$

where $\alpha_3(s) := \xi s$, with ξ such that $||Q_V x|| \geqslant \xi||x||$ for all x, and $\sigma(s) := c_2 s$. The statement then follows from Theorem 2.3. □

Remark 3.3
In Step 1 of Algorithm 3.1, one has to search for a feasible sequence of inputs, which is sufficient for guaranteeing ISS of the closed-loop system, as stated in Theorem 3.2. In other words, *recursive feasibility implies ISS*.

To implement Algorithm 3.1, one has to specify the matrices P_V (the weight of the artificial Lyapunov function) and Q_V (a weight that is related to the decrease of the Lyapunov function). Next, for a known Q_V, we present a procedure for computing the ∞-norm-based artificial Lyapunov function $V(\cdot)$. Let

$$x_{k+1} = Ax_k + Bu_k, \quad k \in \mathbb{Z}_+ \tag{7}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, be a linear approximation of (3a) around $[0\ 0]^\top$. For a given full-column rank matrix Q_V, to compute the matrix P_V, we consider a linear state feedback $u_k = Kx_k, K \in \mathbb{R}^{m\times n}, k \in \mathbb{Z}_+$, and we make use of the following result.

Lemma 3.4 (Lazar et al. [10])
Suppose that a full-column rank matrix $P_V \in \mathbb{R}^{p_v\times n}$ and a gain $K \in \mathbb{R}^{m\times n}$ satisfy

$$1 - ||P_V(A + BK)P_V^{-L}|| - ||Q_V P_V^{-L}|| \geqslant 0 \tag{8}$$

where $P_V^{-L} := (P_V^\top P_V)^{-1} P_V^\top$ is the left Moore–Penrose inverse of P_V. Then, it holds that

$$||P_V(A + BK)x|| - ||P_V x|| \leqslant -||Q_V x|| \quad \text{for all } x$$

DOI: 10.1002/rnc

and the function $V(x)=||P_V x||$ is an ISS Lyapunov function for the system $x_{k+1}=(A+BK)x_k+w_k$.

For ways to find a solution to inequality (8), we refer the interested reader to [10]. Notice that the result of Lemma 3.4 provides a *local* ISS Lyapunov function, i.e. $V(x)=||P_V x||$, for the nonlinear system (3b) in closed loop with an explicit state feedback. Solving online the optimization problem in Step 1 of Algorithm 3.1 amounts to finding a control action that makes $V(\cdot)$ a *global* ISS Lyapunov function for the resulting closed-loop system corresponding to (3b).

3.2. Computational aspects

From a numerical point of view, the proposed ISS sub-optimal MPC scheme has the advantage that the ISS constraint (6b) can be written for any value of $N\in\mathbb{Z}_{\geqslant 1}$ as a finite number of linear inequalities, for the considered class of nonlinear systems (3). Since by definition $||x||_\infty=\max_{i\in\mathbb{Z}_{[1,n]}}|[x]_i|$, for a constraint $||x||_\infty\leqslant c$ with $c>0$ to be satisfied, it is necessary and sufficient to require that $\pm[x]_i\leqslant c$ for all $i\in\mathbb{Z}_{[1,n]}$; in total, this results in $2n$ linear inequalities in x. Therefore, as $x_{0|k}$ in (6b) is just the measured state, which is known at every $k\in\mathbb{Z}_+$, (6b) is equivalent to

$$\pm[P_V(f(x_{0|k})+g(x_{0|k})u_{0|k})]_i\leqslant||P_V x_{0|k}||-||Q_V x_{0|k}|| \quad \forall i\in\mathbb{Z}_{[1,p_v]} \tag{9}$$

which yields $2p_v$ linear inequalities in the control variable $u_{0|k}$.

Interestingly, for $N=1$, the optimization problem that has to be solved at Step 1 of Algorithm 3.1 can be formulated as a single linear program, since the nonlinear system (3) is affine with respect to the input. This problem remains a linear program if one optimizes over the cost (4) defined using infinity norms, which, for $N=1$, is given by

$$J(x_k,u_{0|k})=||P(f(x_{0|k})+g(x_{0|k})u_{0|k})||+||Qx_{0|k}||+||R_u u_{0|k}||$$

Instead of minimizing $J(x_k,u_{0|k})$ one can introduce two auxiliary optimization variables ϵ_1 and ϵ_2 and solve the following optimization problem instead:

$$\begin{aligned}
&\min \epsilon_1+\epsilon_2\\
&\text{ s.t. } (6a),\ (6c),\ (6d),\ (9)\\
&\quad \epsilon_1\geqslant 0,\quad \epsilon_2\geqslant 0\\
&\quad -\epsilon_1\leqslant||P(f(x_{0|k})+g(x_{0|k})u_{0|k})||+||Qx_{0|k}||\leqslant\epsilon_1\\
&\quad -\epsilon_2\leqslant||R_u u_{0|k}||\leqslant\epsilon_2
\end{aligned}$$

Using the technique for rewriting inequalities that include ∞-norms as linear inequalities in combination with the above optimization problem, one can formulate a linear program whose solution minimizes the cost (4) for $N=1$ and it satisfies all constraints in (6). For $N>1$, the corresponding sub-optimal NMPC set-up yields a nonlinear optimization problem subject to linear constraints on top of the prediction constraints, which is still better than additional nonlinear (stabilization) constraints.

4. ONLINE OPTIMIZATION OF THE CLOSED-LOOP ISS GAIN

The ISS sub-optimal NMPC scheme presented in the previous section can be categorized in the class of inherently robust MPC frameworks, as opposed to the min–max MPC set-ups [1]. By this we mean that knowledge about disturbances is not incorporated in the computation of the control signal. Indeed, in case of Algorithm 3.1, the ISS gain of the closed-loop system will only depend on the gain of the $\sigma(\cdot)$ function, i.e. the constant $c_2 = ||P_V||$ (see [7] for details).

However, when bounds on the disturbances are known, it would be desirable to use this knowledge to minimize the ISS gain of $\sigma(\cdot)$ online and, therefore, introduce feedback to disturbances.

A solution for achieving this goal would be to consider a specific type of $\mathscr{K}$-function, for example, $\sigma(s) := \eta_k s$ (here, the gain η_k is now chosen to be a function of time), add the following constraint to Algorithm 3.1:

$$||P_V(f(x_{0|k}) + g(x_{0|k})u_{0|k} + w)|| - ||P_V x_{0|k}|| + ||Q_V x_{0|k}|| - \eta_k ||w|| \leqslant 0 \quad \forall w \in \mathbb{W} \tag{10}$$

and minimize the gain $\eta_k > 0$ at every instant $k \in \mathbb{Z}_+$. Unfortunately, it is difficult to translate the above constraint into a finite number of (linear) inequalities (for example, if $\mathbb{W}$ is a polyhedron, using its vertices) due to the fact that the left-hand term in (10) is not a convex function of w. Indeed, the left-hand term in (10) contains the difference of two convex functions of w, i.e. $||P_V(f(x_{0|k}) + g(x_{0|k})u_{0|k} + w)||$ and $\eta_k||w||$, which is generally not convex.

To incorporate feedback to disturbances and still preserve the computational advantages of Algorithm 3.1, we propose the following modification to Algorithm 3.1, for the case when $\mathbb{W}$ is a compact polyhedron with a non-empty interior containing the origin. Let w^e, $e = 1, \dots, E$, be the vertices of $\mathbb{W}$, suppose that $E > n$ (with n the dimension of the state) and let $\lambda_k^e \geqslant 0$, $k \in \mathbb{Z}_+$, be optimization variables associated with each vertex w^e. We will add the following constraints to the optimization problem in Step 1 of Algorithm 3.1:

$$||P_V(f(x_{0|k}) + g(x_{0|k})u_{0|k} + w^e)|| - ||P_V x_{0|k}|| + ||Q_V x_{0|k}|| - \lambda_k^e \leqslant 0, \quad e = 1, \dots, E \tag{11}$$

and aim at obtaining small values for λ_k^e. Next, consider a finite set of simplices $S_1, \dots, S_M$ with each simplex S_i equal to the convex hull of a subset of the vertices of $\mathbb{W}$ and the origin and such that $\bigcup_{i=1}^M S_i = \mathbb{W}$. More precisely, $S_i = \mathrm{Co}\{0, w^{e_{i,1}}, \dots, w^{e_{i,n}}\}$ and $\{w^{e_{i,1}}, \dots, w^{e_{i,n}}\} \subseteq \{w^1, \dots, w^E\}$ (i.e. $\{e_{i,1}, \dots, e_{i,n}\} \subseteq \{1, \dots, E\}$) with $w^{e_{i,1}}, \dots, w^{e_{i,n}}$ linearly independent. For each simplex S_i, we define the matrix $W_i := [w^{e_{i,1}} \ \dots \ w^{e_{i,n}}] \in \mathbb{R}^{n \times n}$, which is invertible.

Lemma 4.1

If for $k \in \mathbb{Z}_+$ and the measured state $x_k = x_{0|k}$ there exist $u_{0|k}$ and λ_k^e, $e = 1, \dots, E$, such that (6b) and (11) hold, then (10) holds with

$$\eta_k := \max_{i=1,\dots,M} ||\bar{\lambda}_k^i W_i^{-1}|| \tag{12}$$

where $\bar{\lambda}_k^i := [\lambda_k^{e_{i,1}} \dots \lambda_k^{e_{i,n}}] \in \mathbb{R}^{1 \times n}$ and $||\cdot||$ is the corresponding induced matrix norm.

Proof

Let $x_{0|k}$ be given and suppose (11) holds for λ_k^e, $e = 1, \dots, E$. Let $w \in \mathbb{W} = \bigcup_{i=1}^M S_i$. Hence, there exists an i such that $w \in S_i = \mathrm{Co}\{0, w^{e_{i,1}}, \dots, w^{e_{i,n}}\}$, which means that there exist non-negative numbers $\mu_0, \mu_1, \dots, \mu_n$ with $\sum_{j=0,1,\dots,n} \mu_j = 1$ such that $w = \sum_{j=1,\dots,n} \mu_j w^{e_{i,j}} + \mu_0 0 = \sum_{j=1,\dots,n} \mu_j w^{e_{i,j}}$. In matrix notation, we have $w = W_i[\mu_1 \ \dots \ \mu_n]^\top$ and thus $[\mu_1 \ \dots \ \mu_n]^\top = W_i^{-1} w$.

DOI: 10.1002/rnc

By multiplying each inequality in (11) corresponding to the index $e_{i,j}$ and inequality (6b) (which corresponds to 0) with $\mu_j \geqslant 0, j = 0, 1, \ldots, n$, summing up, using $\sum_{j=0,1,\ldots,n} \mu_j = 1$ and the triangle inequality yields

$$||P_V(f(x_{0|k}) + g(x_{0|k})u_{0|k} + w)|| - ||P_V x_{0|k}|| + ||Q_V x_{0|k}|| - \sum_{j=1,\ldots,n} \mu_j \lambda_k^{e_{i,j}} \leqslant 0$$

or equivalently,

$$||P_V(f(x_{0|k}) + g(x_{0|k})u_{0|k} + w)|| - ||P_V x_{0|k}|| + ||Q_V x_{0|k}|| - \bar{\lambda}_k^i [\mu_1 \ \ldots \ \mu_n]^\top \leqslant 0$$

Using that $[\mu_1 \ \ldots \ \mu_n]^\top = W_i^{-1} w$, $\mu_j \geqslant 0$ and $\lambda_k^{e_{i,j}} \geqslant 0$, we obtain (10) for η_k as in (12). □

Note that, according to Theorem 2.3, if $\eta_k \leqslant \eta^*$ for all $k \geqslant k_0$, for some $k_0 \in \mathbb{Z}_+$, an ISS gain is guaranteed. Since η_k is coupled to λ_k^e, $e = 1, \ldots, E$, *via* (12), small λ_k^e, $e = 1, \ldots, E$, will result in a small ISS gain of the closed-loop system and thus, in optimized robustness to disturbances. As η_k is minimized online at each instant $k \in \mathbb{Z}_+$, *via* the variables λ_k^e, $e = 1, \ldots, E$, constraint (11) introduces feedback to disturbances. Define $\Lambda_k := [\lambda_k^1 \ \ldots \ \lambda_k^E]^\top$ and let R_λ be a known full-column rank matrix of appropriate dimensions. Relation (12) can be used to provide insight into how to choose R_λ. Now consider the following cost:

$$J(x_k, \mathbf{u}_k, \Lambda_k) := ||P x_{N|k}|| + \sum_{i=0}^{N-1} \{||Q x_{i|k}|| + ||R_u u_{i|k}||\} + ||R_\lambda \Lambda_k|| \tag{13}$$

Algorithm 4.2 (Feedback ISS sub-optimal NMPC)

Step 1: At time $k \in \mathbb{Z}_+$ measure the state x_k. Let $x_{0|k} := x_k$ and find a control sequence $\mathbf{u}_k = (u_{0|k}, \ldots, u_{N-1|k})$ and optimization variables $\lambda_k^1, \ldots, \lambda_k^E$ that minimize the cost (13) and satisfy

$$x_{i+1|k} = f(x_{i|k}) + g(x_{i|k}) u_{i|k}, \quad i = 0, \ldots, N-1 \tag{14a}$$

$$||P_V(f(x_{0|k}) + g(x_{0|k})u_{0|k})|| - ||P_V x_{0|k}|| \leqslant -||Q_V x_{0|k}|| \tag{14b}$$

$$||P_V(f(x_{0|k}) + g(x_{0|k})u_{0|k} + w^e)|| - ||P_V x_{0|k}|| + ||Q_V x_{0|k}|| - \lambda_k^e \leqslant 0, \quad e = 1, \ldots, E \tag{14c}$$

$$x_{i|k} \in \mathbb{X}, \quad i = 1, \ldots, N \tag{14d}$$

$$u_{i|k} \in \mathbb{U}, \quad i = 0, \ldots, N-1 \tag{14e}$$

$$\lambda_k^e \geqslant 0, \quad e = 1, \ldots, E \tag{14f}$$

Step 2: Let $\Pi(x_k) := \{\mathbf{u} \in \{\mathbb{R}^m\}^N | \mathbf{u} \text{ satisfies (14)}\}$ and let $\pi(x_k) := \{\mathbf{u}(0) \in \mathbb{R}^m | \mathbf{u} \in \Pi(x_k)\}$. Select a feasible sequence of inputs $\bar{\mathbf{u}}_k := (\bar{u}_{0|k}, \bar{u}_{1|k}, \ldots, \bar{u}_{N-1|k}) \in \Pi(x_k)$ and apply the input $\bar{u}_{0|k} \in \pi(x_k)$ to the perturbed system (3b), increment k by one and go to Step 1.

Remark 4.3

Algorithm 3.1 alone is beneficial as it focusses on performance and it also provides an ISS guarantee. In Algorithm 4.2 we make a trade-off between robustness (suppressing disturbances

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adequately) *via* a small η_k on one hand and performance on the other. Besides enhancing robustness, constraint (14c) also ensures that Algorithm 4.2 recovers performance if the state of the closed-loop system approaches the origin. Roughly speaking, when $x_{0|k} \approx 0$, Algorithm 4.2 will produce a control action $u_{0|k} \approx 0$ (because of constraint (14b) and minimization of cost (13)) and constraint (14c) yields $||P_V w^e|| - \lambda_k^e \leqslant 0$, $e = 1, \ldots, E$. Thus, Algorithm 4.2 will not minimize each variable λ_k^e below the corresponding value $||P_V w^e||$, $e = 1, \ldots, E$, which is already implied by (14b) (see proof of Theorem 3.2). Hence, Algorithm 4.2 approaches Algorithm 3.1 for small $x_{0|k}$. This property is desirable, since it is known from min–max MPC [1] that considering a worst-case disturbance scenario in the MPC algorithm leads to poor performance when the disturbance is small or vanishes in reality.

Remark 4.4

The additional feedback ISS constraints (14c) can still be specified *via* a finite number of linear inequalities in the variables $u_{0|k}, \lambda_k^1, \ldots, \lambda_k^E$ and thus, for $N = 1$, Algorithm 4.2 can be formulated as a single linear program.

5. APPLICATION TO THE CONTROL OF DC–DC CONVERTERS

5.1. *Buck-Boost DC–DC power converters*

DC–DC converters are extensively used in power supplies for electronic equipment to control the energy flow between two DC systems. Buck-Boost DC–DC converters are currently used in a wide variety of relevant processes, including electric and hybrid vehicles, solar plants, DC motor drives, switched-mode DC power supplies and many more [5]. In Figure 1 a schematic representation of an ideal Buck-Boost circuit (i.e. neglecting the parasitic components) is drawn.

The following discrete-time nonlinear-averaged model of the converter, which was developed in [11], is used as prediction model:

$$x_{k+1}^m = \begin{bmatrix} x_{1,k}^m + \frac{T}{L} x_{2,k}^m - \frac{T}{L}(x_{2,k}^m - V_{\text{in}})u_k^m \\ -\frac{T}{C} x_{1,k}^m + \frac{T}{C} x_{1,k}^m u_k^m + \left(1 - \frac{T}{RC}\right) x_{2,k}^m \end{bmatrix}, \quad k \in \mathbb{Z}_+ \tag{15}$$

where $x_k^m = [x_{1,k}^m \; x_{2,k}^m]^\top \in \mathbb{R}^2$ and $u_k^m \in \mathbb{R}$ are the state and the input, respectively. The state x_1^m represents the current flowing through the inductor (i_L), x_2^m represents the output voltage (v_o)

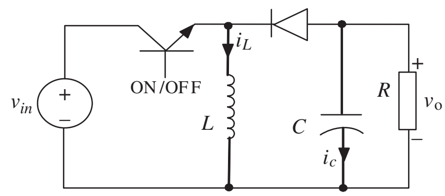

Figure 1. A schematic view of a Buck-Boost converter.

DOI: 10.1002/rnc

and u^m represents the duty cycle (i.e. the fraction of the sampling period during which the transistor is kept ON). The sampling period is $T = 0.65$ ms. The parameters of the circuit are the inductance $L = 4.2$ mH, the capacitance $C = 2200\,\mu$F, the load resistance $R = 165\,\Omega$ and the source input voltage v_{in}, with nominal value $V_{\text{in}} = 15$ V.

5.2. Control goal and MPC scheme set-up

The control objective is twofold: at start-up, a desired value of the output voltage, i.e. x_2^{ss}, should be reached as fast as possible and with minimum overshoot; after the output voltage reaches the desired value, it must be kept close to the operating point, i.e. within a range of $\pm 3\%$ around x_2^{ss} (the industrial operating margin for DC–DC converters) despite changes in the load R (within a 50% range around the nominal value) and disturbances. Note that for a desired output voltage value x_2^{ss} one can obtain the steady-state duty cycle and inductor current as

$$u^{\text{ss}} = \frac{x_2^{\text{ss}}}{x_2^{\text{ss}} - V_{\text{in}}}, \quad x_1^{\text{ss}} = \frac{x_2^{\text{ss}}}{R(u^{\text{ss}} - 1)} \tag{16}$$

Furthermore, the following physical constraints must be fulfilled at all times $k \in \mathbb{Z}_+$:

$$x_{1,k}^m \in [0.01, 5], \quad x_{2,k}^m \in [-20, 0], \quad u_k^m \in [0.1, 0.9] \tag{17}$$

To implement the NMPC algorithms, we first perform the following coordinate transformation on (15):

$$x_{1,k} = x_{1,k}^m - x_1^{\text{ss}}, \quad x_{2,k} = x_{2,k}^m - x_2^{\text{ss}}, \quad u_k = u_k^m - u^{\text{ss}} \tag{18}$$

We obtain the following system description:

$$x_{k+1} = \begin{bmatrix} x_{1,k} + \alpha x_{2,k} + \left(\beta - \dfrac{T}{L} x_{2,k}\right) u_k \\ \left(\dfrac{T}{C} x_{1,k} + \gamma\right) u_k + \left(1 - \dfrac{T}{RC}\right) x_{2,k} + \delta x_{1,k} \end{bmatrix} \tag{19}$$

where the constants α, β, γ and δ depend on the fixed steady-state value x_2^{ss} as follows:

$$\alpha = \frac{T}{L}\left(1 - \frac{x_2^{\text{ss}}}{x_2^{\text{ss}} - V_{\text{in}}}\right), \quad \beta = \frac{T}{L}(V_{\text{in}} - x_2^{\text{ss}}), \quad \gamma = \frac{T}{RCV_{\text{in}}} x_2^{\text{ss}}(x_2^{\text{ss}} - V_{\text{in}}), \quad \delta = \frac{T}{C}\left(\frac{x_2^{\text{ss}}}{x_2^{\text{ss}} - V_{\text{in}}} - 1\right)$$

Using (16) and (18), the constraints given in (17) can be converted to

$$x_{1,k} \in [\underline{b}^{x_1}, \bar{b}^{x_1}], \quad x_{2,k} \in [\underline{b}^{x_2}, \bar{b}^{x_2}], \quad u_k \in [\underline{b}^u, \bar{b}^u] \tag{20}$$

for suitable constants, see [7] for details.

The control objective can now be formulated as to robustly stabilize (19) around the equilibrium $[0\ 0]^\top$ while fulfilling the constraints given in (20).

To apply Algorithm 3.1, we have to compute an ∞-norm-based artificial Lyapunov function. We will use Lemma 3.4 and linearize system (19) around the equilibrium $[0\ 0]^\top$ (for zero input $u_k = 0 \in [\underline{b}^u, \bar{b}^u]$). The linearized equations are

$$\Delta x_{k+1} = A\Delta x_k + B\Delta u_k \tag{21}$$

DOI: 10.1002/rnc

where Δx_k and Δu_k represent 'small' deviations from the equilibrium $[0\ 0]^\top$ and zero input $u_k = 0$, respectively. The matrices A and B are given by

$$A := \frac{\partial f}{\partial x}\bigg|_{\substack{x=0,\\u=0}} = \begin{bmatrix} 1 & \alpha \\ \delta & 1 - \dfrac{T}{RC} \end{bmatrix}, \quad B := \frac{\partial f}{\partial u}\bigg|_{\substack{x=0,\\u=0}} = \begin{bmatrix} \beta \\ \gamma \end{bmatrix}$$

For the linear model corresponding to a steady-state output voltage $x_2^{ss} = -4$ V (which yields $u^{ss} = 0.2105$ and $x_1^{ss} = 0.0307$ A), we apply the method of [10] to find the matrix P_V and the feedback gain K satisfying (8) for $Q_V = 0.001\left[\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right]$. This gives

$$P_V = \begin{bmatrix} 0.9197 & -0.6895 \\ -0.5815 & 1.8109 \end{bmatrix} \quad \text{and} \quad K = [-0.4648\ \ 0.4125]$$

The MPC cost matrices have been chosen to ensure a good performance: $P = \left[\begin{smallmatrix}1 & 0\\0 & 4\end{smallmatrix}\right]$, $Q = \left[\begin{smallmatrix}1 & 0\\0 & 2\end{smallmatrix}\right]$ and $R_u = 0.1$, and independent of P_V and Q_V.

To test robustness, during the simulation we perturb the system with an additive disturbance on the inductor current and we perform a load change. The disturbance set is $\mathbb{W} := \{w \in \mathbb{R}^2 | w = [w_1\ \ w_2]^\top,\ -0.1 \leqslant w_1, w_2 \leqslant 0\}$. For simplicity, we kept the second element of the disturbance vector equal to zero at all times and we employed a single feedback optimization variable λ_k in Algorithm 4.2, corresponding to the vertex $w = [-0.1\ 0]^\top$. The corresponding weight matrix was taken as $R_\lambda = 1$.

To assess the real-time applicability of the developed theory for this type of (very) fast systems with a sampling period well below 1 ms, we chose $N = 1$ and we formulated the optimization problems in Step 1 of Algorithm 3.1 and Algorithm 4.2 as linear programming (LP) problems, *via* the approach of Section 3.2. The LP problem corresponding to Algorithm 3.1 has 3 optimization variables and 14 constraints, while the LP problem corresponding to Algorithm 4.2 has 5 optimization variables and 20 constraints. Here, we did not count the lower and upper bounds on the optimization variables, which are given directly as arguments of the LP solver.

5.3. Simulation results

In one simulation, we first tested the start-up behaviour (see Figure 2) and then, after reaching the desired operating point, we tested the disturbance rejection (see Figure 3).

Note that, although the simulations were performed for the transformed system (19), we chose to plot all variables in the original coordinates corresponding to system (15), which have more physical meaning.

During start-up, when no disturbance acts on the system and the value of the load remains unchanged, the differences between the feedback ISS sub-optimal NMPC scheme and the inherently ISS sub-optimal NMPC scheme are very small, as expected. Both schemes provide a very good start-up response.

However, the difference in performance is significant in the second part of the simulation, when the dynamics were simultaneously affected by an asymptotically decreasing (in norm) additive disturbance of the form $w = [w_1\ \ 0]^\top$ (see Figure 4 for a plot of w_1 *versus* time) and a 50% drop of the load (i.e. $R = 82.5\,\Omega$) for $k = 80$ (0.052 s), $81, \ldots, 120$ (0.078 s). For $k > 120$ (0.078 s) the disturbance was set equal to zero and the load was set to its nominal value (i.e. $R = 165\,\Omega$) to show that the closed-loop system is ISS, i.e. that the asymptotic stability is recovered when the disturbance input vanishes.

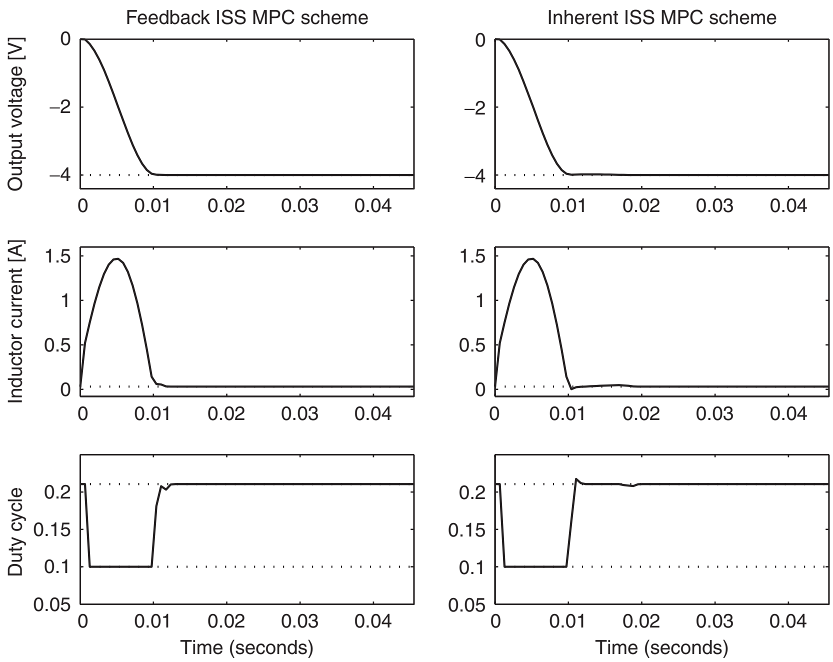

Figure 2. Start-up: State trajectories and sub-optimal NMPC input histories for $N=1$—solid lines, desired steady-state values and constraints—dotted lines.

While the inherently ISS NMPC scheme does not manage to keep the output voltage within the desired operating range, the feedback ISS NMPC scheme achieves very good performance in spite of significant additive and parametric disturbances (changes in the load R).

The smooth transition between activation/deactivation of the feedback to disturbances works as explained in Remark 4.3 (see Figure 4 for a plot of λ_k). One can observe in Figure 4 that when the state reaches the desired operating point, λ_k satisfies $\lambda_k \geqslant ||P_V[-0.1\;\;0]^\top|| = 0.091$.

5.4. *Evaluation of the computation time*

The LP problems for the sub-optimal NMPC optimization problems in Step 1 of Algorithm 3.1 and Algorithm 4.2 were always solved[‡] within the allowed sampling interval, with an worst-case CPU time over 20 runs of 0.6314 ms. In total, 4000 LPs were solved.

The good closed-loop performance obtained for $N=1$ and the small computational time estimate is encouraging for further development of the real-time application of the presented theory to control DC–DC power converters, especially using faster platforms, such as Digital Signal Processors.

[‡] The simulation platform was Matlab 7.0.4 (R14) (CDD Dual Simplex LP solver) running on a Linux Fedora Core 5 operating system powered by an Intel Pentium 4 with a 3.2 GHz CPU.

DOI: 10.1002/rnc

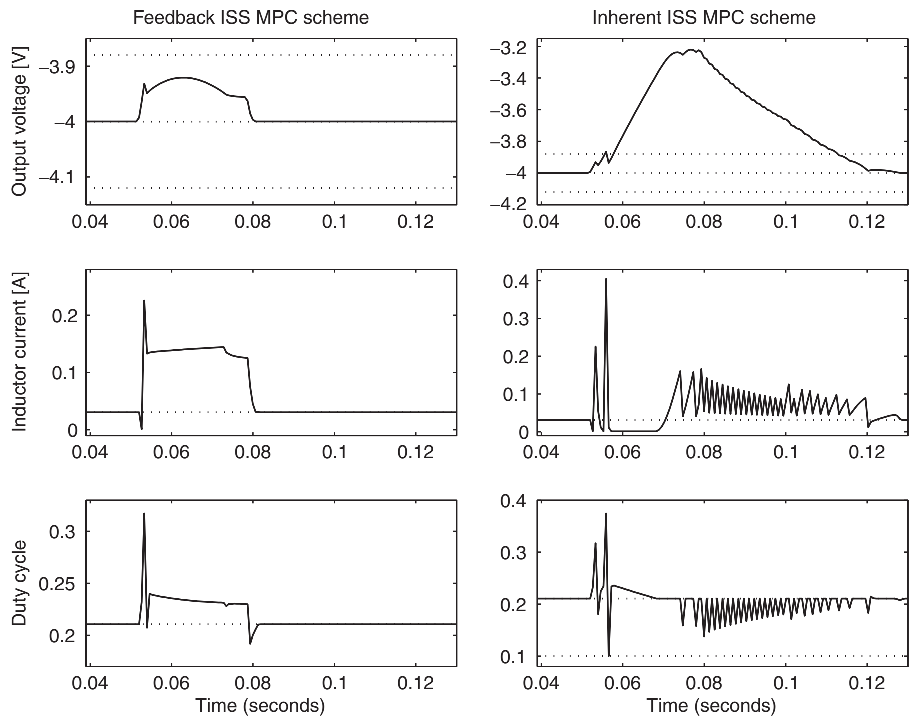

Figure 3. Disturbance rejection: State trajectories and sub-optimal NMPC input histories for $N = 1$—solid lines, desired steady-state values, constraints and industrial operating margins for DC–DC converters ($\pm 3\%$ of the desired output voltage)—dotted lines.

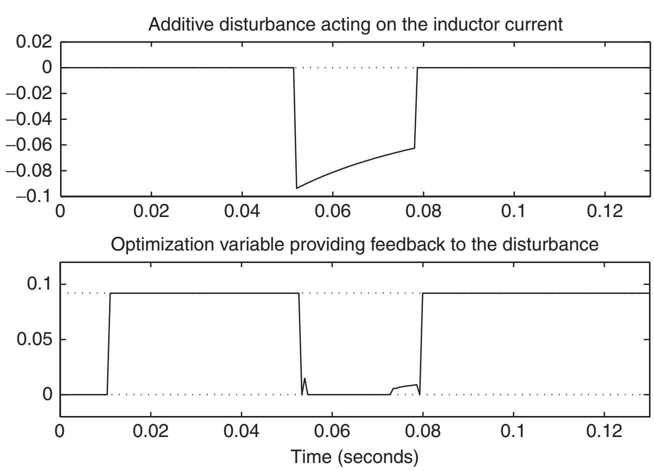

Figure 4. Time history of w_1 and λ_k—solid lines.

DOI: 10.1002/rnc

6. CONCLUSIONS

A Lyapunov-based approach for designing computationally friendly sub-optimal NMPC algorithms with an *a priori* ISS guarantee was presented. The input-to-state stabilization constraints can be written as a finite number of linear inequalities for the class of input affine nonlinear systems. To enhance robust performance, we developed a sub-optimal NMPC scheme that optimizes online over the closed-loop ISS gain. This scheme incorporates feedback to disturbances and results in a better trade-off between robustness and performance. The trade-off varies with the distance to the setpoint: when the distance is large, more robustness is provided; when the distance is small, the scheme 'selects' more performance. A case study on the control of a Buck-Boost DC–DC power converter that includes preliminary real-time numerical data was presented to illustrate the potential of the developed theory for real-time applications.

ACKNOWLEDGEMENTS

This research was supported by the Dutch Science Foundation (STW), Grant 'Model Predictive Control for Hybrid Systems' and the European Community through the Network of Excellence HYCON; contract/grant numbers: DMR. 5675 and FP6-IST-511368.

REFERENCES

1. Mayne DQ, Rawlings JB, Rao CV, Scokaert POM. Constrained model predictive control: stability and optimality. *Automatica* 2000; **36**(6):789–814.
2. Scokaert POM, Mayne DQ, Rawlings JB. Suboptimal model predictive control (feasibility implies stability). *IEEE Transactions on Automatic Control* 1999; **44**(3):648–654.
3. Sontag EB. Further facts about input to state stabilization. *IEEE Transactions on Automatic Control* 1990; **AC-35**:473–476.
4. Jiang Z-P, Wang Y. Input-to-state stability for discrete-time nonlinear systems. *Automatica* 2001; **37**:857–869.
5. Mohan N, Undeland TM, Robbins WP. *Power Electronics*: *Converters*, *Applications and Design*. Wiley: New York, 1989.
6. Magni L, Raimondo DM, Scattolini R. Regional input-to-state stability for nonlinear model predictive control. *IEEE Transactions on Automatic Control* 2006; **51**(9):1548–1553.
7. Lazar M. Model predictive control of hybrid systems: stability and robustness. *Ph.D. Dissertation*, Eindhoven University of Technology, The Netherlands, 2006.
8. Limon D, Alamo T, Salas F, Camacho EF. Input to state stability of min–max MPC controllers for nonlinear systems with bounded uncertainties. *Automatica* 2006; **42**:797–803.
9. Keerthi SS, Gilbert EG. An existence theorem for discrete-time infinite-horizon optimal control problems. *IEEE Transactions on Automatic Control* 1985; **AC-30**(9):907–909.
10. Lazar M, Heemels WPMH, Weiland S, Bemporad A. Stabilizing model predictive control of hybrid systems. *IEEE Transactions on Automatic Control* 2006; **51**(11):1813–1818.
11. Lazar M, De Keyser R. Nonlinear predictive control of a DC-to-DC converter. *Symposium on Power Electronics*, *Electrical Drives*, *Automation and Motion—SPEEDAM*, Capri, Italy, 2004.