Infinity Norms as Lyapunov functions for Model Predictive Control of Constrained PWA Systems

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Abstract. In this paper we develop *a priori* stabilization conditions for infinity norm based hybrid MPC in the terminal cost and constraint set fashion. Closed-loop stability is achieved using infinity norm inequalities that guarantee that the value function corresponding to the MPC cost is a Lyapunov function of the controlled system. We show that Lyapunov asymptotic stability can be achieved even though the MPC value function may be discontinuous. One of the advantages of this hybrid MPC scheme is that the terminal constraint set can be directly obtained as a sublevel set of the calculated terminal cost, which is also a local piecewise linear Lyapunov function. This yields a new method to obtain positively invariant sets for PWA systems.

1 Introduction

Hybrid systems provide a unified framework for modeling complex processes that include both continuous and discrete dynamics. The large variety of practical situations where hybrid systems are encountered (e.g., physical processes interacting with discrete actuators) led to an increasing interest in modeling and control of hybrid systems. Several modeling formalisms have been developed for describing hybrid systems, such as Mixed Logical Dynamical (MLD) systems [1] or Piecewise Affine (PWA) systems [2], and several control strategies have been proposed for relevant classes of hybrid systems. Many of the control schemes for hybrid systems are based on optimal control, e.g., like the ones in [3], [4], or on Model Predictive Control (MPC), e.g., as the ones in [1], [5], [6], [7]. In this paper we focus on the implementation of MPC for constrained PWA systems. This is motivated by the fact that PWA systems can model a broad class of hybrid systems, as shown in [8].

The implementation of MPC for hybrid systems faces two difficult problems: how to reduce the computational complexity of the constrained optimization problem that has to be solved on-line and how to guarantee closed-loop stability. Most of the MPC algorithms are based on the optimization of a cost function which is defined using either quadratic forms or infinity norms. If a quadratic form is used to define the cost function, the MPC constrained optimization problem becomes a Mixed Integer Quadratic Programming (MIQP) problem, e.g., see [1] for details. This choice has led to fruitful results with respect to the stability problem of hybrid MPC, mainly due to the fact that in this case, the stabilization conditions can be reduced to a set of Linear Matrix Inequalities (LMI). Such results have been initially derived in the context of state feedback stabilization of PWA systems, as done in [3], [9]. The extension of the terminal cost and constraint set method for guaranteeing stability in MPC (e.g., see [10] for details) to the class of constrained PWA systems has been worked out in [7]. The terminal weight is calculated in [7] using semi-definite programming and the terminal state is constrained to a polyhedral positively invariant set in order to guarantee stability.

In the case when the infinity norm is used to define the cost function, the MPC constrained optimization problem leads to a Mixed Integer Linear Programming (MILP) problem, as pointed out in [5]. A piecewise affine explicit solution to this problem can be obtained using multi-parametric programming, as shown in [5], [6], [11], which may result in a reduction of the on-line computational complexity (one still has to check in which state space region the measured state resides). Regarding the stability problem, an a priori heuristic test for guaranteeing stability of infinity norm based MPC of PWA systems has been developed in [5]. Recently, an *a posteriori* procedure for guaranteeing stability of hybrid systems with a linear performance index has been derived in [12] by analyzing the explicit PWA closed-loop system. Another option to guarantee stability is to impose a terminal *equality* constraint, as done in [1] for hybrid MPC based on a quadratic form. However, this method has the disadvantage that the system must be brought to the origin in finite time, over the prediction horizon (this requires that the PWA system is controllable, while stabilizability should be sufficient in general). As a result, a longer prediction horizon may be needed for ensuring feasibility of the MPC optimization problem (fact which increases the computational burden). Also, the terminal equality constraint is only proven to guarantee attractivity. Lyapunov stability [13], next to attractivity, is a desirable property from a practical point of view.

In this paper we guarantee asymptotic stability (including Lyapunov stability) for infinity norm based hybrid MPC in the terminal cost and constraint set fashion. *A priori* stabilization conditions are developed using infinity norm inequalities, in contrast with the *a posteriori* verification proposed in [12]. If the considered infinity norm inequalities are satisfied, then the value function of the MPC cost is a Lyapunov function of the controlled PWA system. We show that Lyapunov asymptotic stability can be achieved even though the MPC value function may be discontinuous. This fact has been pointed out in [14] for nonlinear discrete-time systems and it has been used in [9] to derive a state-feedback based stabilizing controller for discrete-time PWA systems. We calculate the terminal weight by solving off-line an optimization problem. Several two-step methods to transform this problem into a Linear Programming (LP) problem are also presented. The terminal constraint set can be automatically obtained as a polyhedron (or as a finite union of polyhedra) by simply taking one of the sublevel sets of the calculated terminal cost, which is a local piecewise linear Lyapunov function. Then the MPC constrained optimization problem that has to be solved on-line still leads to a MILP problem.

The paper is organized as follows. Section 2 deals with preliminary notions and Section 3 provides a precise problem formulation. The main result concerning infinity norms as Lyapunov functions for MPC of constrained PWA systems is presented in Section 4. Several possibilities to obtain the terminal weight matrix are indicated in Section 5 and relaxations are developed in Section 6. The conclusions are summarized in Section 7.

2 Preliminaries

Consider the time-invariant discrete-time autonomous nonlinear system described by

$$x_{k+1} = g(x_k), \tag{1}$$

where $g(\cdot)$ is an arbitrary nonlinear function.

Definition 1. *Given λ, $0 \leq \lambda \leq 1$, a set $\mathcal{P} \subset \mathbb{R}^n$ is a* λ-contractive set *for system* (1) *if for all $x \in \mathcal{P}$ it holds that $g(x) \in \lambda\mathcal{P}$. For $\lambda = 1$ a λ-contractive set is called a* positively invariant set*.*

A polyhedron is a convex set obtained as the intersection of a finite number of open and/or closed half-spaces. Moreover, a convex and compact set in $\mathbb{R}^n$ that contains the origin in its interior is called a C-set [15].

For a vector $x \in \mathbb{R}^n$ we define $\|x\|_\infty := \max_{i=1,\ldots,n} |x_i|$, where x_i is the i-th component of x, and for a matrix $Z \in \mathbb{R}^{m\times n}$ we define

$$\|Z\|_\infty \triangleq \sup_{x\neq 0} \frac{\|Zx\|_\infty}{\|x\|_\infty}.$$

It is well known [16] that $\|Z\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |Z^{\{ij\}}|$, where $Z^{\{ij\}}$ is the ij-th entry of Z. Also, for a matrix $Z \in \mathbb{R}^{m\times n}$ with full-column rank we define $Z^{-L} := (Z^\top Z)^{-1} Z^\top$, which is a left inverse of Z (i.e. $Z^{-L}Z = I_n$).

3 Problem statement

Consider the time-invariant discrete-time PWA system [2] described by equations of the form

$$x_{k+1} = A_j x_k + B_j u_k + f_j \quad \text{when} \quad x_k \in \Omega_j. \tag{2}$$

Here, $x_k \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state and $u_k \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control input at the discrete-time instant $k \geq 0$. $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$, $f_j \in \mathbb{R}^n$, $j \in \mathcal{S}$

with $\mathcal{S} := \{1, 2, \ldots, s\}$ and s denoting the number of discrete modes. The sets $\mathbb{X}$ and $\mathbb{U}$ specify state and input constraints and it is assumed that they are polyhedral C-sets. The collection $\{\Omega_j \mid j \in \mathcal{S}\}$ defines a partition of $\mathbb{X}$, meaning that $\cup_{j\in\mathcal{S}}\Omega_j = \mathbb{X}$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Each Ω_j is assumed to be a polyhedron (not necessarily closed). Let $\mathcal{S}_0 := \{j \in \mathcal{S} \mid 0 \in \mathrm{cl}(\Omega_j)\}$ and let $\mathcal{S}_1 := \{j \in \mathcal{S} \mid 0 \notin \mathrm{cl}(\Omega_j)\}$, where $\mathrm{cl}(\Omega_j)$ denotes the closure of Ω_j. Note that $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$. In the sequel we assume that the origin is an equilibrium state for (2) with $u = 0$, and therefore, we require that

$$f_j = 0 \text{ for all } j \in \mathcal{S}_0. \tag{3}$$

Note that the class of hybrid systems described by (2)-(3) contains PWA systems which *may be discontinuous over the boundaries* and which are PWL in the regions whose closure contains the origin. The goal of this paper is to develop for system (2) an *asymptotically stabilizing* infinity norm based MPC scheme that leads to a MILP problem. For a fixed $N \in \mathbb{N}$, $N \geq 1$, let $\mathbf{x}_k(x_k, \mathbf{u}_k) = (x_{k+1}, \ldots, x_{k+N})$ denote a state sequence generated by system (2) from initial state x_k and by applying the input sequence $\mathbf{u}_k := (u_k, \ldots, u_{k+N-1}) \in \mathbb{U}^N$. Furthermore, let $\mathbb{X}_N \subseteq \mathbb{X}$ denote a desired target set that contains the origin.

Definition 2. *The class of* admissible input sequences *defined with respect to* $\mathbb{X}_N$ *and state* $x_k \in \mathbb{X}$ *is* $\mathcal{U}_N(x_k) := \{\mathbf{u}_k \in \mathbb{U}^N \mid \mathbf{x}_k(x_k, \mathbf{u}_k) \in \mathbb{X}^N,\ x_{k+N} \in \mathbb{X}_N\}$.

Stated differently, the input sequence $\mathbf{u}_k \in \mathbb{U}^N$ is admissible with respect to $\mathbb{X}_N$ and $x_k \in \mathbb{X}$ if the following conditions are satisfied:

$$x_{k+1+i} = A_j x_{k+i} + B_j u_{k+i} + f_j \text{ when } x_{k+i} \in \Omega_j, \tag{4a}$$
$$u_{k+i} \in \mathbb{U}, \quad x_{k+i} \in \mathbb{X} \quad \text{for } i = 0, \ldots, N-1, \tag{4b}$$
$$x_{k+N} \in \mathbb{X}_N. \tag{4c}$$

Now consider the following problem.

Problem 1. At time $k \geq 0$ let $x_k \in \mathbb{X}$ be given. Minimize the cost function

$$J(x_k, \mathbf{u}_k) \triangleq \|Px_{k+N}\|_\infty + \sum_{i=0}^{N-1} \|Qx_{k+i}\|_\infty + \|Ru_{k+i}\|_\infty \tag{5}$$

over all input sequences $\mathbf{u}_k \in \mathcal{U}_N(x_k)$.

Here, N denotes the prediction horizon, and $P \in \mathbb{R}^{p\times n}$, $Q \in \mathbb{R}^{q\times n}$ and $R \in \mathbb{R}^{r\times m}$ are matrices which have full-column rank. The rank condition is necessary in order to ensure that $\|Px\|_\infty \neq 0$ for $x \neq 0$. We call an initial state $x_k \in \mathbb{X}$ *feasible* if $\mathcal{U}_N(x_k) \neq \emptyset$. Similarly, Problem 1 is said to be feasible (or *solvable*) for $x_k \in \mathbb{X}$ if $\mathcal{U}_N(x_k) \neq \emptyset$. Let

$$V_{\mathrm{MPC}}(x_k) \triangleq \min_{\mathbf{u}_k \in \mathcal{U}_N(x_k)} J(x_k, \mathbf{u}_k) \tag{6}$$

denote the value function corresponding to (5) and consider an optimal sequence of controls calculated for state $x_k \in \mathbb{X}$ by solving Problem 1, i.e.,

$$\mathbf{u}_k^* \triangleq (u_k^*, u_{k+1}^*, \ldots, u_{k+N-1}^*), \tag{7}$$

which minimizes (5). Let $\mathbf{u}_k^*(1)$ denote the first element of the sequence (7). According to the receding horizon strategy, the MPC control law is defined as

$$u_k^{\text{MPC}} = \mathbf{u}_k^*(1); \quad k \in \mathbb{Z}_+. \tag{8}$$

A more precise problem formulation can now be stated as follows.

Problem 2. Given Q, R and system (2) the objective is to determine P, N and $\mathbb{X}_N$ such that system (2) in closed-loop with the MPC control (8) is asymptotically stable in the Lyapunov sense and Problem 1 leads to a MILP problem.

Note that many of the hybrid MPC schemes only guarantee attractivity, e.g., see [1], [5], and not Lyapunov stability, which is important in practice (we thank the reviewer for this remark).

Remark 1. A partial solution to Problem 2 has been presented in [5], where a test criterion has been developed to *a priori* guarantee attractivity of the origin for the closed-loop system. Unfortunately, the results of [5] did not yield a systematic way for calculating the matrix P, but only a heuristic procedure. Another option to guarantee stability in infinity norm based hybrid MPC is to perform an *a posteriori* check of stability, after computing (8) as an explicit PWA control law, as it has been done in [12].

4 Infinity norms as Lyapunov functions for hybrid MPC

In order to solve Problem 2 we aim at using the value function (6) as a candidate Lyapunov function for the closed-loop system (2)-(8) and we employ a terminal cost and constraint set method [10]. We also consider an auxiliary PWL control action of the form

$$\tilde{u}_k \triangleq K_j x_k, \quad x_k \in \Omega_j, \quad k \in \mathbb{Z}_+, \quad K_j \in \mathbb{R}^{m \times n}, \quad j \in \mathcal{S}. \tag{9}$$

Let $\mathbb{X}_{\mathbb{U}} := \cup_{j \in \mathcal{S}} \{x \in \Omega_j \mid K_j x \in \mathbb{U}\}$ denote the safe set with respect to *state and input* constraints for this controller and let $\mathbb{X}_N \subseteq \mathbb{X}_{\mathbb{U}}$ be a positively invariant set for the PWA system (2) in closed-loop with (9). In the sequel we require that $\mathbb{X}_N$ contains the origin in its interior. Now consider the following inequalities:

$$\|P(A_j + B_j K_j)P^{-L}\|_\infty + \|QP^{-L}\|_\infty + \|RK_j P^{-L}\|_\infty \leq 1 - \gamma_j, \quad j \in \mathcal{S} \tag{10}$$

and

$$\|Pf_j\|_\infty \leq \gamma_j \|Px\|_\infty, \quad \forall x \in \mathbb{X}_N \cap \Omega_j, \quad j \in \mathcal{S}, \tag{11}$$

where $\{\gamma_j \mid j \in \mathcal{S}\}$ are scaling factors that satisfy $0 \leq \gamma_j < 1$ for all $j \in \mathcal{S}$. Note that, because of (3), (11) trivially holds if $\mathcal{S} = \mathcal{S}_0$.

Theorem 1. *Suppose* (10)-(11) *is solvable in* (P, K_j, γ_j) *for* P *with full-column rank and* $j \in \mathcal{S}$, $\mathbb{X}_N \subseteq \mathbb{X}_\mathbb{U}$ *is a positively invariant set for the closed-loop system* (2)-(9) *that contains the origin in its interior and fix* $N \geq 1$. *Then it holds that*

1. *If Problem 1 is feasible at time* $k \in \mathbb{Z}_+$ *for state* $x_k \in \Omega_j$, *then Problem 1 is feasible at time* $k+1$ *for state* $x_{k+1} = A_j x_k + B_j \mathbf{u}_k^*(1) + f_j$.
2. *The MPC control* (8) *asymptotically stabilizes the PWA system* (2) *for all feasible initial states (including the set* $\mathbb{X}_N$*), while satisfying the state and input constraints* (4).
3. *The origin of the PWA system* (2) *in closed-loop with feedback* (9) *is locally asymptotically stable, while satisfying the state and input constraints* (4).
4. *If* $\mathbb{X} = \mathbb{R}^n$, $\mathbb{U} = \mathbb{R}^m$ *and* (11) *holds for* $\mathbb{X}_N = \mathbb{R}^n$, *then the origin of the PWA system* (2) *in closed-loop with feedback* (9) *is globally asymptotically stable.*

In order to prove Theorem 1 we will need the following result, the proof of which can be found in the appendix.

Lemma 1. *Consider the closed-loop PWA system* (2)-(9):

$$x_{k+1} = (A_j + B_j K_j)x_k + f_j \quad \text{when} \quad x_k \in \Omega_j, \quad j \in \mathcal{S}. \tag{12}$$

Assume that (11) *is solvable for some* P *with full-column rank. Then for any* $l = 0, 1, 2, \ldots$ *there exists an* $\alpha_l > 0$ *such that for all* $x_k \in \mathbb{X}_N$

$$\|x_{k+l}\|_\infty \leq \alpha_l \|x_k\|_\infty, \tag{13}$$

if $(x_k, x_{k+1}, \ldots, x_{k+l})$ *is a solution of* (12).

Now we prove Theorem 1.

Proof. Consider (7) and the shifted sequence of controls

$$\mathbf{u}_{k+1} \triangleq (u_{k+1}^*, u_{k+2}^*, \ldots, u_{k+N-1}^*, \tilde{u}_{k+N}), \tag{14}$$

where the auxiliary control $\tilde{u}_{k+N}$ denotes the control law (9) at time $k+N$.

1) If Problem 1 is feasible at time $k \in \mathbb{Z}_+$ for state $x_k \in \Omega_j$ then there exists $\mathbf{u}_k^* \in \mathcal{U}_N(x_k)$ that solves Problem 1. Then x_{k+N} satisfies constraint (4c). Since $\mathbb{X}_N \subseteq \mathbb{X}_\mathbb{U}$ is positively invariant for system (2) in closed-loop with (9), it follows that $\mathbf{u}_{k+1} \in \mathcal{U}_N(x_{k+1})$. Hence, Problem 1 is feasible for state $x_{k+1} = A_j x_k + B_j \mathbf{u}_k^*(1) + f_j$. Moreover, all states in the set $\mathbb{X}_N \subseteq \mathbb{X}_\mathbb{U}$ are feasible with respect to Problem 1, as the PWL feedback (9) can be applied for any N.

2) In order to achieve stability we require for all *feasible* initial conditions $x_0 \in \mathbb{X}\backslash\{0\}$ that the forward difference $\Delta V_{\mathrm{MPC}}(x_k) := V_{\mathrm{MPC}}(x_{k+1}) - V_{\mathrm{MPC}}(x_k)$ is strictly negative for all $k \in \mathbb{Z}_+$, which can be written as:

$$\begin{aligned} \Delta V_{\mathrm{MPC}}(x_k) &= J(x_{k+1}, \mathbf{u}_{k+1}^*) - J(x_k, \mathbf{u}_k^*) \leq J(x_{k+1}, \mathbf{u}_{k+1}) - J(x_k, \mathbf{u}_k^*) = \\ &= -\|Qx_k\|_\infty - \|Ru_k^*\|_\infty + \|Px_{k+N+1}\|_\infty + \|R\tilde{u}_{k+N}\|_\infty \\ &+ \|Qx_{k+N}^*\|_\infty - \|Px_{k+N}^*\|_\infty < 0, \quad \forall x_{k+N}^* \in \mathbb{X}_N\backslash\{0\}. \end{aligned} \tag{15}$$

Here, $x_k \in \Omega_j$ is the measured state at the sampling instant k and $x^*_{k+1} = A_j x_k + B_j u^*_k + f_j$. Hence, it suffices to determine the matrix P such that there exists $\tilde{u}_{k+N}$ with

$$\|Px_{k+N+1}\|_\infty + \|R\tilde{u}_{k+N}\|_\infty + \|Qx^*_{k+N}\|_\infty - \|Px^*_{k+N}\|_\infty \leq 0, \, \forall x^*_{k+N} \in \mathbb{X}_N, \tag{16}$$

in order to guarantee that $\Delta V_{\text{MPC}}(x_k) \leq -\|Qx_k\|_\infty$ for all *feasible* initial conditions $x_0 \in \mathbb{X}\backslash\{0\}$. Since Q has full-column rank, there exists a positive number τ such that $\|Qx\|_\infty \geq \tau\|x\|_\infty$ for all $x \in \mathbb{X}$. Hence, it follows that (16) implies that V_{MPC} possesses a *negative definite forward difference* (see [13] for details). Substituting (9) at time $k+N$ and (2) in (16) yields the equivalent

$$\begin{aligned} &\|P(A_j + B_j K_j)x^*_{k+N} + Pf_j\|_\infty + \|RK_j x^*_{k+N}\|_\infty \\ &\qquad + \|Qx^*_{k+N}\|_\infty - \|Px^*_{k+N}\|_\infty \leq 0, \quad \forall x^*_{k+N} \in \mathbb{X}_N \cap \Omega_j, \quad j \in \mathcal{S}. \end{aligned} \tag{17}$$

Now we prove that if (10)-(11) holds, then (17) holds. Since P and $\{K_j \mid j \in \mathcal{S}\}$ satisfy (10) we have that

$$\|P(A_j + B_j K_j)P^{-L}\|_\infty + \|QP^{-L}\|_\infty + \|RK_j P^{-L}\|_\infty + \gamma_j - 1 \leq 0, \quad j \in \mathcal{S}. \tag{18}$$

Right multiplying the inequality (18) with $\|Px^*_{k+N}\|_\infty$ and using the inequality (11) yields:

$$\begin{aligned} 0 &\geq \|P(A_j + B_j K_j)P^{-L}\|_\infty \|Px^*_{k+N}\|_\infty + \|QP^{-L}\|_\infty \|Px^*_{k+N}\|_\infty \\ &\quad + \gamma_j \|Px^*_{k+N}\|_\infty + \|RK_j P^{-L}\|_\infty \|Px^*_{k+N}\|_\infty - \|Px^*_{k+N}\|_\infty \geq \\ &\geq \|P(A_j + B_j K_j)P^{-L}Px^*_{k+N}\|_\infty + \|QP^{-L}Px^*_{k+N}\|_\infty \\ &\quad + \|Pf_j\|_\infty + \|RK_j P^{-L}Px^*_{k+N}\|_\infty - \|Px^*_{k+N}\|_\infty \geq \\ &\geq \|P(A_j + B_j K_j)x^*_{k+N} + Pf_j\|_\infty + \|RK_j x^*_{k+N}\|_\infty \\ &\quad + \|Qx^*_{k+N}\|_\infty - \|Px^*_{k+N}\|_\infty. \end{aligned} \tag{19}$$

Hence, inequality (17) holds and consequently $\Delta V_{\text{MPC}}(x_k) \leq -\tau\|x_k\|_\infty$. Next, we show that $V_{\text{MPC}}(x_k)$ is a positive definite, radially unbounded and decrescent function [13]. From (6) and (5) we have that

$$V_{\text{MPC}}(x_k) \geq \|Qx_k\|_\infty \geq \tau\|x_k\|_\infty, \quad \forall N \geq 1, \quad \forall x \in \mathbb{X}. \tag{20}$$

Hence, $V_{\text{MPC}}(x_k)$ is a *positive definite* and *radially unbounded* function.

For $x_k \in \mathbb{X}_N$ we have that the control law $\tilde{u}_k = K_j x_k$ when $x_k \in \mathbb{X}_N \cap \Omega_j$ is admissible. Then it follows that the control sequence $\tilde{\mathbf{u}}_k := (\tilde{u}_k, \ldots, \tilde{u}_{k+N-1}) \in \mathbb{U}^N$ is contained in $\mathcal{U}_N(x_k)$. Since there always exist some positive constants γ_P and γ_Q such that $\|Px_k\|_\infty \leq \gamma_P\|x_k\|_\infty$ and $\|Qx_k\|_\infty \leq \gamma_Q\|x_k\|_\infty$ (e.g.,

$\gamma_P = \|P\|_\infty$ and $\gamma_Q = \|Q\|_\infty$), we have that

$$V_{\mathrm{MPC}}(x_k) \leq J(x_k, \tilde{\mathbf{u}}_k) = \|Px_{k+N}\|_\infty + \sum_{i=0}^{N-1} \|Qx_{k+i}\|_\infty + \|RK_{j_i}x_{k+i}\|_\infty \leq$$
$$\leq \gamma_P \|x_{k+N}\|_\infty + (\gamma_Q + \kappa) \sum_{i=0}^{N-1} \|x_{k+i}\|_\infty, \quad \forall x_k \in \mathbb{X}_N, \tag{21}$$

where $\kappa = \max_{j\in\mathcal{S}} \|RK_j\|_\infty$ and $j_i \in \mathcal{S}$ is such that $x_{k+i} \in \Omega_{j_i}$. From Lemma 1 it follows that there exist constants $\alpha_i > 0$ such that $\|x_{k+i}\|_\infty \leq \alpha_i \|x_k\|_\infty$, $i = 1, \ldots, N$, and by letting $\beta := \gamma_P \alpha_N + (\gamma_Q + \kappa)(1 + \sum_{i=1}^{N-1} \alpha_i)$ it follows that

$$V_{\mathrm{MPC}}(x_k) \leq \beta \|x_k\|_\infty, \quad \forall x_k \in \mathbb{X}_N.$$

Hence, $V_{\mathrm{MPC}}(x_k)$ is a *decrescent* function [13] on $\mathbb{X}_N$ (note that $\mathbb{X}_N$ contains the origin in its interior). Since $V_{\mathrm{MPC}}(x_k)$ is also positive definite it follows that

$$\tau \|x_k\|_\infty \leq V_{\mathrm{MPC}}(x_k) \leq \beta \|x_k\|_\infty, \quad \forall x_k \in \mathbb{X}_N. \tag{22}$$

Then, by applying the reasoning used in the proof of Theorem 3 and Theorem 4 from [9] (note that for any $\epsilon > 0$ we can choose $\delta = (\tau/\beta)\epsilon < \epsilon$ and hence, continuity of $V_{\mathrm{MPC}}(x_k)$ is not necessary, see [9] and [13] for details) it follows that the infinity norm inequalities (10)-(11) are sufficient for guaranteeing Lyapunov asymptotic stability [13] for the PWA system (2) in closed-loop with the MPC control (8).

3) Since $\{(P, K_j) \mid j \in \mathcal{S}\}$ satisfy (17) we have that

$$\|P(A_j + B_jK_j)x_k + Pf_j\|_\infty - \|Px_k\|_\infty \leq -\|Qx_k\|_\infty < 0, \quad \forall x_k \in \mathbb{X}_N \setminus \{0\},\ j \in \mathcal{S}. \tag{23}$$

Then it follows that $V(x) := \|Px\|_\infty$, which is a radially unbounded, positive definite and decrescent function, possesses a negative definite forward difference. Hence, $V(x_k)$ is a common polyhedral Lyapunov function for the dynamics $x_{k+1} = (A_j + B_jK_j)x_k + f_j$, $j \in \mathcal{S}$. Then, the origin of the PWA system (2) with feedback (9) is asymptotically stable on some region of attraction, e.g., the *polyhedral* sublevel set given by the largest $\varphi > 0$ for which $\{x \in \mathbb{X} \mid V(x) \leq \varphi\}$ is contained in $\mathbb{X}_{\mathbb{U}}$.

4) For the PWA system (2) with $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{U} = \mathbb{R}^m$ we have that $\mathbb{X}_{\mathbb{U}} = \mathbb{R}^n$. Since (23) holds for $\mathbb{X}_N = \mathbb{R}^n$, it follows that the origin of the PWA system (2) with feedback (9) is globally asymptotically stable. □

It follows from Theorem 1 that a terminal set $\mathbb{X}_N$ can be easily obtained as a sublevel set

$$\mathbb{X}_N \triangleq \{x \in \mathbb{X} \mid \|Px\|_\infty \leq \varphi^*\}, \tag{24}$$

where $\varphi^* = \sup_\varphi \{\{x \in \mathbb{X} \mid \|Px\|_\infty \leq \varphi\} \subset \mathbb{X}_{\mathbb{U}}\}$. Since this set is a polyhedron, Problem 1 leads to a MILP problem, which can be solved by standard tools developed in the context of infinity norm based hybrid MPC [4].

Remark 2. We have shown that Lyapunov asymptotic stability can be guaranteed for the closed-loop system (2)-(8) and all *feasible* initial states, even though the MPC value function and the PWA dynamics (2) may be discontinuous. This results from the fact that V_{MPC} is radially unbounded, it possesses a negative definite forward difference and the inequality (22) holds on $\mathbb{X}_N$ (note that $V_{\mathrm{MPC}}(0) = 0$ and (22) implies that $V_{\mathrm{MPC}}(x)$ is continuous at $x = 0$). Moreover, it follows from Theorem 2 of [14] that the origin of the closed-loop system (2)-(8) is locally *exponentially stable* (i.e. this property holds for all states in $\mathbb{X}_N$).

Remark 3. The set of feasible initial states with respect to Problem 1 depends on the value of the prediction horizon N, due to the terminal constraint (4c). The larger N, the larger the set of feasible states is. For a given terminal constraint set $\mathbb{X}_N$ and an assigned set of initial conditions, one can perform a reachability (controllability) analysis in order to obtain the minimum prediction horizon needed to achieve feasibility of Problem 1 for the desired set of initial states. A procedure that can be employed to solve this problem for constrained PWA systems is given in [6].

Finding the matrix P and the feedback matrices $\{K_j \mid j \in \mathcal{S}\}$ that satisfy the infinity norm inequality (10) amounts to solving an optimization problem subject to the constraint $\mathrm{rank}(P) = n$. Note that this constraint can be replaced by the convex constraint $P^\top P > 0$. Once a matrix P satisfying (10) has been found, one still has to check that P also satisfies inequality (11), provided that $\mathcal{S} \neq \mathcal{S}_0$. For example, this can be verified by checking the inequality

$$\|Pf_j\|_\infty \leq \gamma_j \min_{x \in \mathbb{X}_N \cap \Omega_j} \|Px\|_\infty, \quad j \in \mathcal{S}(\mathbb{X}_N),$$

where $\mathcal{S}(\mathbb{X}_N) := \{j \mid \mathbb{X}_N \cap \Omega_j \neq \emptyset\} \cap \mathcal{S}_1$. In order not to perform this additional check, the inequality (11) can be removed by requiring that $\mathbb{X}_N \subseteq \cup_{j \in \mathcal{S}_0} \Omega_j$ is a positively invariant set only for the PWL sub-system of the closed-loop PWA system (2)-(9), i.e., for the system $x_{k+1} = (A_j + B_j K_j)x_k$ when $x_k \in \Omega_j$, $j \in \mathcal{S}_0$, as done in [7] for hybrid MPC based on quadratic forms. Note that the auxiliary control action (9) defines now a local state feedback, instead of a global state feedback, as in Theorem 1. In this case Theorem 1 can be reformulated as follows.

Corollary 1. *Suppose that the inequality*

$$\|P(A_j + B_j K_j)P^{-L}\|_\infty + \|QP^{-L}\|_\infty + \|RK_j P^{-L}\|_\infty \leq 1 \qquad (25)$$

is solvable in (P, K_j) for P with full-column rank and $j \in \mathcal{S}_0$. Let $\mathbb{X}_N \subseteq \mathbb{X}_{\mathbb{U}} \cap \{\cup_{j \in \mathcal{S}_0} \Omega_j\}$ be a positively invariant set for the closed-loop system $x_{k+1} = (A_j + B_j K_j)x_k$ when $x_k \in \Omega_j$, $j \in \mathcal{S}_0$ and assume that $\mathbb{X}_N$ contains the origin in its interior. Fix $N \geq 1$. Then the first three statements of Theorem 1 hold.

Proof. From the fact that the terminal state is constrained to lie in $\mathbb{X}_N \subseteq \mathbb{X}_{\mathbb{U}} \cap \{\cup_{j \in \mathcal{S}_0} \Omega_j\}$ and from (3) we have that $f_j = 0$ for all $j \in \mathcal{S}_0$. Then it follows that (11) holds with equality for $\gamma_j = 0$, $\forall j \in \mathcal{S}_0$. Since (P, K_j) satisfy (25) for all $j \in \mathcal{S}_0$ it follows that (P, K_j) satisfy (10)-(11) for all $j \in \mathcal{S}_0$. Then the results follow from Theorem 1. □

Example 1. Consider the following third order chain of integrators with a varying sampling rate:

$$x_{k+1} = \begin{cases} A_1 x_k + B_1 u_k & \text{if } [0\ 1\ 1]x_k \leq 0\ ,\ [1\ 0\ 0]x_k < 4\ ,\ [-1\ 0\ 0]x_k < 4 \\ A_2 x_k + B_2 u_k & \text{if } [0\ 1\ 1]x_k > 0\ ,\ [1\ 0\ 0]x_k < 4\ ,\ [-1\ 0\ 0]x_k < 4 \\ A_3 x_k + B_3 u_k + f & \text{otherwise} \end{cases} \tag{26}$$

subject to the constraints $x_k \in \mathbb{X} = [-15, 15]^3$ and $u_k \in \mathbb{U} = [-1, 1]$, where

$$A_1 = \begin{bmatrix} 1 & 0.4 & 0.08 \\ 0 & 1 & 0.4 \\ 0 & 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0.7 & 0.245 \\ 0 & 1 & 0.7 \\ 0 & 0 & 1 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & 0.8 & 0.32 \\ 0 & 1 & 0.8 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.0107 \\ 0.08 \\ 0.4 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.0572 \\ 0.245 \\ 0.7 \end{bmatrix},\ B_3 = \begin{bmatrix} 0.0853 \\ 0.32 \\ 0.8 \end{bmatrix},\ f = \begin{bmatrix} 0.3 \\ 0.1 \\ 0.1 \end{bmatrix}.$$

The MPC tuning parameters are $Q = I_3$ and $R = 0.1$. The following solution to the inequality (10) has been found using a min-max formulation and the Matlab `fmincon` solver (CPU time was 5.65 seconds on a Pentium III at 700MHz):

$$P = \begin{bmatrix} 24.1304 & 20.3234 & 4.9959 \\ 20.3764 & 35.9684 & 10.5832 \\ 6.3709 & 9.21 & 9.9118 \end{bmatrix},\ K_3 = \begin{bmatrix} -0.8434 & -2.063 & -1.9809 \end{bmatrix},\ \gamma = 0.174,$$

$$K_1 = \begin{bmatrix} -2.3843 & -4.5862 & -3.1858 \end{bmatrix},\ K_2 = \begin{bmatrix} -0.8386 & -2.1077 & -2.1084 \end{bmatrix}. \tag{27}$$

$\mathbb{X}_N$ has been obtained as in (24) for $\varphi^* = 2.64$. Due to the input constraints we have that $\mathbb{X}_N \subset \cup_{j \in \mathcal{S}_0} \Omega_j$ for system (26). However, inequality (11) holds for system (26) and all $x \in \mathbb{X}$. The initial state is $x_0 = [3\ -1\ 2]^\top$ and the prediction horizon of $N = 8$ is obtained as in Remark 3 for the matrices P, Q and R given above. The simulation results are plotted in Figure 1 for system (26) in closed-loop with the MPC control (8). As guaranteed by Theorem 1, the MPC control law (8) stabilizes the unstable system (26) while satisfying the state and input constraints. □

5 Solving the stabilization conditions

This section gives some techniques to approach the computationally challenging problem associated with inequality (10). All these methods start from the fact that if the matrix P is known in (10), then the optimization problem can be recast as an LP problem. In the sequel we will indicate three ways to find an educated guess of P. The first two methods are based on the observation that a matrix P that satisfies (10)-(11) (for some K_j, $j \in \mathcal{S}$) has the property that $V(x) = \|Px\|_\infty$ is a common polyhedral Laypunov function of the PWA system (2) in closed-loop with some PWL feedback (9). Using this observation, an educated guess of P is now based on functions $V(x) = \|Px\|_\infty$ that satisfy this necessary condition and thus, induce positively invariant sets for the closed-loop system (2)-(9).

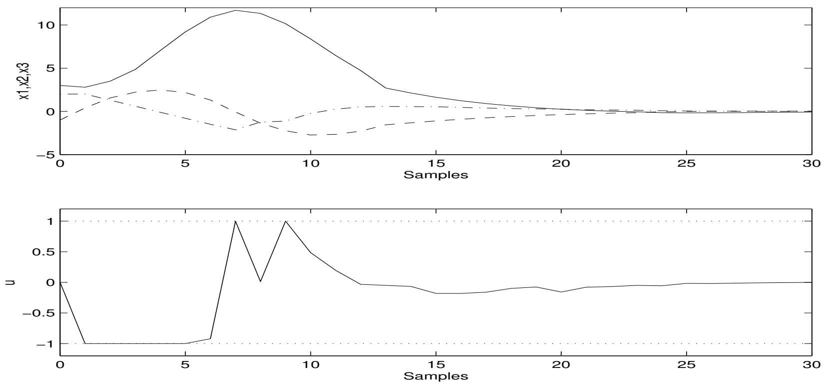

Fig. 1. Example 1: State trajectory and Input history.

5.1 A quadratic approach

One possibility to fix the terminal weight matrix is to use the approach of [7] to calculate a polyhedral positively invariant set for the PWL sub-system of the PWA system (2)-(9), i.e., for the system $x_{k+1} = (A_j + B_j K_j)x_k$ when $x_k \in \Omega_j$, $j \in \mathcal{S}_0$. If the algorithm of [7] terminates in finite time and the resulting polyhedral set is symmetric, then a good choice for P is the matrix that induces this polyhedral set, i.e. $\{x \in \mathbb{X} \mid \|Px\|_\infty \leq c\}$ for some $c > 0$. Note that this type of approach to obtain P is based on the fact that the feedback matrices $\{K_j \mid j \in \mathcal{S}_0\}$ are already known, e.g., in [7] they are calculated via semi-definite programming in order to obtain a common quadratic Lyapunov function. The approach of [9] can also be used to compute the feedbacks that guarantee quadratic stability and then, the algorithm of [7] can be employed to obtain a polyhedral positively invariant set. Fixing P in (10) and solving in $\{K_j \mid j \in \mathcal{S}_0\}$ (and γ_j) amounts to looking for a different state feedback control law, which not only renders the employed polyhedral set positively invariant, but also ensures that $V_{\text{MPC}}(x_k)$ possesses a negative definite forward difference.

5.2 "Squaring the circle"

Another way to obtain polyhedral (or piecewise polyhedral) positively invariant sets for closed-loop PWA systems that admit a common (or a piecewise) quadratic Lyapunov function has been recently developed in [17]. In this approach, the polyhedral set can be constructed by solving the problem of fitting a polyhedron in between two ellipsoidal sublevel sets of a quadratic Lyapunov function, where one is contained in the interior of the other and the states on the boundary of the outer ellipsoid are mapped by the closed-loop dynamics into the interior of the inner ellipsoid. This problem can be solved using the recent algorithm developed in [18] in the context of DC programming. The polyhedral set is constructed by treating the ellipsoids as sublevel sets of convex functions,

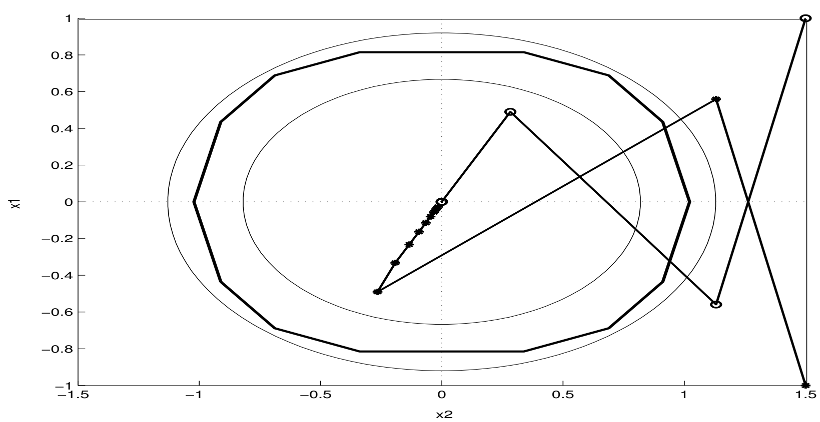

Fig. 2. Example 2: State trajectory.

and by exploiting upper and lower piecewise affine bounds on such functions. Giving additional structure to the algorithm of [18] such that it generates a polyhedron with a finite number of facets, a polyhedral positively invariant set is obtained for system (2) and then P can be chosen as the matrix that induces this polyhedron.

Example 2. Consider the example proposed in [5], i.e.,

$$x_{k+1} = \begin{cases} A_1 x_k + B u_k & \text{if} \quad [1\ 0]x_k \geq 0 \\ A_2 x_k + B u_k & \text{if} \quad [1\ 0]x_k < 0 \end{cases} \tag{28}$$

subject to the constraints $x_k \in \mathbb{X} = [-5, 5] \times [-5, 5]$, $u_k \in \mathbb{U} = [-1, 1]$ and with

$$A_1 = \begin{bmatrix} 0.35 & -0.6062 \\ 0.6062 & 0.35 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.35 & 0.6062 \\ -0.6062 & 0.35 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The common *quadratic* Lypunov function calculated in [7] for system (28)-(9) with feedback matrices $K_1 = [-0.611\ \ -0.3572]$, $K_2 = [0.611\ \ -0.3572]$ and the algorithm of [18] have been used to compute a polyhedral positively invariant set for system (28). The two ellipsoidal sublevel sets of the quadratic Lyapunov function plotted in Figure 2 are such that all the states on the boundary of the outer ellipsoid go inside the inner ellipsoid in one discrete-time step. The matrix P has been chosen as the matrix that induces the polyhedron plotted in Figure 2. Then (10) has been solved for the MPC tuning parameters $Q = \text{diag}([0.6\ 1])$ and $R = 0.1$, yielding the new state feedback matrices $K_1 = [-0.6897\ \ -0.1416]$ and $K_2 = [0.1454\ \ -0.7461]$. The simulation results obtained for $N = 2$ and the initial states $x_0 = [1\ 1.5]^\top$ (circle line) and $x_0 = [-1\ 1.5]^\top$ (star line) are shown in Figure 2 together with a plot of the polyhedral positively invariant set. $\square$

5.3 Square matrices Q and P

If Q is square and invertible, a different way to simplify (10) is to parameterize the terminal weight as $P = \frac{1}{\epsilon}Q$, where $0 < \epsilon < 1$.

Lemma 2. *Assume that* $\{K_j, \gamma_j \mid j \in \mathcal{S}\}$ *with* $0 \leq \gamma_j < 1$ *and* ϵ *satisfy the inequality*

$$\|Q(A_j + B_jK_j)Q^{-1}\|_\infty + \epsilon\|RK_jQ^{-1}\|_\infty \leq 1 - \epsilon - \gamma_j, \quad j \in \mathcal{S}. \tag{29}$$

Then $P = \frac{1}{\epsilon}Q$ *and* $\{K_j, \gamma_j \mid j \in \mathcal{S}\}$ *satisfy the inequality* (10).

Proof. From the fact that Q is square and invertible it follows that $P = \frac{1}{\epsilon}Q$ is square and invertible. Then the inequality (10) can be written as

$$\|P(A_j + B_jK_j)P^{-1}\|_\infty + \|QP^{-1}\|_\infty + \|RK_jP^{-1}\|_\infty \leq 1 - \gamma_j, \quad j \in \mathcal{S}.$$

By replacing $P = \frac{1}{\epsilon}Q$ and $P^{-1} = \epsilon Q^{-1}$ in the above inequality yields the equivalent inequality (29). □

For a fixed ϵ, finding $\{K_j, \gamma_j \mid j \in \mathcal{S}\}$ that satisfy the inequality (29) amounts to solving an LP problem. Then the matrix P can be simply chosen as $P = \frac{1}{\epsilon}Q$.

6 Relaxations

The *a priori* stabilization conditions for infinity norm based MPC of constrained PWA systems developed in Section 4 amount to searching for a common Lyapunov function and a common polyhedral positively invariant set for all subsystems of (2). Since in general there is no guarantee that such a function and such a set exist, in the sequel we relax the conditions of Section 4 by employing different terminal weight matrices in cost (5), depending on the state space region where the terminal state resides. Now consider the following problem.

Problem 3. At time $k \geq 0$ let $x_k \in \mathbb{X}$ be given. Minimize the cost function

$$J(x_k, \mathbf{u}_k) \triangleq \|P_jx_{k+N}\|_\infty + \sum_{i=0}^{N-1} \|Qx_{k+i}\|_\infty + \|Ru_{k+i}\|_\infty \text{ when } x_{k+N} \in \Omega_j, j \in \mathcal{S} \tag{30}$$

over all input sequences $\mathbf{u}_k \in \mathcal{U}_N(x_k)$.

Let $\mathcal{Q}_{ji} := \{x \in \Omega_j \mid \exists u \in \mathbb{U} : A_jx + B_ju + f_j \in \Omega_i\}$, $(j, i) \in \mathcal{S} \times \mathcal{S}$ and let $\mathcal{X} := \{(j, i) \in \mathcal{S} \times \mathcal{S} \mid \mathcal{Q}_{ji} \neq \emptyset\}$. The set of pairs of indices $\mathcal{X}$ can be determined off-line by performing a one-step rechability analysis for the PWA system (2) (note that the *one-step* rechability analysis does not yield a combinatorial drawback). The set $\mathcal{X}$ contains all discrete mode transitions that can occur in the PWA system

(2), i.e. a transition from Ω_j to Ω_i can occur if and only if $(j,i) \in \mathcal{X}$. The infinity norm inequalities (10) and (11) become:

$$\|P_i(A_j + B_jK_j)P_j^{-L}\|_\infty + \|QP_j^{-L}\|_\infty + \|RK_jP_j^{-L}\|_\infty \leq 1 - \gamma_{ji}, \quad (j,i) \in \mathcal{X} \tag{31}$$

and

$$\|P_if_j\|_\infty \leq \gamma_{ji}\|P_jx\|_\infty, \quad \forall x \in \mathbb{X}_N \cap \Omega_j, \quad (j,i) \in \mathcal{X}, \tag{32}$$

where γ_{ji} are scaling factors that satisfy $0 \leq \gamma_{ji} < 1$, $(j,i) \in \mathcal{X}$. Now Theorem 1 can be extended as follows.

Theorem 2. *Suppose* (31)-(32) *is solvable in* (P_j, K_j, γ_{ji}) *for* P_j *with full-column rank and* $(j,i) \in \mathcal{X}$. *Let* $\mathbb{X}_N \subseteq \mathbb{X}_\mathbb{U}$ *be a positively invariant set for the closed-loop system* (2)-(9) *that contains the origin in its interior. Fix* $N \geq 1$ *and calculate the MPC control* (8) *by solving at each sampling instant Problem 3 instead of Problem 1. Then the four statements of Theorem 1 hold for Problem 3.*

The proof is similar to the proof of Theorem 1 and is omitted here for brevity.

Since $\{(P_j, K_j) \mid j \in \mathcal{S}\}$ satisfy (31) and (32) we have that

$$\|P_i(A_j + B_jK_j)x_k + P_if_j\|_\infty - \|P_jx_k\|_\infty \leq -\|Qx_k\|_\infty < 0, \quad \forall x_k \in \mathbb{X}_N \setminus \{0\}, \quad (j,i) \in \mathcal{X}. \tag{33}$$

Then, it can be proven along the lines of the proof of Theorem 1 that the *discontinuous* function $V(x) := \|P_jx\|_\infty$ when $x \in \Omega_j$ is a (piecewise linear) Lyapunov function for the dynamics $x_{k+1} = (A_j + B_jK_j)x_k + f_j$, $j \in \mathcal{S}$. Hence, the origin of the PWA system (2) with feedback (9) is asymptotically stable on some region of attraction, e.g., the *piecewise polyhedral* sublevel set given by the largest $\varphi > 0$ for which $\{x \in \mathbb{X} \mid V(x) \leq \varphi\}$ is contained in $\mathbb{X}_\mathbb{U}$. The terminal set $\mathbb{X}_N$ can be obtained in this case as

$$\mathbb{X}_N \triangleq \cup_{j \in \mathcal{S}}\{x \in \Omega_j \mid \|P_jx\|_\infty \leq \varphi^*\}, \tag{34}$$

where $\varphi^* = \sup_\varphi\{\{x \in \Omega_j \mid \|P_jx\|_\infty \leq \varphi\} \subset \mathbb{X}_\mathbb{U}\}$. Since this set is a finite union of polyhedra, Problem 3 still leads to a MILP problem, which is a standard tool in the context of infinity norm based hybrid MPC [4].

Note that the methods of Section 5.2 and Section 5.3 can also be applied to reduce the optimization problem associated with the infinity norm inequality (31) to an LP problem.

Remark 4. The sublevel sets of the Lyapunov function $V(x) = \|P_jx\|_\infty$ when $x \in \Omega_j$ with P_j satisfying (33) are λ-contractive sets [15] and they are finite unions of polyhedra (i.e. they are represented by a polyhedron in each region of the PWA system). Hence, this yields a new method to obtain (in finite time) a *piecewise polyhedral* λ*-contractive set* for the class of PWA systems, which takes into account also the affine terms f_j for $j \in \mathcal{S}_1$. If we set $P_j = P$ for all $j \in \mathcal{S}$ (as done in Section 4), this yields a new way to obtain *polyhedral* λ*-contractive sets* for PWA systems.

7 Conclusions

In this paper we have developed *a priori* stabilization conditions for infinity norm based MPC of constrained PWA systems. Stability has been achieved using infinity norm inequalities. If the considered inequalities are satisfied, then the possibly discontinuous value function of the MPC cost is a Lyapunov function of the controlled PWA system. The terminal weight(s) are obtained by solving off-line an optimization problem. Several possibilities to reduce this problem to an LP problem via a two-step procedure have been indicated. The terminal constraint set is simply obtained by taking one of the sublevel sets of the terminal cost, which is a local piecewise linear Lyapunov function. As a by-product we have also obtained a new approach for the calculation of positively invariant sets for PWA systems.

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A Proof of Lemma 1

We will use induction to prove Lemma 1. For $l = 0$, the inequality (13) holds for any $\alpha_0 \geq 1$. Suppose (13) holds for some $l \geq 0$. Now we will prove that (13) holds for $l + 1$. We have that

$$\|x_{k+l+1}\|_\infty = \|(A_j + B_j K_j)x_{k+l} + f_j\|_\infty \quad \text{when} \quad x_{k+l} \in \mathbb{X}_N \cap \Omega_j.$$

Due to the full-column rank of P, there exist positive numbers μ_P and γ_P such that $\mu_P\|z\|_\infty \leq \|Pz\|_\infty \leq \gamma_P\|z\|_\infty$ for all $z \in \mathbb{R}^n$. Then it follows that

$$\begin{aligned}\|x_{k+l+1}\|_\infty &\leq \|(A_j + B_j K_j)x_{k+l}\|_\infty + \|f_j\|_\infty \leq \\ &\leq \eta\|x_{k+l}\|_\infty + \mu_P^{-1}\|Pf_j\|_\infty \leq \eta\|x_{k+l}\|_\infty + \mu_P^{-1}\|Px_{k+l}\|_\infty,\end{aligned}$$

where $\eta = \max_{j\in\mathcal{S}} \|A_j + B_j K_j\|_\infty$ and in the last inequality we used (11) and $0 \leq \gamma_j < 1$ for all $j \in \mathcal{S}$. The above inequality yields

$$\|x_{k+l+1}\|_\infty \leq (\eta + \mu_P^{-1}\gamma_P)\|x_{k+l}\|_\infty.$$

By the induction hypothesis there exists $\alpha_l > 0$ such that (13) holds for x_{k+l} and by letting $\alpha_{l+1} := (\eta + \mu_P^{-1}\gamma_P)\alpha_l > 0$ it follows that

$$\|x_{k+l+1}\|_\infty \leq \alpha_{l+1}\|x_k\|_\infty.$$

□

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