Self-optimizing Robust
Nonlinear Model Predictive Control

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Abstract : This paper presents a novel method for designing robust MPC schemes that are self-optimizing in terms of disturbance attenuation. The method employs convex control Lyapunov functions and disturbance bounds to optimize robustness of the closed-loop system on-line, at each sampling instant - a unique feature in MPC. Moreover, the proposed MPC algorithm is computationally efficient for nonlinear systems that are affine in the control input and it allows for a decentralized implementation.

1 Introduction

Robustness of nonlinear model predictive controllers has been one of the most relevant and challenging problems within MPC, see, e.g., [1, 2, 3, 4, 5]. From a conceptual point of view, three main categories of robust nonlinear MPC schemes can be identified, each with its pros and cons: inherently robust, tightened constraints and min-max MPC schemes, respectively. In all these approaches, the input-to-state stability property [6] has been employed as a theoretical tool for characterizing robustness, or robust stability[1].

The goal of the existing design methods for synthesizing control laws that achieve ISS [7, 8, 9] is to a priori guarantee a *predetermined* closed-loop ISS gain. Consequently, the ISS property, with a predetermined, constant ISS gain, is in this way enforced for all state space trajectories of the closed-loop system and at all time instances. As the existing approaches, which are also employed in the design of MPC schemes that achieve ISS, can lead to overly conservative solutions along particular trajectories, it is of high interest to develop a control (MPC) design method with the explicit goal of adapting the closed-loop ISS gain depending of the evolution of the state trajectory.

In this article we present a novel method for synthesizing robust MPC schemes with this feature. The method employs convex control Lyapunov functions (CLFs) and disturbance bounds to embed the standard ISS conditions of [8] using a finite number of inequalities. This leads to a finite dimensional optimization problem that has to be solved on-line, in a receding horizon fashion. The proposed inequalities govern the evolution of the closed-loop state trajectory through the sublevel sets of the CLF. The unique feature of the proposed robust MPC scheme is to allow for the *simultaneous* on-line (i) computation of a control action that achieves ISS and (ii) minimization of the

[1]Other characterizations of robustness used in MPC, such as ultimate boundedness or stability of a robustly positively invariant set, can be recovered as a particular case of ISS or shown to be related.

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closed-loop ISS gain depending of an actual state trajectory. As a result, the developed nonlinear MPC scheme is self-optimizing in terms of disturbance attenuation. From the computational point of view, following a particular design recipe, the self-optimizing robust MPC algorithm can be implemented as *a single linear program* for discrete-time nonlinear systems that are affine in the control variable and the disturbance input. Furthermore, we demonstrate that the freedom to optimize the closed-loop ISS gain on-line makes self-optimizing robust MPC suitable for decentralized control of networks of nonlinear systems.

2 Preliminary definitions and results

2.1 Basic notions and definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{\geq c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ to denote the sets $\{k \in \mathbb{Z}_+ \mid k \geq c_1\}$ and $\{k \in \mathbb{Z}_+ \mid c_1 < k \leq c_2\}$, respectively, for some $c_1, c_2 \in \mathbb{Z}_+$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\text{int}(\mathcal{S})$ the interior of $\mathcal{S}$. For two arbitrary sets $\mathcal{S} \subseteq \mathbb{R}^n$ and $\mathcal{P} \subseteq \mathbb{R}^n$, let $\mathcal{S} \sim \mathcal{P} := \{x \in \mathbb{R}^n \mid x + \mathcal{P} \subseteq \mathcal{S}\}$ denote their Pontryagin difference. A polyhedron (or a polyhedral set) in $\mathbb{R}^n$ is a set obtained as the intersection of a finite number of open and/or closed half-spaces. The Hölder p-norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\|_p := (|[x]_1|^p + \ldots + |[x]_n|^p)^{\frac{1}{p}}$ for $p \in \mathbb{Z}_{[1,\infty)}$ and $\|x\|_\infty := \max_{i=1,\ldots,n} |[x]_i|$, where $[x]_i$, $i = 1, \ldots, n$, is the *i*-th component of x and $|\cdot|$ is the absolute value. For a matrix $M \in \mathbb{R}^{m\times n}$, let $\|M\|_p := \sup_{x\neq 0} \frac{\|Mx\|_p}{\|x\|_p}$ denote its corresponding induced matrix norm. Then $\|M\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^n |[M]_{ij}|$, where $[M]_{ij}$ is the *ij*-th entry of M. Let $\mathbf{z} := \{z(l)\}_{l\in\mathbb{Z}_+}$ with $z(l) \in \mathbb{R}^o$ for all $l \in \mathbb{Z}_+$ denote an arbitrary sequence. Define $\|\mathbf{z}\| := \sup\{\|z(l)\| \mid l \in \mathbb{Z}_+\}$, where $\|\cdot\|$ denotes an arbitrary p-norm, and $\mathbf{z}_{[k]} := \{z(l)\}_{l\in\mathbb{Z}_{[0,k]}}$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and $\lim_{s\to\infty} \varphi(s) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k\to\infty} \beta(s, k) = 0$.

2.2 ISS definitions and results

Consider the discrete-time nonlinear system

$$x(k + 1) \in \Phi(x(k), w(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state and $w(k) \in \mathbb{R}^l$ is an unknown disturbance input at the discrete-time instant k. The mapping $\Phi : \mathbb{R}^n \times \mathbb{R}^l \hookrightarrow \mathbb{R}^n$ is an arbitrary nonlinear set-valued function. We assume that $\Phi(0, 0) = \{0\}$. Let $\mathbb{W}$ be a subset of $\mathbb{R}^l$.

Definition 2.1 We call a set $\mathcal{P} \subseteq \mathbb{R}^n$ *robustly positively invariant (RPI)* for system (1) with respect to $\mathbb{W}$ if for all $x \in \mathcal{P}$ it holds that $\Phi(x, w) \subseteq \mathcal{P}$ for all $w \in \mathbb{W}$.

Definition 2.2 Let $\mathbb{X}$ with $0 \in \text{int}(\mathbb{X})$ and $\mathbb{W}$ be subsets of $\mathbb{R}^n$ and $\mathbb{R}^l$, respectively. We call system (1) *ISS($\mathbb{X}$, $\mathbb{W}$)* if there exist a $\mathcal{KL}$-function $\beta(\cdot, \cdot)$ and a $\mathcal{K}$-function $\gamma(\cdot)$ such that, for each $x(0) \in \mathbb{X}$ and all $\mathbf{w} = \{w(l)\}_{l\in\mathbb{Z}_+}$ with $w(l) \in \mathbb{W}$ for all $l \in \mathbb{Z}_+$, it holds that all corresponding state trajectories of (1) satisfy $\|x(k)\| \leq \beta(\|x(0)\|, k) + \gamma(\|\mathbf{w}_{[k-1]}\|)$, $\forall k \in \mathbb{Z}_{\geq 1}$. We call the function $\gamma(\cdot)$ an ISS gain of system (1).

Theorem 2.3 *Let $\mathbb{W}$ be a subset of $\mathbb{R}^l$ and let $\mathbb{X} \subseteq \mathbb{R}^n$ be a RPI set for* (1) *with respect to $\mathbb{W}$, with $0 \in \operatorname{int}(\mathbb{X})$. Furthermore, let $\alpha_1(s) := as^\delta$, $\alpha_2(s) := bs^\delta$, $\alpha_3(s) := cs^\delta$ for some $a, b, c, \delta \in \mathbb{R}_{>0}$, $\sigma \in \mathcal{K}$ and let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a function such that:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{2a}$$

$$V(x^+) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|w\|) \tag{2b}$$

for all $x \in \mathbb{X}$, $w \in \mathbb{W}$ and all $x^+ \in \Phi(x, w)$. Then the system (1) *is ISS($\mathbb{X}$,$\mathbb{W}$) with*

$$\beta(s,k) := \alpha_1^{-1}(2\rho^k \alpha_2(s)),\ \gamma(s) := \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right),\ \rho := 1 - \frac{c}{b} \in [0,1). \tag{3}$$

If inequality (2b) *holds for $w = 0$, then the* 0*-input system $x(k+1) \in \Phi(x(k), 0)$, $k \in \mathbb{Z}_+$, is asymptotically stable in $\mathbb{X}$.*

The proof of Theorem 2.3 is similar in nature to the proof given in [8, 10, 11] by replacing the difference *equation* with the difference *inclusion* as in (1).

2.3 Inherent ISS through continuous and convex control Lyapunov functions

Consider the discrete-time constrained nonlinear system

$$x(k+1) = \phi(x(k), u(k), w(k)) := f(x(k), u(k)) + g(x(k))w(k),\ k \in \mathbb{Z}_+, \tag{4}$$

where $x(k) \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $u(k) \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control action and $w(k) \in \mathbb{W} \subset \mathbb{R}^l$ is an unknown disturbance input at the discrete-time instant k. $\phi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times l}$ are arbitrary nonlinear functions with $\phi(0,0,0) = 0$ and $f(0,0) = 0$. Note that we allow that $g(0) \neq 0$. We assume that $0 \in \operatorname{int}(\mathbb{X})$, $0 \in \operatorname{int}(\mathbb{U})$ and $\mathbb{W}$ is bounded. We also assume that $\phi(\cdot,\cdot,\cdot)$ is bounded in $\mathbb{X}$. Next, let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and let $\sigma \in \mathcal{K}$.

Definition 2.4 A function $V : \mathbb{R}^n \to \mathbb{R}_+$ that satisfies (2a) for all $x \in \mathbb{X}$ is called a control Lyapunov function (CLF) for system $x(k+1) = \phi(x(k), u(k), 0)$, $k \in \mathbb{Z}_+$, if for all $x \in \mathbb{X}$, $\exists u \in \mathbb{U}$ such that $V(\phi(x, u, 0)) - V(x) \leq -\alpha_3(\|x\|)$.

Problem 2.5 Let a CLF $V(\cdot)$ be given. At time $k \in \mathbb{Z}_+$ measure the state $x(k)$ and calculate a control action $u(k)$ that satisfies:

$$u(k) \in \mathbb{U},\ \phi(x(k), u(k), 0) \in \mathbb{X}, \tag{5a}$$

$$V(\phi(x(k), u(k), 0)) - V(x(k)) + \alpha_3(\|x(k)\|) \leq 0. \tag{5b}$$

Let $\pi_0(x(k)) := \{u(k) \in \mathbb{R}^m \mid (5)\text{ holds}\}$. Let $x(k+1) \in \phi_0(x(k), \pi_0(x(k))) := \{f(x(k), u) \mid u \in \pi_0(x(k))\}$ denote the difference inclusion corresponding to the 0-input system (4) in "closed-loop" with the set of feasible solutions obtained by solving Problem 2.5 at each instant $k \in \mathbb{Z}_+$.

Theorem 2.6 *Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ of the form specified in Theorem 2.3 and a corresponding CLF $V(\cdot)$ be given. Suppose that Problem 2.5 is feasible for all states x in $\mathbb{X}$. Then:* (i) *The difference inclusion*

$$x(k+1) \in \phi_0(x(k), \pi_0(x(k))), \quad k \in \mathbb{Z}_+, \tag{6}$$

is asymptotically stable in $\mathbb{X}$; (ii) *Consider a perturbed version of* (6)*, i.e.*

$$\tilde{x}(k+1) \in \phi_0(\tilde{x}(k), \pi_0(\tilde{x}(k))) + g(\tilde{x}(k))w(k), \quad k \in \mathbb{Z}_+ \tag{7}$$

and let $\widetilde{\mathbb{X}} \subseteq \mathbb{X}$ *be a RPI set for* (7) *with respect to* $\mathbb{W}$. *If* $\mathbb{X}$ *is compact, the CLF* $V(\cdot)$ *is convex and continuous*[2] *on* $\mathbb{X}$ *and* $\exists M \in \mathbb{R}_{>0}$ *such that* $\|g(x)\| \leq M$ *for all* $x \in \mathbb{X}$, *then system* (7) *is ISS(*$\widetilde{\mathbb{X}}$,$\mathbb{W}$*).*

Proof: (i) Let $x(k) \in \mathbb{X}$ for some $k \in \mathbb{Z}_+$. Then, feasibility of Problem 2.5 ensures that $x(k+1) \in \phi_0(x(k), \pi_0(x(k))) \subseteq \mathbb{X}$ due to constraint (5a). Hence, Problem 2.5 remains feasible and thus, $\mathbb{X}$ is a PI set for system (6). The result then follows directly from Theorem 2.3. (ii) By convexity and continuity of $V(\cdot)$ and compactness of $\mathbb{X}$, $V(\cdot)$ is Lipschitz continuous on $\mathbb{X}$ [12]. Hence, letting $\mathcal{L} \in \mathbb{R}_{>0}$ denote a Lipschitz constant of $V(\cdot)$ in $\mathbb{X}$, one obtains $|V(\phi(x,u,w)) - V(\phi(x,u,0))| = |V(f(x,u) + g(x)w) - V(f(x,u))| \leq \mathcal{L}M\|w\|$ for all $x \in \mathbb{X}$ and all w. From this property, together with inequality (5b) we have that inequality (2b) holds with $\sigma(s) := \mathcal{L}Ms \in \mathcal{K}$. Since $\widetilde{\mathbb{X}}$ is an RPI set for (7) by the hypothesis, ISS($\widetilde{\mathbb{X}}$,$\mathbb{W}$) of the difference inclusion (7) follows from Theorem 2.3. □

3 Problem definition

Theorem 2.6 establishes that all feasible solutions of Problem 2.5 are stabilizing feedback laws which, under additional assumptions even achieve ISS. However, this inherent ISS property of a feedback law calculated by solving Problem 2.5 relies on a fixed, possibly large gain of $\sigma(\cdot)$, which depends on $V(\cdot)$. This gain is explicitly related to the ISS gain of the closed-loop system via (3). To optimize disturbance attenuation for the closed-loop system, at each time instant $k \in \mathbb{Z}_+$ and for a given $x(k) \in \mathbb{X}$, it would be desirable to *simultaneously* compute a control action $u(k) \in \mathbb{U}$ that satisfies:

(i) $$V(\phi(x(k), u(k), w(k))) - V(x(k)) + \alpha_3(\|x\|) - \sigma(\|w(k)\|) \leq 0, \ \forall w(k) \in \mathbb{W} \quad (8)$$

and some function $\sigma(s) := \eta(k)s^{\delta}$ and (ii) minimize $\eta(k)$ ($\eta(k), \delta \in \mathbb{R}_{>0}, \forall k \in \mathbb{Z}_+$).

Remark 3.1 It is not possible to directly include (8) in Problem 2.5, as it leads to an infinite dimensional optimization problem. If $\mathbb{W}$ is a compact polyhedron, a possibility to resolve this issue would be to evaluate the inequality (8) only for $w(k)$ taking values in the set of vertices of $\mathbb{W}$. However, this does not guarantee that (8) holds for all $w(k) \in \mathbb{W}$ due to the fact that the left-hand term in (8) is not necessarily a convex function of $w(k)$, i.e. it contains the difference of two, possibly convex, functions of $w(k)$. This makes the considered problem challenging and interesting. □

4 Main results

In what follows we present a solution to the problem stated in Section 3. More specifically, we demonstrate that by considering continuous and convex CLFs and compact polyhedral sets $\mathbb{X}$, $\mathbb{U}$, $\mathbb{W}$ (that contain the origin in their interior) a solution to inequality (8) can be obtained via a finite set of inequalities that only depend on the vertices of $\mathbb{W}$. The standing assumption throughout the remainder of the article is that the considered system, i.e. (4), is affine in the disturbance input w.

[2] Continuity of $V(\cdot)$ alone is sufficient, but it requires a somewhat more complex proof.

4.1 Optimized ISS through continuous and convex CLFs

Let w^e, $e = 1, ..., E$, be the vertices of $\mathbb{W}$. Next, consider a finite set of simplices $S_1, \ldots, S_M$ with each simplex S_i equal to the convex hull of a subset of the vertices of $\mathbb{W}$ and the origin, and such that $\cup_{i=1}^M S_i = \mathbb{W}$. More precisely, $S_i = \mathrm{Co}\{0, w^{e_{i,1}}, \ldots, w^{e_{i,l}}\}$ and $\{w^{e_{i,1}}, \ldots, w^{e_{i,l}}\} \subset \{w^1, \ldots, w^E\}$ (i.e. $\{e_{i,1}, \ldots, e_{i,l}\} \subset \{1, \ldots, E\}$) with $w^{e_{i,1}}, \ldots, w^{e_{i,l}}$ linearly independent. For each simplex S_i we define the matrix $W_i := [w^{e_{i,1}} \ \ldots \ w^{e_{i,l}}] \in \mathbb{R}^{l \times l}$, which is invertible. Let $\lambda_e(k)$, $k \in \mathbb{Z}_+$, be optimization variables associated with each vertex w^e. Let $\alpha_3 \in \mathcal{K}_\infty$, suppose that $x(k)$ at time $k \in \mathbb{Z}_+$ is given and consider the following set of inequalities depending on $u(k)$ and $\lambda_1(k), \ldots, \lambda_E(k)$:

$$V(\phi(x(k), u(k), 0)) - V(x(k)) + \alpha_3(\|x(k)\|) \leq 0, \tag{9a}$$

$$V(\phi(x(k), u(k), w^e)) - V(x(k)) + \alpha_3(\|x(k)\|) - \lambda_e(k) \leq 0, \ \forall e = \overline{1, E}. \tag{9b}$$

Theorem 4.1 *Let $V(\cdot)$ be a convex CLF. If for $\alpha_3 \in \mathcal{K}_\infty$ and $x(k)$ at time $k \in \mathbb{Z}_+$ there exist $u(k)$ and $\lambda_e(k)$, $e = 1, \ldots, E$, such that* (9a) *and* (9b) *hold, then* (8) *holds for the same $u(k)$, with $\sigma(s) := \eta(k)s$ and*

$$\eta(k) := \max_{i=1,\ldots,M} \|\bar{\lambda}_i(k) W_i^{-1}\|, \tag{10}$$

where $\bar{\lambda}_i(k) := [\lambda_{e_{i,1}}(k) \ \ldots \ \lambda_{e_{i,l}}(k)] \in \mathbb{R}^{1 \times l}$.

Proof: Let $\alpha_3 \in \mathcal{K}_\infty$ and $x(k)$ be given and suppose (9b) holds for some $\lambda_e(k)$, $e = 1, \ldots, E$. Let $w \in \mathbb{W} = \bigcup_{i=1}^M S_i$. Hence, there exists an i such that $w \in S_i = \mathrm{Co}\{0, w^{e_{i,1}}, \ldots, w^{e_{i,l}}\}$, which means that there exist non-negative numbers $\mu_0, \mu_1, \ldots, \mu_l$ with $\sum_{j=0}^l \mu_j = 1$ such that $w = \sum_{j=1}^l \mu_j w^{e_{i,j}} + \mu_0 0 = \sum_{j=1}^l \mu_j w^{e_{i,j}}$. In matrix notation we have that $w = W_i[\mu_1 \ \ldots \ \mu_l]^\top$ and thus $[\mu_1 \ \ldots \ \mu_l]^\top = W_i^{-1} w$. Multiplying each inequality in (9b) corresponding to the index $e_{i,j}$ and the inequality (9a) with $\mu_j \geq 0$, $j = 0, 1, \ldots, l$, summing up and using $\sum_{j=0}^l \mu_j = 1$ yield:

$$\mu_0 V(\phi(x(k), u(k), 0)) + \sum_{j=1}^{l} \mu_j V(\phi(x(k), u(k), w^{e_{i,j}})) \\ - V(x(k)) + \alpha_3(\|x(k)\|) - \sum_{j=1}^{l} \mu_j \lambda_{e_{i,j}}(k) \leq 0.$$

Furthermore, using $\phi(x(k), u(k), w^{e_{i,j}}) = f(x(k), u(k)) + g(x(k)) w^{e_{i,j}}$, convexity of $V(\cdot)$ and $\sum_{j=0}^l \mu_j = 1$ yields

$$V(\phi(x(k), u(k), \sum_{j=1}^{l} \mu_j w^{e_{i,j}})) - V(x(k)) + \alpha_3(\|x(k)\|) - \sum_{j=1}^{l} \mu_j \lambda_{e_{i,j}}(k) \leq 0,$$

or equivalently

$$V(\phi(x(k), u(k), w)) - V(x(k)) + \alpha_3(\|x(k)\|) - \bar{\lambda}_i(k)[\mu_1 \ \ldots \ \mu_l]^\top \leq 0.$$

Using that $[\mu_1 \ \ldots \ \mu_l]^\top = W_i^{-1} w$ we obtain (8) with $w(k) = w$ for $\sigma(s) = \eta(k)s$ and $\eta(k) \geq 0$ as in (10). □

4.2 Self-optimizing robust nonlinear MPC

For any $x \in \mathbb{X}$ let $\mathbb{W}_x := \{g(x)w \mid w \in \mathbb{W}\} \subset \mathbb{R}^n$ (note that $0 \in \mathbb{W}_x$) and assume that $\mathbb{X} \sim \mathbb{W}_x \neq \emptyset$. Let $\bar{\lambda} := [\lambda_1, \ldots, \lambda_E]^\top$ and let $J(\bar{\lambda}) : \mathbb{R}^E \to \mathbb{R}_+$ be a function that satisfies $\alpha_4(\|\bar{\lambda}\|) \leq J(\bar{\lambda}) \leq \alpha_5(\|\bar{\lambda}\|)$ for some $\alpha_4, \alpha_5 \in \mathcal{K}_\infty$; for example, $J(\bar{\lambda}) := \max_{i=1,\ldots,M} \|\bar{\lambda}_i W_i^{-1}\|$.

Problem 4.2 Let $\alpha_3 \in \mathcal{K}_\infty$, $J(\cdot)$ and a CLF $V(\cdot)$ be given. At time $k \in \mathbb{Z}_+$ measure the state $x(k)$ and minimize the cost $J(\lambda_1(k), \ldots, \lambda_E(k))$ over $u(k), \lambda_1(k), \ldots, \lambda_E(k)$, subject to the constraints

$$u(k) \in \mathbb{U},\ \lambda_e(k) \geq 0,\ f(x(k), u(k)) \in \mathbb{X} \sim \mathbb{W}_{x(k)}, \tag{11a}$$

$$V(\phi(x(k), u(k), 0)) - V(x(k)) + \alpha_3(\|x(k)\|) \leq 0, \tag{11b}$$

$$V(\phi(x(k), u(k), w^e)) - V(x(k)) + \alpha_3(\|x(k)\|) - \lambda_e(k) \leq 0,\ \forall e = \overline{1, E}. \tag{11c}$$

□

Let $\pi(x(k)) := \{u(k) \in \mathbb{R}^m \mid (11) \text{ holds}\}$ and let

$$x(k+1) \in \phi_{\text{cl}}(x(k), \pi(x(k)), w(k)) := \{\phi(x(k), u, w(k)) \mid u \in \pi(x(k))\}$$

denote the difference inclusion corresponding to system (4) in "closed-loop" with the set of feasible solutions obtained by solving Problem 4.2 at each $k \in \mathbb{Z}_+$.

Theorem 4.3 *Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ of the form specified in Theorem 2.3, a continuous and convex CLF $V(\cdot)$ and a cost $J(\cdot)$ be given. Suppose that Problem 4.2 is feasible for all states x in $\mathbb{X}$. Then the difference inclusion*

$$x(k+1) \in \phi_{\text{cl}}(x(k), \pi(x(k)), w(k)), \quad k \in \mathbb{Z}_+ \tag{12}$$

is ISS($\mathbb{X}$, $\mathbb{W}$).

Proof: Let $x(k) \in \mathbb{X}$ for some $k \in \mathbb{Z}_+$. Then, feasibility of Problem 4.2 ensures that $x(k+1) \in \phi_{\text{cl}}(x(k), \pi(x(k)), w(k)) \subseteq \mathbb{X}$ for all $w(k) \in \mathbb{W}$, due to $g(x(k))w(k) \in \mathbb{W}_{x(k)}$ and constraint (11a). Hence, Problem 4.2 remains feasible and thus, $\mathbb{X}$ is a RPI set with respect to $\mathbb{W}$ for system (12). From Theorem 4.1 we also have that $V(\cdot)$ satisfies (2b) with $\sigma(s) := \eta(k)s$ and $\eta(k)$ as in (10). Let

$$\lambda^* := \sup_{x \in \mathbb{X}, u \in \mathbb{U}, e=1,\ldots,E} \{V(\phi(x, u, w^e)) - V(x) + \alpha_3(\|x\|)\}.$$

Due to continuity of $V(\cdot)$, compactness of $\mathbb{X}$, $\mathbb{U}$ and boundedness of $\phi(\cdot,\cdot,\cdot)$, λ^* exists and is finite (the sup above is a max if $\phi(\cdot,\cdot,\cdot)$ is continuous in x and u). Hence, inequality (11c) is always satisfied for $\lambda_e(k) = \lambda^*$ for all $e = 1, \ldots, E$, $k \in \mathbb{Z}_+$, and for all $x \in \mathbb{X}$, $u \in \mathbb{U}$. This in turn, via (10) ensures the existence of a $\eta^* \in \mathbb{R}_{>0}$ such that $\eta(k) \leq \eta^*$ for all $k \in \mathbb{Z}_+$. Hence, we proved that inequality (8) holds for all $x \in \mathbb{X}$ and all $w \in \mathbb{W}$. Then, since $\mathbb{X}$ is RPI, ISS($\mathbb{X}$,$\mathbb{W}$) follows directly from Theorem 2.3. □

Remark 4.4 An alternative proof to Theorem 4.3 can be obtained by simply applying the reasoning used in the proof of Theorem 2.6. Hence, inherent ISS can be established directly from constraint (11b). Also, notice that in the proof of Theorem 4.3 we used a worst case evaluation of $\lambda_e(k)$ to prove ISS. However, it is important to observe that compared to Problem 2.5, nothing is lost in terms of feasibility, while Problem 4.2, *although it inherently guarantees a constant ISS gain, it provides freedom to optimize the ISS gain of the closed-loop system, by minimizing the variables* $\lambda_1(k), \dots, \lambda_E(k)$ *via the cost* $J(\cdot)$. As such, in reality the gain $\eta(k)$ of the function $\sigma(\cdot)$ can be much smaller for $k \geq k_0$, for some $k_0 \in \mathbb{Z}_+$, depending on the state trajectory $x(k)$. □

In Theorem 4.3 we assumed for simplicity that Problem 4.2 is feasible for all $x \in \mathbb{X}$; in other words, *feasibility implies ISS*. Whenever Problem 4.2 can be solved explicitly (see the implementation paragraph below), it is possible to calculate the maximal RPI set for the closed-loop dynamics that is contained within the explicit set of feasible solutions. Alternatively, we establish next an easily verifiable sufficient condition under which any sublevel set of $V(\cdot)$ contained in $\mathbb{X}$ is a RPI subset of the set of feasible solutions of Problem 4.2.

Lemma 4.5 *Given a CLF* $V(\cdot)$ *that satisfies the hypothesis of Theorem 4.3, let* $\mathcal{V}_\Delta := \{x \in \mathbb{R}^n \mid V(x) \leq \Delta\}$. *Then, for any* $\Delta \in \mathbb{R}_{>0}$ *such that* $\mathcal{V}_\Delta \subseteq \mathbb{X}$, *if* $\lambda^* \leq (1-\rho)\Delta$, *with* ρ *as defined in* (3), *Problem 4.2 is feasible for all* $x \in \mathcal{V}_\Delta$ *and remains feasible for all resulting closed-loop trajectories that start in* $\mathcal{V}_\Delta$.

Proof: From the proof of Theorem 4.3 we know that inequalities (11c) are feasible for all $x(k) \in \mathbb{X}$, $u(k) \in \mathbb{U}$ and $e = \overline{1,E}$ by taking $\lambda(k) = \lambda^*$ for all $k \in \mathbb{Z}_+$. Thus, for any $x(k) \in \mathcal{V}_\Delta \subseteq \mathbb{X}$, $\Delta \in \mathbb{R}_{\geq 0}$, we have that:

$$
\begin{aligned}
V(\phi(x(k),u(k),w(k))) &\leq V(x(k)) - \alpha_3(\|x(k)\|) + \lambda^* \leq \rho V(x(k)) + \lambda^* \\
&\leq \rho\Delta + \lambda^* \leq \rho\Delta + (1-\rho)\Delta = \Delta,
\end{aligned}
$$

which yields $\phi(x(k),u(k),w(k)) \in \mathcal{V}_\Delta \subseteq \mathbb{X}$. This in turn ensures feasibility of (11a), while (11b) is feasible by definition of the CLF $V(\cdot)$, which concludes the proof. □

Remark 4.6 The result of Theorem 4.3 holds for all inputs $u(k)$ for which Problem 4.2 is feasible. To select on-line one particular control input from the set $\pi(x(k))$ and to improve closed-loop performance (in terms of settling time) it is useful to also penalize the state and the input. Let $F : \mathbb{R}^n \to \mathbb{R}_+$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $F(0) = L(0,0) = 0$ be arbitrary nonlinear functions. For $N \in \mathbb{Z}_{\geq 1}$ let $\bar{\mathbf{u}}(k) := (\bar{u}(k), \bar{u}(k+1), \dots, \bar{u}(k+N-1)) \in \mathbb{U}^N$ and $J_{\text{RHC}}(x(k), \bar{\mathbf{u}}(k)) := F(\bar{x}(k+N)) + \sum_{i=0}^{N-1} L(\bar{x}(k+i), \bar{u}(k+i))$, where $\bar{x}(k+i+1) := f(\bar{x}(k+i), \bar{u}(k+i))$ for $i = \overline{0, N-1}$ and $\bar{x}(k) := x(k)$. Then one can add this cost to Problem 4.2, i.e. at time $k \in \mathbb{Z}_+$ measure the state $x(k)$ and minimize $J_{\text{RHC}}(x(k), \bar{\mathbf{u}}(k)) + J(\lambda_1(k), \dots, \lambda_E(k))$ over $\bar{\mathbf{u}}(k), \lambda_1(k), \dots, \lambda_E(k)$, subject to constraints (11) and $\bar{x}(k+i) \in \mathbb{X}$, $i = \overline{2,N}$. Observe that the optimum needs not to be attained at each sampling instant to achieve ISS, which is appealing for practical reasons but also in the case of a possibly discontinuous value function. □

Remark 4.7 Besides enhancing robustness, the constraints (11b)-(11c) also ensure that Problem 4.2 recovers performance (in terms of settling time) when the state of the closed-loop system approaches the origin. Loosely speaking, when $x(k) \approx 0$, solving Problem 4.2 will produce a control action $u(k) \approx 0$ (because of constraint (11b) and the fact that the cost $J_{\text{RHC}}(\cdot) + J(\cdot)$ is minimized). This yields $V(\phi(0,0,w^e)) - \lambda_e(k) \leq 0$, $e = \overline{1,E}$, due to constraint (11c). Thus, solving Problem 4.2 with the above cost will not optimize each variable $\lambda_e(k)$ below the corresponding value $V(\phi(0,0,w^e))$, $e = \overline{1,E}$, when the state reaches the equilibrium. This property is desirable, since it is known from min-max MPC [11] that considering a worst case disturbance scenario leads to poor performance when the real disturbance is small or vanishes. □

4.3 Decentralized formulation

In this paragraph we give a brief outline of how the proposed self-optimizing MPC algorithm can be implemented in a decentralized fashion. We consider a connected directed graph $\mathcal{G} = (\mathcal{S}, \mathcal{E})$ with a finite number of vertices $\mathcal{S}$ and a set of directed edges $\mathcal{E} \subseteq \{(i,j) \in \mathcal{S} \times \mathcal{S} \mid i \neq j\}$. A dynamical system is assigned to each vertex $i \in \mathcal{S}$, with the dynamics governed by the following equation:

$$x_i(k+1) = \phi_i(x_i(k), u_i(k), v_i(x_{\mathcal{N}_i}(k)), w_i(k)), \quad k \in \mathbb{Z}_+. \tag{13}$$

In (13), $x_i \in \mathbb{X}_i \subset \mathbb{R}^{n_i}$, $u_i \in \mathbb{U}_i \subset \mathbb{R}^{m_i}$ are the state and the control input of the i-th system, and $w_i \in \mathbb{W}_i \subset \mathbb{R}^{l_i}$ is an exogenous disturbance input that directly affects only the i-th system. With each directed edge $(j,i) \in \mathcal{E}$ we associate a function $v_{ij} : \mathbb{R}^{n_j} \to \mathbb{R}^{n_i}$, which defines the interconnection signal $v_{ij}(x_j(k)), k \in \mathbb{Z}_+$, between system j and system i, i.e. $v_{ij}(\cdot)$ characterizes how the states of system j influence the dynamics of system i. The set $\mathcal{N}_i := \{j \mid (j,i) \in \mathcal{E}\}$ denotes the set of direct neighbors (observe that $j \in \mathcal{N}_i \not\Leftrightarrow i \in \mathcal{N}_j$) of the system i. For simplicity of notation we use $x_{\mathcal{N}_i}(k)$ and $v_i(x_{\mathcal{N}_i}(k))$ to denote $\{x_j(k)\}_{j \in \mathcal{N}_i}$ and $\{v_{ij}(x_j(k))\}_{j \in \mathcal{N}_i}$, respectively. Both $\phi_i(\cdot,\cdot,\cdot,\cdot)$ and $v_{ij}(\cdot)$ are arbitrary nonlinear, possibly discontinuous functions that satisfy $\phi_i(0,0,0,0) = 0$, $v_{ij}(0) = 0$ for all $(i,j) \in \mathcal{S} \times \mathcal{N}_i$. For all $i \in \mathcal{S}$ we assume that $\mathbb{X}_i$, $\mathbb{U}_i$ and $\mathbb{W}_i$ are compact sets that contain the origin in their interior.

Assumption 4.8 The value of all interconnection signals $v_{ij}(x_j(k))$ is known at all discrete-time instants $k \in \mathbb{Z}_+$ for any system $i \in \mathcal{S}$.

From a technical point of view, Assumption 4.8 is satisfied, e.g., if all interconnection signals $v_{ij}(x_j(k))$ are directly measurable at all $k \in \mathbb{Z}_+$ or, if all directly neighboring systems $j \in \mathcal{N}_i$ are able to communicate their local measured state $x_j(k)$ to system $i \in \mathcal{S}$. Consider next the following decentralized version of Problem 4.2, where the notation and definitions employed so far are carried over *mutatis mutandis*.

Problem 4.9 For system $i \in \mathcal{S}$ let $\alpha_3^i \in \mathcal{K}_\infty$, $J_i(\cdot)$ and a CLF $V_i(\cdot)$ be given. At time $k \in \mathbb{Z}_+$ measure the local state $x_i(k)$ and the interconnection signals $v_i(x_{\mathcal{N}_i}(k))$ and minimize the cost $J_i(\lambda_1^i(k), \ldots, \lambda_{E_i}^i(k))$ over $u_i(k), \lambda_1^i(k), \ldots, \lambda_{E_i}^i(k)$, subject to the constraints

$$u_i(k) \in \mathbb{U},\ \lambda_e^i(k) \geq 0,\ \phi_i(x_i(k), u_i(k), v_i(x_{\mathcal{N}_i}(k)), 0) \in \mathbb{X}_i \sim \mathbb{W}_{x_i(k)}, \tag{14a}$$

$$V_i(\phi_i(x_i(k), u_i(k), v_i(x_{\mathcal{N}_i}(k)), 0)) - V_i(x_i(k)) + \alpha_3^i(\|x_i(k)\|) \leq 0, \tag{14b}$$

$$V_i(\phi_i(x_i(k), u_i(k), v_i(x_{\mathcal{N}_i}(k)), w_i^e)) - V_i(x_i(k)) + \alpha_3^i(\|x_i(k)\|) - \lambda_e^i(k) \leq 0,$$
$$\forall e = \overline{1, E_i}. \tag{14c}$$

□

Let $\pi_i(x_i(k), v_i(x_{\mathcal{N}_i}(k))) := \{u_i(k) \in \mathbb{R}^{m_i} \mid (14) \text{ holds}\}$ and let

$$\begin{aligned} x_i(k+1) \in &\phi_i^{\mathrm{cl}}(x_i(k), \pi_i(x_i(k), v_i(x_{\mathcal{N}_i}(k)), v_i(x_{\mathcal{N}_i}(k)), w_i(k)) \\ &:= \{\phi_i(x_i(k), u, v_i(x_{\mathcal{N}_i}(k)), w_i(k)) \mid u \in \pi_i(x_i(k), v_i(x_{\mathcal{N}_i}(k)))\} \end{aligned}$$

denote the difference inclusion corresponding to system (13) in "closed-loop" with the set of feasible solutions obtained by solving Problem 4.9 at each $k \in \mathbb{Z}_+$.

Theorem 4.10 *Let, $\alpha_1^i, \alpha_2^i, \alpha_3^i \in \mathcal{K}_\infty$ of the form specified in Theorem 2.3, continuous and convex CLFs $V_i(\cdot)$ and costs $J_i(\cdot)$ be given for all systems indexed by $i \in \mathcal{S}$. Suppose Assumption 4.8 holds and Problem 4.9 is feasible for each system $i \in \mathcal{S}$ and for all states x_i in $\mathbb{X}_i$ and all corresponding $v_i(x_{\mathcal{N}_i})$. Then the interconnected dynamically coupled nonlinear system described by the collection of difference inclusions*

$$x_i(k+1) \in \phi_i^{cl}(x_i(k), \pi_i(x_i(k), v_i(x_{\mathcal{N}_i}(k)), v_i(x_{\mathcal{N}_i}(k)), w_i(k)), \ i \in \mathcal{S}, \ k \in \mathbb{Z}_+ \tag{15}$$

is ISS($\mathbb{X}_1 \times \ldots \times \mathbb{X}_S, \mathbb{W}_1 \times \ldots \times \mathbb{W}_S$).

The proof is omitted due to space limitations. Its central argument is that each continuous and convex CLF $V_i(x_i)$ is in fact Lipschitz continuous on $\mathbb{X}_i$ [12], which makes $\sum_{i\in\mathcal{S}} V_i(x_i) =: V(\{x_i\}_{i\in\mathcal{S}})$ a Lipschitz continuous CLF for the global interconnected system. The result then follows similarly to the proof of Theorem 2.6-(ii). Theorem 4.10 guarantees a constant ISS gain for the global closed-loop system, while the ISS gain of each closed-loop system $i \in \mathcal{S}$ can still be optimized on-line.

Remark 4.11 Problem 4.9 defines a set of *decoupled* optimization problems, implying that the computation of control actions can be performed in completely decentralized fashion, i.e. with no communication among controllers (if each $v_{ij}(\cdot)$ is measurable at all $k \in \mathbb{Z}_+$). Inequality (14b) can be further significantly relaxed by replacing the zero on the righthand side with an optimization variable $\tau_i(k)$ and adding the *coupling constraint* $\sum_{i\in\mathcal{S}} \tau_i(k) \leq 0$ for all $k \in \mathbb{Z}_+$. Using the dual decomposition method, see e.g. [13], it is then possible to devise a *distributed* control scheme, which yields an optimized ISS-gain of the global interconnected system in the sense that $\sum_{i\in\mathcal{S}} J_i(\cdot)$ is minimized. Further relaxations can be obtained by asking that the sum of $\tau_i(k)$ is non-positive over a finite horizon, rather than at each time step. □

4.4 Implementation issues

In this section we briefly discuss the ingredients, which make it possible to implement Problem 4.2 (or its corresponding decentralized version Problem 4.9) as a single linear or quadratic program. Firstly, we consider nonlinear systems of the form (4) that are affine in control. Then it makes sense that there exist functions $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ with $f_1(0) = 0$ and $f_2 : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ such that:

$$x(k+1) = \phi(x(k), u(k), w(k)) := f_1(x(k)) + f_2(x(k))u(k) + g(x(k))w(k). \tag{16}$$

Secondly, we restrict our attention to CLFs defined using the ∞-norm, i.e. $V(x) := \|Px\|_\infty$, where $P \in \mathbb{R}^{p\times n}$ is a matrix (to be determined) with full-column rank. We refer to [14] for techniques to compute CLFs based on norms.

Then, the first step is to show that the ISS inequalities (11b)-(11c) can be specified, without introducing conservatism, via a finite number of linear inequalities. Since by definition $\|x\|_\infty = \max_{i\in\mathbb{Z}_{[1,n]}} |[x]_i|$, for a constraint $\|x\|_\infty \leq c$ with $c > 0$ to be satisfied, it is *necessary and sufficient* to require that $\pm[x]_i \leq c$ for all $i \in \mathbb{Z}_{[1,n]}$. Therefore, as $x(k)$ in (11) is the measured state, which is known at every $k \in \mathbb{Z}_+$, for (11b)-(11c) to be satisfied it is necessary and sufficient to require that:

$$\begin{aligned}
&\pm [P(f_1(x(k)) + f_2(x(k))u(k))]_i - V(x(k)) + \alpha_3(\|x(k)\|) \leq 0 \\
&\pm [P(f_1(x(k)) + f_2(x(k))u(k) + g(x(k))w^e)]_i - V(x(k)) + \alpha_3(\|x(k)\|) - \lambda_e(k) \leq 0, \\
&\forall i \in \mathbb{Z}_{[1,p]}, \ e = \overline{1,E},
\end{aligned}$$

which yields $2p(E+1)$ linear inequalities in the variables $u(k), \lambda_1(k), \ldots, \lambda_E(k)$. If the sets $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{W}_{x(k)}$ are polyhedra, which is a reasonable assumption, then clearly the inequalities in (11a) are also linear in $u(k), \lambda_1(k), \ldots, \lambda_E(k)$. Thus, a solution to Problem 4.2, *including minimization of the cost* $J_{RHC}(\cdot) + J(\cdot)$ *for any* $N \in \mathbb{Z}_{\geq 1}$, can be obtained by solving a nonlinear optimization problem subject to linear constraints.

Following some straightforward manipulations [10], the optimization problem to be solved on-line can be further simplified as follows. If the model is (i) piecewise affine or (ii) affine and the cost functions $J_{\text{RHC}}(\cdot)$ and $J(\cdot)$ are defined using quadratic forms or infinity norms, then a solution to Problem 4.2 (with the cost $J_{\text{RHC}}(\cdot) + J(\cdot)$) can be obtained by solving (i) a single mixed integer quadratic or linear program (MIQP - MILP), or (ii) a single QP - LP, respectively, for any $N \in \mathbb{Z}_{\geq 1}$. Alternatively, for $N = 1$ and quadratic or ∞-norm based costs, Problem 4.2 can be formulated as a single QP or LP for any discrete-time nonlinear model that is affine in the control variable and the disturbance input.

5 Illustrative example

Consider the nonlinear system (13) with $\mathcal{S} = \{1, 2\}$, $\mathcal{N}_1 = \{2\}$, $\mathcal{N}_2 = \{1\}$, $\mathbb{X}_1 = \mathbb{X}_2 = \{\xi \in \mathbb{R}^2 \mid \|\xi\|_\infty \leq 5\}$, $\mathbb{U}_1 = \mathbb{U}_2 = \{\xi \in \mathbb{R} \mid |\xi| \leq 2\}$ and $\mathbb{W}_1 = \mathbb{W}_2 = \{\xi \in \mathbb{R}^2 \mid \|\xi\|_1 \leq 0.2\}$. The dynamics are given by:

$$\phi_1(x_1, u_1, v_1(x_{\mathcal{N}_1}), w_1) := \begin{bmatrix} 1 & 0.7 \\ 0 & 1 \end{bmatrix} x_1 + \begin{bmatrix} \sin([x_1]_2) \\ 0 \end{bmatrix} + \begin{bmatrix} 0.245 \\ 0.7 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ ([x_2]_1)^2 \end{bmatrix} + w_1, \tag{17a}$$

$$\phi_2(x_2, u_2, v_2(x_{\mathcal{N}_2}), w_2) := \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} x_2 + \begin{bmatrix} \sin([x_2]_2) \\ 0 \end{bmatrix} + \begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ [x_1]_1 \end{bmatrix} + w_2. \tag{17b}$$

The technique of [14] was used to compute the weights $P_1, P_2 \in \mathbb{R}^{2\times 2}$ of the CLFs $V_1(x) = \|P_1 x\|_\infty$ and $V_2(x) = \|P_2 x\|_\infty$ for $\alpha_3^1(s) = \alpha_3^2(s) := 0.01s$ and the linearizations of (17a), (17b), respectively, around the origin, in closed-loop with $u_1(k) := K_1 x_1(k)$, $u_2(k) := K_2 x_2(k)$, $K_1, K_2 \in \mathbb{R}^{2\times 1}$, yielding

$$P_1 = \begin{bmatrix} 1.3204 & 0.6294 \\ 0.5629 & 2.0811 \end{bmatrix}, \ K_1 = \begin{bmatrix} -0.2071 & -1.2731 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1.1356 & 0.5658 \\ 0.7675 & 2.1356 \end{bmatrix}, \ K_2 = \begin{bmatrix} -0.3077 & -1.4701 \end{bmatrix}.$$

Note that the control laws $u_1(k) = K_1 x(k)$ and $u_2(k) = K_2 x_2(k)$ are only employed off-line, to calculate the weight matrices P_1, P_2 and they are never used for controlling the system. To optimize robustness, 4 optimization variables $\lambda_1^i(k), \ldots, \lambda_4^i(k)$ were introduced for each system, each one assigned to a vertex of the set $\mathbb{W}_i$, $i = 1, 2$, respectively. The following cost functions were employed in the optimization problem, as specified in Remark 4.6: $J_{\text{RHC}}^i(x_i(k), u_i(k)) := \|Q_1^i \phi_i(x_i, u_i, v_i(x_{\mathcal{N}_i}), 0)\|_\infty + \|Q^i x_i(k)\|_\infty + \|R^i u_i(k)\|_\infty$, $J_i(\lambda_1^i(k), \ldots, \lambda_4^i(k)) := \Gamma^i \sum_{j=1}^4 |\lambda_j^i(k)|$, where $i = 1, 2$, $Q_1^1 = Q_1^2 = 4I_2$, $Q^1 = Q^2 = 0.1I_2$, $R^1 = R^2 = 0.4$, $\Gamma^1 = 1$ and $\Gamma^2 = 0.1$. For each system, the resulting linear program has 7 optimization variables and 42 constraints. During the simulations, *the worst case computational time required by the CPU* (Pentium 4, 3.2GHz, 1GB RAM) *over* 400 *runs was* 5 *milliseconds*, which shows the potential for controlling networks of fast nonlinear systems. In the simulation scenario we tested the closed-loop system response for $x_1(0) = [3, \ -1]^\top$,

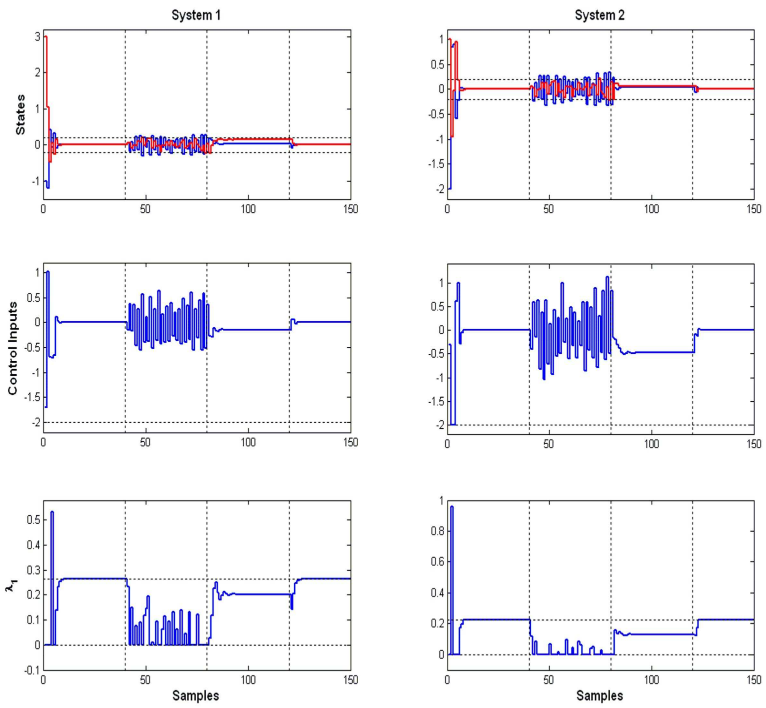

Figure 1: States, inputs and first optimization variable histories for each system.

$x_2(0) = [1,\ -2]^\top$ and for the following disturbance scenarios: $w_1(k) = w_2(k) = [0,\ 0]^\top$ for $k \in \mathbb{Z}_{[0,40]}$ (nominal stabilization), $w_i(k)$ takes random values in $\mathbb{W}_i$, $i = 1, 2$, for $k \in \mathbb{Z}_{[41,80]}$ (robustness to random inputs), $w_1(k) = w_2(k) = [0,\ 0.1]^\top$ for $k \in \mathbb{Z}_{[81,120]}$ (robustness to constant inputs) and $w_1(k) = w_2(k) = [0,\ 0]^\top$ for $k \in \mathbb{Z}_{[121,160]}$ (to show that asymptotic stability is recovered for zero inputs).

In Figure 1 the time history of the states, control input and the optimization variables $\lambda_1^1(k)$ and $\lambda_1^2(k)$, assigned to $w_1^1 = w_2^1 = [0,\ 0.2]^\top$, are depicted for each system. In the state trajectories plots, the dashed horizontal lines give an approximation of the bounded region in which the system's states remain despite disturbances, i.e. approximately within the interval $[-0.2,\ 0.2]$. In the input trajectory plots the dashed line shows the input constraints. In all plots, the dashed vertical lines delimit the time intervals during which one of the four disturbance scenarios is active. One can observe that the feedback to disturbances is provided actively, resulting in good robust performance, while state and input constraints are satisfied at all times, despite the strong nonlinear coupling present. In the λ_1 plot, one can see that whenever the disturbance is acting on the system, or when the state is far from the origin (in the first disturbance scenario), these variables act to optimize the decrease of each $V_i(\cdot)$ and to counteract the influence of the interconnecting signal. Whenever the equilibrium is reached, the optimization variables satisfy the constraint $V_i(\phi_i(0, 0, w_i^e)) \leq \lambda_e^i(k),\ e = 1, \ldots, 4$, as explained in Remark 4.7. In Figure 1, the λ_1 plot, the values $V_1(\phi_1(0, 0, w_1^1)) = 0.2641$ and $V_2(\phi_2(0, 0, w_2^1)) = 0.2271$ are depicted with dashed horizontal lines.

6 Conclusions

In this article we studied the design of robust MPC schemes with focus on adapting the closed-loop ISS gain on-line, in a receding horizon fashion. Exploiting convex CLFs and disturbance bounds, we were able to construct a finite dimensional optimization problem that allows for the simultaneous on-line (i) computation of a control action that achieves ISS, and (ii) minimization of the ISS gain of the resulting closed-loop system depending on the actual state trajectory. As a consequence, the proposed robust nonlinear MPC algorithm is self-optimizing in terms of disturbance attenuation. Solutions for establishing recursive feasibility and for decentralized implementation have also been briefly presented. Furthermore, we indicated a design recipe that can be used to implement the developed self-optimizing MPC scheme as a single linear program, for nonlinear systems that are affine in the control variable and the disturbance input. This brings the application to (networks of) fast nonlinear systems within reach.

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