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Stabilization Conditions for Model Predictive Control of Constrained PWA Systems

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***Abstract*— Model Predictive Control (MPC) has recently been applied to several relevant classes of hybrid systems with promising results. These developments generated an increasing interest towards issues such as stability and computational problems that arise in hybrid MPC. Stability aspects have been addressed only marginally. In this paper we present an extension of the terminal cost and constraint set method for guaranteeing stability in MPC to the class of constrained piecewise affine systems. Semidefinite programming is used to calculate the employed terminal weight matrix that ensures stability for quadratic cost based MPC. A procedure for computing a robust positively invariant set for piecewise linear systems is also developed. The implementation of the proposed method is illustrated by an example.**

***Index Terms*— Piecewise affine systems, Model predictive control, Stability, Positively invariant sets, Linear matrix inequalities.**

I. INTRODUCTION

Recently, research has focused on questions related to the optimal control and stabilization of hybrid systems in general and Piecewise Affine (PWA) systems in particular. This is motivated by the fact that PWA systems can model a broad class of hybrid systems [11], [21]. Several results have been reported in this framework, e.g., see [6], [18], [19], [22] and the references therein. Extension of Model Predictive Control (MPC) to this class of systems led to successful implementations such as the ones reported in [2], [9], [13], [15]. However, all the implementations mentioned above faced two serious drawbacks. Firstly, the on-line computational load caused by the Mixed Integer Quadratic (or linear) Programming (MIQP) problem prevents real-time implementation. Secondly, closed-loop stability is not guaranteed *a priori*.

The first solution for guaranteeing stability of hybrid model based receding horizon control has been presented in [1] for Mixed Logical Dynamical (MLD) systems. This approach is based on enforcing a terminal state equality constraint. However, this method may require a long prediction horizon to guarantee feasibility for all initial states of interest, especially when input constraints are present. As a result, a large sampling time is required for real-time implementation. Most of the other MPC schemes mentioned above handle stability by keeping the state within a controllable (reachable) path (a sequence of controllable sets computed with respect to a desired target set) and by assuming positive invariance of the predefined target set. A notable exception is [2], where an extension of the results obtained for (linear) constrained LP-based receding horizon control [3] has been pursued. Unfortunately, this infinity norm based MPC approach did not yield conclusive stabilization conditions, but only a heuristic stabilization criterion.

Another option is to determine stability *a posteriori* by obtaining the explicit PWA solution of the MPC constrained optimization problem and then analyzing the stability of the closed-loop system using piecewise quadratic Lyapunov functions.

In this paper we develop *a priori* stabilization conditions for quadratic cost based MPC of constrained PWA systems. The proposed method is an extension of the *terminal cost and constraint set* approach [17] for guaranteeing stability in linear or nonlinear MPC. The procedure for deriving the stabilization conditions is based on Lyapunov arguments, which yield, after non-trivial transformations, a set of Linear Matrix Inequalities (LMI). The feasibility of the resulting LMI implies that the value function of the MPC cost is a Lyapunov function of the controlled PWA system. The terminal weight on the state variables is obtained from the solution of the developed LMI and the terminal state has to be constrained to a positively invariant set containing the origin in order to ensure stability. A procedure for computing a positively invariant set for PWL systems is also presented. If this set is polyhedral then the MPC constrained optimization problem can be transformed into an MIQP problem.

II. PRELIMINARY DEFINITIONS

Consider the time-invariant discrete-time autonomous nonlinear system described by

$$x_{k+1} = f(x_k) \tag{1}$$

and the switched nonlinear system

$$x_{k+1} = f_j(x_k); \quad j \in \mathcal{S}, \tag{2}$$

where $f(\cdot)$ and $f_j(\cdot)$ are smooth nonlinear functions and $\mathcal{S}$ is a finite set of indices.

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Definition II.1 A set $\mathcal{P} \subset \mathbb{R}^n$ is *positively invariant* for system (1) if for all $x \in \mathcal{P}$ it holds that $f(x) \in \mathcal{P}$.

Definition II.2 A set $\mathcal{P} \subset \mathbb{R}^n$ is *positively invariant* for system (2) with arbitrary switching if for all $x \in \mathcal{P}$ and all $j \in \mathcal{S}$ it holds that $f_j(x) \in \mathcal{P}$.

Definition II.3 A positively invariant set $\mathcal{P}$ is called *maximal with respect to a set* $\mathcal{X}$ if $\mathcal{P} \subset \mathcal{X}$ and every positively invariant set $\tilde{\mathcal{P}}$ contained in $\mathcal{X}$ is also contained in $\mathcal{P}$.

Note that Definition II.3 applies to both positively invariant sets in the sense of Definition II.1 and positively invariant sets in the sense of Definition II.2.

A polyhedron is a convex set obtained as the intersection of a finite number of open and/or closed half-spaces. Moreover, a convex and compact set in $\mathbb{R}^n$ that contains the origin in its interior is called a C-set [4].

III. PROBLEM FORMULATION

Consider the time-invariant discrete-time PWA system described by equations of the form [20]:

$$x_{k+1} = A_j x_k + B_j u_k + f_j \quad \text{when } x_k \in \Omega_j. \tag{3}$$

Here, $x_k \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state vector and $u_k \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control input vector at the discrete-time instant $k \geq 0$. $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$, $f_j \in \mathbb{R}^n$, $j \in \mathcal{S}$ with $\mathcal{S} := \{1, 2, \ldots, s\}$ and s denoting the number of discrete modes. The sets $\mathbb{X}$ and $\mathbb{U}$ specify state and input constraints and it is assumed that they are polyhedral C-sets. If there are no constraints, then $\mathbb{X}$ is equal to $\mathbb{R}^n$ and $\mathbb{U}$ is equal to $\mathbb{R}^m$. The collection $\{\Omega_j \mid j \in \mathcal{S}\}$ defines a partition of $\mathbb{X}$, meaning that $\cup_{j\in\mathcal{S}}\Omega_j = \mathbb{X}$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Ω_j is assumed to be a convex polyhedron (not necessarily closed) for all $j \in \mathcal{S}$. Let $\mathcal{S}_0 := \{j \in \mathcal{S} \mid 0 \in \mathrm{cl}(\Omega_j)\}$ and let $\mathcal{S}_1 := \{j \in \mathcal{S} \mid 0 \notin \mathrm{cl}(\Omega_j)\}$ so that $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$, where $\mathrm{cl}(\Omega_j)$ denotes the closure of Ω_j.

The purpose is to regulate the state of system (3) to the origin and we assume that the origin is an equilibrium state for $u = 0$. To have this, we require that

$$f_j = 0 \text{ for all } j \in \mathcal{S}_0. \tag{4}$$

The goal of this paper is to develop for the PWA system (3) a *stabilizing* quadratic cost based MPC scheme that leads to an MIQP problem. For a fixed $N \in \mathbb{N}$, let $\mathbf{x}_k(x_k, \mathbf{u}_k) = (x_{k+1}, \ldots, x_{k+N})$ denote a state sequence generated by system (3) from initial state x_k and by applying the input sequence $\mathbf{u}_k := (u_k, \ldots, u_{k+N-1}) \in \mathbb{U}^N$. Furthermore, let $\mathbb{X}_N \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$ denote a desired target set that contains the origin.

Definition III.1 The class of *admissible input sequences* defined with respect to $\mathbb{X}_N$ and state $x_k \in \mathbb{X}$ is $\mathcal{U}_N(x_k) := \{\mathbf{u}_k \in \mathbb{U}^N \mid \mathbf{x}_k(x_k, \mathbf{u}_k) \in \mathbb{X}^N,\ x_{k+N} \in \mathbb{X}_N\}$.

Stated differently, the input sequence $\mathbf{u}_k \in \mathbb{U}^N$ is contained in $\mathcal{U}_N(x_k)$ if the following conditions are satisfied:

$$x_{k+1+i} = A_j x_{k+i} + B_j u_{k+i} + f_j \text{ when } x_{k+i} \in \Omega_j, \tag{5a}$$
$$u_{k+i} \in \mathbb{U}, \quad x_{k+i} \in \mathbb{X} \quad \text{for } i = 0, \ldots, N-1, \tag{5b}$$
$$x_{k+N} \in \mathbb{X}_N, \tag{5c}$$

where $x_k \in \mathbb{X}$ is given. Now consider the following problem.

Problem III.2 At time $k \geq 0$ let $x_k \in \mathbb{X}$ be given. Minimize the quadratic cost

$$J(x_k, \mathbf{u}_k) := x_{k+N}^\top P x_{k+N} + \sum_{i=0}^{N-1} x_{k+i}^\top Q x_{k+i} + u_{k+i}^\top R u_{k+i} \tag{6}$$

over all input sequences $\mathbf{u}_k \in \mathcal{U}_N(x_k)$.

Here, N denotes the prediction horizon and P, Q and R are positive definite and symmetric matrices. We call an initial state $x_k \in \mathbb{X}$ *feasible* if $\mathcal{U}_N(x_k) \neq \emptyset$. Similarly, Problem III.2 is said to be feasible (or *solvable*) for $x_k \in \mathbb{X}$ if $\mathcal{U}_N(x_k) \neq \emptyset$. Let

$$V(x_k) := \min_{\mathbf{u}_k \in \mathcal{U}_N(x_k)} J(x_k, \mathbf{u}_k) \tag{7}$$

denote the value function corresponding to (6) and consider an optimal sequence of controls calculated for state x_k by solving Problem III.2, i.e.,

$$\mathbf{u}_k^* := (u_k^*, u_{k+1}^*, \ldots, u_{k+N-1}^*). \tag{8}$$

According to the receding horizon strategy, the MPC control is obtained as

$$u_k = \mathbf{u}_k^*(1); \quad k \in \mathbb{Z}_+. \tag{9}$$

Note that the optimal sequence of controls (8) may not be unique, but this fact does not affect the stability analysis that follows.

A more precise problem formulation can now be stated as follows: given Q, R and system (3) the objective is to determine P, N and $\mathbb{X}_N$ such that system (3) in closed-loop with the MPC control (9) is asymptotically stable. Moreover, if possible, Problem III.2 should lead to an MIQP problem, as this is a standard tool in the context of hybrid MPC [1].

Remark III.3 A possible way to ensure stability is to use the terminal equality constraint method in MPC [17], which is feasible for PWA systems. Although this method can be applied straightforwardly and is conceptually simple, it has the disadvantage that the system must be brought to the origin in finite time, over the prediction horizon (this requires that the PWA system is controllable, while stabilizability should be sufficient in general). Also, the terminal equality constraint approach may require a long prediction horizon for ensuring feasibility of Problem III.2.

IV. LMI BASED STABILIZATION CONDITIONS

In order to achieve stability, we aim at using the value function (7) as a candidate Lyapunov function for the closed-loop system (3)-(9) and we consider a local PWL controller of the form

$$u_k := K_j x_k \text{ when } x_k \in \Omega_j,\ K_j \in \mathbb{R}^{m\times n},\ j \in \mathcal{S}_0. \quad (10)$$

Let $\mathbb{X}_{\mathbb{U}} := \cup_{j\in\mathcal{S}_0}\{x \in \Omega_j \mid K_j x \in \mathbb{U}\}$ denote the safe set with respect to *state and input* constraints for this local controller. Now consider the following nonlinear matrix inequality

$$P-(A_j+B_jK_j)^\top P(A_j+B_jK_j)-Q-K_j^\top RK_j > 0 \quad (11)$$

in the unknowns (P, K_j), $j \in \mathcal{S}_0$, where the matrix P is the terminal weight employed in cost (6).

Theorem IV.1 *Assume that* $\{(P, K_j) \mid j \in \mathcal{S}_0\}$ *with* $P > 0$ *satisfy* (11) *and let* $\mathbb{X}_N \subseteq \mathbb{X}_{\mathbb{U}}$ *be a positively invariant set for system* (3) *in closed-loop with* (10)*, i.e., for system* $x_{k+1} = (A_j + B_jK_j)x_k$ *when* $x_k \in \Omega_j$, $j \in \mathcal{S}_0$*. Then it holds that*

1) *Problem III.2 is feasible at time* $k \in \mathbb{Z}_+$ *for state* $x_k \in \Omega_j$ *implies that Problem III.2 is feasible at time* $k + 1$ *for state* $x_{k+1} = A_jx_k + B_j\mathbf{u}_k^*(1) + f_j$.
2) *The MPC control* (9) *asymptotically stabilizes the PWA system* (3) *for all feasible initial states, while satisfying the state and input constraints* (5).
3) *The origin of the PWA system* (3) *in closed-loop with feedback* (10) *is locally asymptotically stable.*

Proof: Consider (8) and the shifted sequence of controls

$$\mathbf{u}_{k+1} := (u_{k+1}^*, u_{k+2}^*, \ldots, u_{k+N-1}^*, u_{k+N}), \quad (12)$$

where the auxiliary control u_{k+N} is the PWL state-feedback (10).

1) If Problem III.2 is feasible at time $k \in \mathbb{Z}_+$ for state $x_k \in \Omega_j$ then $\exists \mathbf{u}_k^* \in \mathcal{U}_N(x_k)$ that solves Problem III.2. Then x_{k+N} satisfies constraint (5c). Since $\mathbb{X}_N \subseteq \mathbb{X}_{\mathbb{U}}$ is positively invariant for system (3) in closed-loop with (10), it follows that $\mathbf{u}_{k+1} \in \mathcal{U}_N(x_{k+1})$. Hence, Problem III.2 is feasible for state $x_{k+1} = A_jx_k + B_j\mathbf{u}_k^*(1) + f_j$.

2) In order to achieve stability we require for all *feasible* initial conditions $x_0 \in \mathbb{X}\backslash\{0\}$ (note that all the states in the set $\mathbb{X}_N \subseteq \mathbb{X}_{\mathbb{U}}$ are feasible with respect to Problem III.2) that

$$V(x_{k+1}) - V(x_k) < 0; \quad k \in \mathbb{Z}_+, \quad (13)$$

which can be written as

$$\begin{aligned} V(x_{k+1}) - V(x_k) &= \\ &= J(x_{k+1}, \mathbf{u}_{k+1}^*) - J(x_k, \mathbf{u}_k^*) \leq \\ &\leq J(x_{k+1}, \mathbf{u}_{k+1}) - J(x_k, \mathbf{u}_k^*) = -x_k^{*\top}Qx_k^* \\ &- u_k^{*\top}Ru_k^* + x_{k+N+1}^\top Px_{k+N+1} + u_{k+N}^\top Ru_{k+N} \\ &- x_{k+N}^{*\top}(P-Q)x_{k+N}^* < 0 \quad ,\forall x_k \in \mathbb{X}\backslash\{0\}. \end{aligned} \quad (14)$$

Here, $x_k^* = x_k \in \Omega_j$ is the measured state at the sampling instant k and $x_{k+1} = A_jx_k + B_ju_k^* + f_j$. Since the first two terms of the last inequality in (14) are always negative, it suffices to determine the matrix P such that there exists u_{k+N} with

$$\begin{aligned} x_{k+N+1}^\top Px_{k+N+1} - x_{k+N}^{*\top}(P-Q)x_{k+N}^* \\ + u_{k+N}^\top Ru_{k+N} \leq 0 \quad ,\forall x_k \in \mathbb{X}\backslash\{0\} \end{aligned} \quad (15)$$

for condition (13) to hold. Next, we substitute $x_{k+N+1} = A_jx_{k+N}^* + B_ju_{k+N}$ when $x_{k+N}^* \in \Omega_j$, $j \in \mathcal{S}_0$ and (10) in (15), yielding the equivalent

$$\begin{aligned} x_{k+N}^{*\top}\big(P - (A_j + B_jK_j)^\top P(A_j + B_jK_j) \\ -Q - K_j^\top RK_j\big)\, x_{k+N}^* > 0 \end{aligned}$$

for all $j \in \mathcal{S}_0$. Since $\{(P, K_j) \mid j \in \mathcal{S}_0\}$ satisfy (11) for all $j \in \mathcal{S}_0$ it follows that (13) holds and then the value function (7) is a Lyapunov function for the closed-loop system (3)-(9), thereby proving asymptotic stability.

3) Since $\{(P, K_j) \mid j \in \mathcal{S}_0\}$ satisfy (11) we have that

$$\begin{cases} P > 0 \\ (A_j + B_jK_j)^\top P(A_j + B_jK_j) - P < 0 \end{cases} ,\ j \in \mathcal{S}_0. \quad (16)$$

Therefore, it directly follows that the function $\tilde{V}(x) := x^\top Px$ is a common quadratic Lyapunov function for the matrices $(A_j + B_jK_j)$, $j \in \mathcal{S}_0$, and then the origin of the PWA system (3) with feedback (10) is asymptotically stable on some region of attraction, e.g., the level set given by the largest $\gamma > 0$ for which $\{x \in \mathbb{X} \mid \tilde{V}(x) \leq \gamma\}$ is contained in $\cup_{j\in\mathcal{S}_0}\Omega_j$. ■

For obvious reasons it would be useful to transform the nonlinear matrix inequality (11) into an LMI. A solution to transform the matrix inequality (11) without the terms $Q + K_j^\top RK_j$ into an LMI has been presented in [18], where state-feedback stabilization of PWA systems has been investigated. Note that the approach of [18] no longer works for (11) due to the extra terms. In order to transform (11) into an LMI, we employ the following Schur complements [7] based technique. Consider the variables

$$Z := P^{-1} \text{ and } Y_j := K_jP^{-1} \quad ,j \in \mathcal{S}_0 \quad (17)$$

and the LMI

$$\Delta_j > 0 \quad ,j \in \mathcal{S}_0, \quad (18)$$

where

$$\Delta_j := \begin{pmatrix} Z & Z & Y_j^\top & (A_jZ + B_jY_j)^\top \\ Z & Q^{-1} & 0 & 0 \\ Y_j & 0 & R^{-1} & 0 \\ (A_jZ + B_jY_j) & 0 & 0 & Z \end{pmatrix}$$

Theorem IV.2 *Suppose that for* $j \in \mathcal{S}_0$ *the variables* (P, K_j) *and* (Z, Y_j) *are related according to* (17)*. Then* (11) *and* $P > 0$ *are feasible if and only* (18) *is feasible.*

Proof: Given in [16]. ■

Remark IV.3 An alternative solution to rewrite matrix inequalities of the form (11) as an LMI has been developed in [14] for uncertain linear systems, in the context of robust MPC.

If the LMI (18) is feasible then, by Theorem IV.2, the terminal weight and the feedback gains are recovered as

$$P = Z^{-1} \text{ and } K_j := Y_j Z^{-1} \text{ for } j \in \mathcal{S}_0. \qquad (19)$$

The main result of the paper can now be formulated as follows.

Theorem IV.4 *Assume that the LMI* (18) *is feasible and let* $\{(Z, Y_j) \mid j \in \mathcal{S}_0\}$ *be a solution. Calculate* P *and* K_j *as in* (19) *and let* $\mathbb{X}_N \subseteq \mathbb{X}_\mathbb{U}$ *be a positively invariant set for system* (3) *in closed-loop with* (10), *i.e., for system* $x_{k+1} = (A_j + B_j K_j)x_k$ *when* $x_k \in \Omega_j$, $j \in \mathcal{S}_0$. *Then it holds that*

1) *Problem III.2 is feasible at time* $k \in \mathbb{Z}_+$ *for state* $x_k \in \Omega_j$ *implies that Problem III.2 is feasible at time* $k+1$ *for state* $x_{k+1} = A_j x_k + B_j \mathbf{u}_k^*(1) + f_j$.
2) *The MPC control* (9) *asymptotically stabilizes the PWA system* (3) *for all feasible initial states, while satisfying the state and input constraints* (5).
3) *The origin of the PWA system* (3) *in closed-loop with feedback* (10) *is locally asymptotically stable.*

Remark IV.5 Theorem IV.1 and Theorem IV.4 require that a common terminal weight matrix P should satisfy (11) for all sub-models of the PWA system (3) corresponding to the indices in $\mathcal{S}_0$. One possibility to relax this condition is to use different terminal weights for each sub-model in (6) or to employ the S-procedure [7] with respect to (11). The implementation of these approaches is given in [16].

V. POSITIVELY INVARIANT SETS FOR PWL SYSTEMS

In order to implement the stabilization conditions given in Theorem IV.4 one has to compute a positively invariant set (i.e., $\mathbb{X}_N \subseteq \mathbb{X}_\mathbb{U}$) for the autonomous PWL system

$$x_{k+1} = (A_j + B_j K_j)x_k =: A_j^{cl} x_k \text{ when } x_k \in \Omega_j, j \in \mathcal{S}_0, \qquad (20)$$

where the feedback gains K_j are calculated as in (19).

It follows from Theorem IV.4 that the most obvious choice of $\mathbb{X}_N$ would be

$$\mathbb{X}_N := \{x \in \mathbb{X} \mid x^\top P x \leq \gamma^*\},$$

where $\gamma^* = \sup_\gamma \{\{x \in \mathbb{X} \mid x^\top P x \leq \gamma\} \subset \mathbb{X}_\mathbb{U}\}$. However, this set is an ellipsoid, which implies that constraint (5c) becomes quadratic and Problem III.2 is no longer linear in the constraints. This is an undesirable property because Problem III.2 does not lead to an MIQP problem [1].

Another option is to calculate for the PWL system (20) the maximal positively invariant set in the sense of Definition II.1. Unfortunately, this set is not convex in general, but only a union (possibly infinite) of convex sets [12]. Only in the (rare) case when this set is a finite union of polyhedral sets, Problem 1 can be put in an MIQP form. However, this approach suffers from two other drawbacks. Firstly, it considerably complicates the solution of the MIQP problem that has to be solved on-line. Secondly, calculating the maximal positively invariant set for a PWL system in the sense of Definition II.1 leads to a combinatorial explosion of possibilities.

That is why it would be preferable that $\mathbb{X}_N$ is a *polyhedral positively invariant set* for system (20). Systematic ways of computing such sets for PWA systems or hybrid systems are not yet available. A possible solution for solving this problem is developed in the sequel, thereby enabling the results of the previous sections for application.

In order to obtain a polyhedral positively invariant set for the PWL system (20) we consider the autonomous switched linear system corresponding to (20), i.e.,

$$x_{k+1} = A_j^{cl} x_k, j \in \mathcal{S}_0, \qquad (21)$$

and we derive the following result.

Theorem V.1 *Positive invariance for system* (21) *in the sense of Definition II.2 implies positive invariance for system* (20) *in the sense of Definition II.1.*

Proof: This follows directly from the fact that $f(x) = f_j(x)$ for at least one $j \in \mathcal{S}_0$ at any time, where $f_j(x) = \{A_j^{cl} x\}$ and $f(x) = \{A_j^{cl} x \text{ when } x \in \Omega_j\}$. ■

Let $\mathbb{X}_T$ denote an arbitrary target set and let

$$\mathcal{Q}_j^1(\mathbb{X}_T) := \{x \in \mathbb{X} \mid A_j^{cl} x \in \mathbb{X}_T\}.$$

Proposition V.2 *[4] If* $\mathbb{X}_T$ *is compact, then* $\mathcal{Q}_j^1(\mathbb{X}_T)$ *is closed. If* $\mathbb{X}_T$ *is convex, then* $\mathcal{Q}_j^1(\mathbb{X}_T)$ *is convex. If* $\mathbb{X}_T$ *is a polyhedron, then* $\mathcal{Q}_j^1(\mathbb{X}_T)$ *is a polyhedron.*

Since we require that $\mathbb{X}_N \subseteq \mathbb{X}_\mathbb{U}$ and $\mathbb{X}_\mathbb{U}$ is not convex in general, we consider in the followings a new safe set, $\tilde{\mathbb{X}}_\mathbb{U}$, taken as a reasonably large polyhedral set (that contains the origin) inside $\mathbb{X}_\mathbb{U}$. For instance, if $\mathbb{X}_\mathbb{U}$ is a polyhedron we might choose $\tilde{\mathbb{X}}_\mathbb{U} = \mathbb{X}_\mathbb{U}$ or, if $\bigcup_{j \in \mathcal{S}_0} \Omega_j$ is a polyhedron we could take $\tilde{\mathbb{X}}_\mathbb{U} = \{x \in \bigcup_{j \in \mathcal{S}_0} \mid K_j x \in \mathbb{U}, \forall j \in \mathcal{S}_0\}$.

Consider now the following sequence of sets:

$$\mathcal{X}_0 = \tilde{\mathbb{X}}_\mathbb{U}, \; \mathcal{X}_i = \bigcap_{j \in \mathcal{S}_0} \mathcal{X}_{i-1}^j, \; i = 1, 2, \ldots, \qquad (22)$$

where $\mathcal{X}_{i-1}^j := \mathcal{Q}_j^1(\mathcal{X}_{i-1}) \bigcap \mathcal{X}_{i-1}, \; i = 1, 2, \ldots$.

Theorem V.3 *The maximal positively invariant set contained in the safe set* $\tilde{\mathbb{X}}_\mathbb{U}$, *calculated for system* (21) *with arbitrary switching, is a convex set that contains the origin and is given by*

$$\mathcal{P} = \bigcap_{i=0}^{\infty} \mathcal{X}_i = \lim_{i \to \infty} \mathcal{X}_i. \qquad (23)$$

Proof: It results from (22) that $\mathcal{X}_i \subseteq \mathcal{X}_{i-1}$ for all $i > 0$. If $x \in \mathcal{P}$ then $x \in \mathcal{X}_i$ for all i. Hence, we have that

$A_j^{cl}x \in \mathcal{X}_{i-1}$ for all $j \in \mathcal{S}_0$ and all i. Then $A_j^{cl}x \in \mathcal{P}$ for all $j \in \mathcal{S}_0$. So, $\mathcal{P}$ is a positively invariant set for system (21) in the sense of Definition II.2.

In order to prove that the set $\mathcal{P}$ is maximal let $\tilde{\mathcal{P}} \subset \tilde{\mathbb{X}}_{\mathbb{U}} = \mathcal{X}_0$ be a postively invariant set for system (21) with arbitrary switching. In order to use induction, we assume that $\tilde{\mathcal{P}} \subset \mathcal{X}_i$ for some i. For any $x \in \tilde{\mathcal{P}}$ we have that $A_j^{cl}x \in \tilde{\mathcal{P}}, \forall j \in \mathcal{S}_0$, yielding $\bigcup_{j\in\mathcal{S}_0}\{A_j^{cl}x\} \subset \tilde{\mathcal{P}} \subset \mathcal{X}_i$ and hence, $x \in \mathcal{X}_{i+1}$. Thus, $\tilde{\mathcal{P}} \subset \mathcal{X}_{i+1}$ and by induction $\tilde{\mathcal{P}} \subset \mathcal{X}_i$ for all i, which yields $\tilde{\mathcal{P}} \subset \bigcap_{i=0}^{\infty}\mathcal{X}_i = \mathcal{P}$.

Now we prove that $\mathcal{P}$ is a convex set. Assume that $\mathcal{P}$ is the maximal positively invariant set for system (21) with arbitrary switching. Then we have that $\mathcal{P}$ is a positively invariant set for any linear system in (21) and then it follows from [10] that the convex hull of $\mathcal{P}$ is also a positively invariant set for any linear system in (21). Hence, the convex hull of $\mathcal{P}$ is a positively invariant set for system (21) in the sense of Definition II.2. Since $\tilde{\mathbb{X}}_{\mathbb{U}}$ is a convex set, it follows that the convex hull of $\mathcal{P}$ is included in $\tilde{\mathbb{X}}_{\mathbb{U}}$. By maximality, the convex hull of $\mathcal{P}$ is also included in $\mathcal{P}$ and thus, $\mathcal{P}$ is convex.

As the origin is an equilibrium for $x_{k+1} = A_j^{cl}x, \forall j \in \mathcal{S}_0$, $\mathcal{P}$ contains the origin. ■

Corollary V.4 *If there exists a finite* i^* *such that* $\mathcal{X}_{i^*} = \mathcal{X}_{i^*+1}$, *then* $\mathcal{X}_i = \mathcal{X}_{i^*}$ *for all* $i \geq i^*$, $\mathcal{P} = \mathcal{X}_{i^*}$ *and* $\mathcal{P}$ *is polyhedral.*

Proof: If there exists a finite i^* such that $\mathcal{X}_{i^*} = \mathcal{X}_{i^*+1}$ it follows directly from $\mathcal{X}_i \subseteq \mathcal{X}_{i-1}$ for all $i > 0$ that $\mathcal{X}_i = \mathcal{X}_{i^*}$ for all $i \geq i^*$ and $\mathcal{P} = \mathcal{X}_{i^*}$. From Proposition V.2 and from the fact that the intersection of (convex) polyhedra produces (convex) polyhedra the sets $\mathcal{X}_0^j := \mathcal{Q}_j^1(\tilde{\mathbb{X}}_{\mathbb{U}}) \bigcap \tilde{\mathbb{X}}_{\mathbb{U}}$ are polyhedra for all $j \in \mathcal{S}_0$. Then it follows that the set $\mathcal{X}_1$ is a polyhedral set and, for the same reason, $\mathcal{X}_i$, $i = 2, 3, \ldots$, are polyhedral sets. Then, by hypothesis, $\mathcal{P}$ is also a polyhedral set. ■

Corollary V.5 *The set* $\mathcal{P}$ *defined as in* (23) *generated by the sequence of sets* (22) *is a positively invariant set for the PWL system* (20).

Proof: This follows directly from Theorem V.1 and Theorem V.3. ■

Remark V.6 If Corollary V.4 holds, then the set (23) generated using (22) is a polyhedral set calculated in finite time. Due to the fact that this set is not the maximal positively invariant set for the PWL system (20), the *backward procedure* of [5] can be employed to enlarge $\mathcal{P}$. However, this procedure must be implemented in the sense of Definition II.1 and then the new regions added to $\mathcal{P}$ must be such that the resulting set remains a polyhedron.

Remark V.7 For any solution of the LMI (18) a different sequence $\{(P, K_j) \mid j \in \mathcal{S}_0\}$ and a different positively invariant set are obtained. Hence, it would be interesting to investigate if certain conditions can be added to the LMI (18) such that the resulting positively invariant set is "maximized"(under constraints).

VI. EXAMPLE

Consider the example proposed in [2], i.e.,

$$x_{k+1} = \begin{cases} A_1 x_k + B u_k & \text{if} \quad [1\ 0]x_k \geq 0 \\ A_2 x_k + B u_k & \text{if} \quad [1\ 0]x_k < 0 \end{cases} \tag{24}$$

subject to the constraints

$$x_k \in \mathbb{X} = [-5, 5] \times [-5, 5], \qquad u_k \in \mathbb{U} = [-1, 1], \tag{25}$$

where

$$A_1 = \begin{bmatrix} 0.35 & -0.6062 \\ 0.6062 & 0.35 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.35 & 0.6062 \\ -0.6062 & 0.35 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In [2] a heuristic procedure is employed to guarantee stability for system (24) in closed-loop with an MILP based MPC controller. Here we use an MIQP based MPC algorithm and a systematic method to *a priori* guarantee stability. The LMI (18) has been solved using the Matlab LMI Control Toolbox [8] for the tuning parameters $Q = I_2$, $R = 0.4$, yielding the following terminal weight matrix and feedback gains:

$$P = \begin{bmatrix} 1.4876 & 0 \\ 0 & 2.2434 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} -0.611 & -0.3572 \end{bmatrix},\ K_2 = \begin{bmatrix} 0.611 & -0.3572 \end{bmatrix}. \tag{26}$$

We take the safe set with respect to state and input constraints as $\tilde{\mathbb{X}}_{\mathbb{U}} = \{x \in \mathbb{X} \mid |K_1 x| \leq 1, |K_2 x| \leq 1\}$. The corresponding polyhedral positively invariant set is

$$\mathbb{X}_N = \left\{ x \in \tilde{\mathbb{X}}_{\mathbb{U}} \mid \begin{bmatrix} -0.2121 & 0.373 \\ 0.2121 & -0.373 \\ 0.2121 & 0.373 \\ -0.2121 & -0.373 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}. \tag{27}$$

The simulation results are plotted in Figure 1 for system (24) with initial state $x_0 = \begin{bmatrix} 4 & -4 \end{bmatrix}^\top$ in closed-loop with the MPC control (9) calculated for $N = 3$. A plot of the set (27) is also depicted.

The MPC algorithm based on Problem III.2 with the terminal weight given in (26), calculated as in (19), successfully stabilizes system (24) while fulfilling the state and input constraints specified in (25).

VII. CONCLUSIONS

In this paper we have derived *a priori* stabilization conditions for quadratic cost based MPC of constrained PWA systems using a terminal cost and constraint set method. An LMI set-up has been developed to calculate the employed terminal weight matrix and local feedback gains such that the value function of the MPC cost is a Lyapunov function of the PWA system in closed-loop with

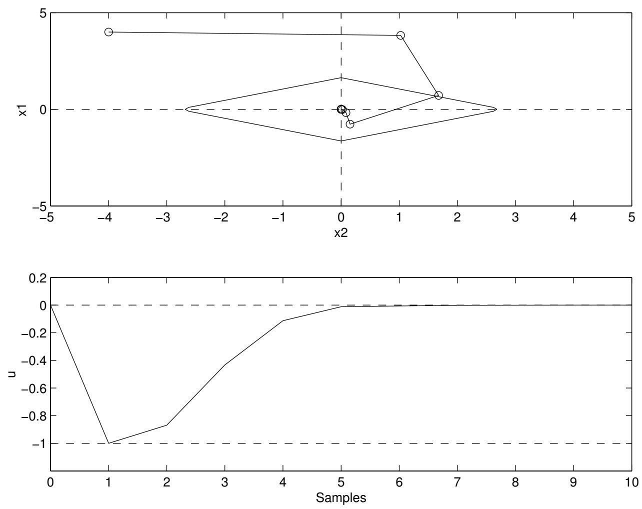

Fig. 1. Simulation results: State trajectory and input history.

the predictive controller. In order to guarantee stability, the terminal state has to be constrained to a positively invariant set containing the origin. A procedure for constructing a robust positively invariant set for PWL systems has also been developed, which is based on a new concept of positive invariance for switched systems. If this set is polyhedral, then the MPC optimization problem leads to an MIQP problem, which is a standard problem in hybrid MPC. The implementation of the stabilization conditions has been illustrated by an example.

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