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Brief paper

Predictive control of hybrid systems: Input-to-state stability results for sub-optimal solutions[☆]

M. Lazar [*], W.P.M.H. Heemels

Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

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ABSTRACT

This article presents a novel model predictive control (MPC) scheme that achieves input-to-state stabilization of constrained discontinuous nonlinear and hybrid systems. Input-to-state stability (ISS) is guaranteed when an optimal solution of the MPC optimization problem is attained. Special attention is paid to the effect that sub-optimal solutions have on ISS of the closed-loop system. This issue is of interest as firstly, the infimum of MPC optimization problems does not have to be attained and secondly, numerical solvers usually provide only sub-optimal solutions. An explicit relation is established between the deviation of the predictive control law from the optimum and the resulting deterioration of the ISS property of the closed-loop system. By imposing stronger conditions on the sub-optimal solutions, ISS can even be attained in this case.

1. Introduction

Discrete-time discontinuous systems form a powerful and general modeling class for the approximation of hybrid and nonlinear phenomena, which also includes the class of piecewise affine (PWA) systems (Heemels, De Schutter, & Bemporad, 2001). The modeling capability of the latter class of systems has already been shown in several applications, including switched power converters, automotive systems and systems biology. As a consequence, there is an increasing interest in developing synthesis techniques for the robust control of discrete-time hybrid systems. The model predictive control (MPC) methodology (Mayne, Rawlings, Rao, & Scokaert, 2000) has proven to be one of the most successful frameworks for this task, see, for example, Bemporad and Morari (1999), Kerrigan and Mayne (2002) and Lazar, Heemels, Weiland, and Bemporad (2006) and the references therein.

In this paper we are interested in input-to-state stability (ISS) (Jiang & Wang, 2001) as a property to characterize the robust stability of hybrid systems in closed-loop with MPC. More precisely, we consider systems that are piecewise continuous and affected by additive disturbances. It is known (see, for example, Lazar (2006)) that for such discontinuous systems most of the results obtained for smooth nonlinear MPC (Grimm, Messina, Tuna, & Teel, 2007; Limon, Alamo, & Camacho, 2002; Mayne et al., 2000; Roset, Heemels, Lazar, & Nijmeijer, 2008) do not necessarily apply. The min–max MPC methodology (see, e.g., Lazar, Muñoz de la Peña, Heemels, and Alamo (2008) and the references therein) might be applicable, but its prohibitive computational complexity prevents implementation even for linear systems. As such, computationally feasible input-to-state stabilizing predictive controllers are widely unavailable.

In what follows we propose a tightened constraints MPC scheme for *discontinuous systems* along with conditions for ISS of the resulting closed-loop system, assuming that optimal MPC control sequences are implemented. These results provide advances to the existing works on tightened constraints MPC (Grimm et al., 2007; Limon et al., 2002), where continuity of the system dynamics is assumed, towards discontinuous and hybrid systems. Guaranteeing robust stability and feasibility in the presence of discontinuities is difficult and requires an innovative usage of tightened constraints, which is conceptually different from the approaches in Limon et al. (2002) and Grimm et al. (2007). Therein tightened constraints are employed for *robust feasibility* only. However, by carefully matching the new tightening approach with the discontinuities in the system dynamics, we achieve *both robust feasibility and ISS* for the optimal case. Another issue that is neglected in MPC of hybrid systems is the effect of sub-optimal implementations. In particular, an important result was presented in Spjøtvold, Kerrigan, Rakovic, Mayne, and Johansen (2007), where it was shown that in the case of optimal control of discontinuous PWA systems

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[*] Corresponding author. Tel.: +31 40 2473795; fax: +31 40 2434582.
E-mail addresses: m.lazar@tue.nl (M. Lazar), m.heemels@tue.nl (W.P.M.H. Heemels).

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it is not uncommon that there does not exist a control law that attains the infimum. Moreover, numerical solvers usually provide only sub-optimal solutions. As a consequence, for hybrid systems it is necessary to study if and how stability results for optimal predictive control change in the case of sub-optimal implementations, which forms one the main topics in this paper.

To cope with MPC control sequences (obtained by solving MPC optimization problems) that are not optimal, but within a margin $\delta \geq 0$ from the optimum, we introduce the notion of ε-ISS as a particular case of the input-to-state practical stability (ISpS) property (Jiang, Mareels, & Wang, 1996). Next, we establish an analytic relation between the *optimality margin* δ of the solution of the MPC optimization problem and the *ISS margin* $\varepsilon(\delta)$. While the ISS results presented in this paper require the use of a specific robust MPC problem formulation (i.e. based on tightened constraints), we also show that nominal asymptotic stability can be guaranteed for sub-optimal MPC of hybrid systems without any modification to the standard MPC set-up presented in Mayne et al. (2000). Compared to classical sub-optimal MPC (Scokaert, Mayne, & Rawlings, 1999), where an explicit constraint on the MPC cost function is employed, this provides a fundamentally different approach.

1.1. Notation and basic definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{\geq c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ to denote the sets $\{k \in \mathbb{Z} \mid k \geq c_1\}$ and $\{k \in \mathbb{Z} \mid c_1 < k \leq c_2\}$, respectively, for some $c_1, c_2 \in \mathbb{Z}$. For $x \in \mathbb{R}^n$ let $\|x\|$ denote an arbitrary norm and for $Z \in \mathbb{R}^{m\times n}$, let $\|Z\|$ denote the corresponding induced matrix norm. We will use both $(z(0), z(1), \ldots)$ and $\{z(l)\}_{l\in\mathbb{Z}_+}$ with $z(l) \in \mathbb{R}^n$, $l \in \mathbb{Z}_+$, to denote a sequence of real vectors. For a sequence $\mathbf{z} := \{z(l)\}_{l\in\mathbb{Z}_+}$ let $\|\mathbf{z}\| := \sup\{\|z(l)\| \mid l \in \mathbb{Z}_+\}$ and let $\mathbf{z}_{[k]} := \{z(l)\}_{l\in\mathbb{Z}_{[0,k]}}$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\text{int}(\mathcal{S})$ the interior of $\mathcal{S}$. For two arbitrary sets $\mathcal{S} \subseteq \mathbb{R}^n$ and $\mathcal{P} \subseteq \mathbb{R}^n$, let $\mathcal{S} \sim \mathcal{P} := \{x \in \mathbb{R}^n \mid x + \mathcal{P} \subseteq \mathcal{S}\}$ denote their Pontryagin difference. For any $\mu > 0$ we define $\mathcal{B}_\mu$ as $\{x \in \mathbb{R}^n \mid \|x\| \leq \mu\}$. A polyhedron (or a polyhedral set) is a set obtained as the intersection of a finite number of open and/or closed half-spaces. A real-valued function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k\to\infty} \beta(s, k) = 0$.

2. Preliminaries

Consider the discrete-time system of the form

$$x(k+1) \in G(x(k), w(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{W} \subseteq \mathbb{R}^l$ is an unknown input at discrete-time instant $k \in \mathbb{Z}_+$ and $G : \mathbb{R}^n \times \mathbb{R}^l \to 2^{(\mathbb{R}^n)}$ is an arbitrary nonlinear, possibly discontinuous, set-valued function. For simplicity of notation, we assume that the origin is an equilibrium in (1) for zero input, i.e. $G(0, 0) = \{0\}$.

Definition 1. *RPI* We call a set $\mathcal{P} \subseteq \mathbb{R}^n$ *robustly positively invariant (RPI)* for system (1) with respect to $\mathbb{W}$ if for all $x \in \mathcal{P}$ and all $w \in \mathbb{W}$ it holds that $G(x, w) \subseteq \mathcal{P}$.

Definition 2. *ε-ISS* Let $\mathbb{X}$ with $0 \in \text{int}(\mathbb{X})$ and $\mathbb{W}$ be subsets of $\mathbb{R}^n$ and $\mathbb{R}^l$, respectively. For a given $\varepsilon \in \mathbb{R}_+$, the perturbed system (1) is called *ε-ISS in $\mathbb{X}$ for inputs in* $\mathbb{W}$ if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that, for each $x(0) \in \mathbb{X}$ and all $\mathbf{w} = \{w(l)\}_{l\in\mathbb{Z}_+}$ with $w(l) \in \mathbb{W}$ for all $l \in \mathbb{Z}_+$, it holds that all state trajectories of (1) with initial state $x(0)$ and input sequence $\mathbf{w}$ satisfy

$$\|x(k)\| \leq \beta(\|x(0)\|, k) + \gamma(\|\mathbf{w}_{[k-1]}\|) + \varepsilon, \quad \forall k \in \mathbb{Z}_{\geq 1}.$$

We call system (1) *ISS in $\mathbb{X}$ for inputs in* $\mathbb{W}$ if (1) is 0-ISS in $\mathbb{X}$ for inputs in $\mathbb{W}$.

Definition 3. *ε-AS* For a given $\varepsilon \in \mathbb{R}_+$, the 0-input system (1), i.e. $x(k + 1) \in G(x(k), 0)$, $k \in \mathbb{Z}_+$, is called *ε-asymptotically stable (ε-AS) in* $\mathbb{X}$ if there exists a $\mathcal{KL}$-function β such that, for each $x(0) \in \mathbb{X}$ it holds that all state trajectories with initial state $x(0)$ satisfy $\|x(k)\| \leq \beta(\|x(0)\|, k) + \varepsilon$, $\forall k \in \mathbb{Z}_{\geq 1}$. We call the 0-input system (1) *AS in* $\mathbb{X}$ if it is 0-AS in $\mathbb{X}$.

We refer to ε by the term *ISS (AS) margin*.

Theorem 4. *Let d_1, d_2 be non-negative reals, a, b, c, λ be positive reals with $c \leq b$ and $\alpha_1(s) := as^\lambda$, $\alpha_2(s) := bs^\lambda$, $\alpha_3(s) := cs^\lambda$ and $\sigma \in \mathcal{K}$. Furthermore, let $\mathbb{X}$ be a RPI set for system (1) with respect to $\mathbb{W}$ and let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a function such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) + d_1, \tag{2a}$$

$$V(x^+) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|w\|) + d_2 \tag{2b}$$

for all $x \in \mathbb{X}$, $w \in \mathbb{W}$ and all $x^+ \in G(x, w)$. Then the system (1) is ε-ISS in $\mathbb{X}$ for inputs in $\mathbb{W}$ with

$$\beta(s, k) := \alpha_1^{-1}(3\rho^k\alpha_2(s)), \qquad \gamma(s) := \alpha_1^{-1}\left(\frac{3\sigma(s)}{1-\rho}\right),$$

$$\varepsilon := \alpha_1^{-1}\left(3\left(d_1 + \frac{d_2}{1-\rho}\right)\right), \qquad \rho := 1 - \frac{c}{b} \in [0, 1). \tag{3}$$

If the inequalities (2) hold for $d_1 = d_2 = 0$, the system (1) is ISS in $\mathbb{X}$ for inputs in $\mathbb{W}$.

The proof of Theorem 4 is similar in nature to the proof given in Lazar et al. (2008) by replacing the difference *equation* by a difference *inclusion* as in (1) and is omitted here. We call a function $V(\cdot)$ that satisfies the hypothesis of Theorem 4 an *ε-ISS function*.

3. MPC scheme set-up

Consider the piecewise continuous (PWC) system

$$x(k+1) = g(x(k), u(k), w(k)) := g_j(x(k), u(k)) + w(k)$$
$$\text{if } x(k) \in \Omega_j, \quad k \in \mathbb{Z}_+, \tag{4}$$

where each $g_j : \Omega_j \times \mathbb{U} \to \mathbb{R}^n$, $j \in \mathcal{S}$, is a continuous, possibly nonlinear function in x and $\mathcal{S} := \{1, 2, \ldots, s\}$ is a finite set of indices. We assume that x and u are constrained in some sets $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ that contain the origin in their interior. The collection $\{\Omega_j \subseteq \mathbb{R}^n \mid j \in \mathcal{S}\}$ defines a partition of $\mathbb{X}$, meaning that $\cup_{j\in\mathcal{S}} \Omega_j = \mathbb{X}$ and $\Omega_i \cap \Omega_j = \emptyset$, with the sets Ω_j not necessarily closed. We also assume that w takes values in the set $\mathbb{W} := \mathcal{B}_\mu$ with $\mu \in \mathbb{R}_{>0}$ sufficiently small as determined later.

Assumption 5. For each fixed $j \in \mathcal{S}$, $g_j(\cdot, \cdot)$ satisfies a continuity condition in the first argument in the sense that there exists a $\mathcal{K}$-function $\eta_j(\cdot)$ such that

$$\|g_j(x, u) - g_j(y, u)\| \leq \eta_j(\|x - y\|), \ \forall x, y \in \Omega_j, \ \forall u \in \mathbb{U},$$

and $\exists j_0 \in \mathcal{S}$ such that $0 \in \text{int}(\Omega_{j_0})$ and $g_{j_0}(0, 0) = 0$.

As we allow $g(\cdot, \cdot, \cdot)$ to be discontinuous in x over the switching boundaries, discontinuous PWA systems are a sub-class of PWC systems as given in (4). For a fixed $N \in \mathbb{Z}_{\geq 1}$, let $(\phi(1), \ldots, \phi(N))$ denote a state sequence generated by the *unperturbed* system (4), i.e.

$$\phi(i+1) := g_j(\phi(i), u(i)) \quad \text{if } \phi(i) \in \Omega_j, \tag{5}$$

for $i = 0, \ldots, N-1$, from initial condition $\phi(0) := x(k)$ and by applying an input sequence $\mathbf{u}_{[N-1]} = (u(0), \ldots, u(N-1)) \in \mathbb{U}^N := \mathbb{U} \times \cdots \times \mathbb{U}$. Let $\mathbb{X}_T \subseteq \mathbb{X}$ denote a set with $0 \in \text{int}(\mathbb{X}_T)$. Define $\eta(s) := \max_{j \in \mathcal{S}} \eta_j(s)$. As the maximum of a finite number of $\mathcal{K}$-functions is a $\mathcal{K}$-function, $\eta \in \mathcal{K}$. Let $\eta^{[p]}(s)$ denote the p-times function composition with $\eta^{[0]}(s) := s$ and $\eta^{[k]}(s) = \eta(\eta^{[k-1]}(s))$ for $k \in \mathbb{Z}_{\geq 1}$. For any $\mu > 0$ and $i \in \mathbb{Z}_{\geq 1}$, define

$$\mathcal{L}_\mu^i := \left\{ \zeta \in \mathbb{R}^n \,\middle|\, \|\zeta\| \leq \sum_{p=0}^{i-1} \eta^{[p]}(\mu) \right\}.$$

Define the *set of admissible input sequences* for $x \in \mathbb{X}$ as:

$$\mathcal{U}_N(x) := \{\mathbf{u}_{[N-1]} \in \mathbb{U}^N \mid \quad \phi(i) \in \mathbb{X}_i, \quad i = 1, \ldots, N-1, \\ \phi(0) = x, \ \phi(N) \in \mathbb{X}_T\}, \tag{6}$$

where $\mathbb{X}_i := \cup_{j \in \mathcal{S}} \{\Omega_j \sim \mathcal{L}_\mu^i\} \subset \mathbb{X}, \quad \forall i = 1, \ldots, N-1$. The purpose of the above set of input sequences will be made clear in Lemma 11. For a given $N \in \mathbb{Z}_{\geq 1}$, notice that $\mu > 0$ has to be sufficiently small so that $0 \in \text{int}(\Omega_{j_0} \sim \mathcal{L}_\mu^{N-1}) \neq \emptyset$. Let $F : \mathbb{R}^n \to \mathbb{R}_+$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $F(0) = L(0, 0) = 0$ be arbitrary nonlinear mappings.

Problem 6 (*MPC Optimization Problem*). Let $\mathbb{X}_T \subseteq \mathbb{X}$ and $N \in \mathbb{Z}_{\geq 1}$ be given. At time $k \in \mathbb{Z}_+$ let $x(k) \in \mathbb{X}$ be given and *infimize* the cost $J(x(k), \mathbf{u}_{[N-1]}) := F(\phi(N)) + \sum_{i=0}^{N-1} L(\phi(i), u(i))$ over all sequences $\mathbf{u}_{[N-1]}$ in $\mathcal{U}_N(x(k))$.

We call a state $x \in \mathbb{X}$ *feasible* if $\mathcal{U}_N(x) \neq \emptyset$. Problem 6 is said to be *feasible* for $x \in \mathbb{X}$ if $\mathcal{U}_N(x) \neq \emptyset$. Let $\mathbb{X}_f(N) \subseteq \mathbb{X}$ denote the set of *feasible states* for Problem 6. Let $V^*(x) := \inf_{\mathbf{u}_{[N-1]} \in \mathcal{U}_N(x)} J(x, \mathbf{u}_{[N-1]})$. Since $J(\cdot, \cdot)$ is lower bounded by 0, the infimum exists. As such, $V^*(x)$ is well defined for all $x \in \mathbb{X}_f(N)$. However, as shown in Spjøtvold et al. (2007), the infimum is not necessarily attainable. Therefore, we will consider the following set of sub-optimal control sequences. For any $x \in \mathbb{X}_f(N)$ and $\delta \geq 0$, we define

$$\Pi_\delta(x) := \{\mathbf{u}_{[N-1]} \in \mathcal{U}_N(x) \mid J(x, \mathbf{u}_{[N-1]}) \leq V^*(x) + \delta\}$$

and $\pi_\delta(x) := \{u(0) \in \mathbb{R}^m \mid \mathbf{u}_{[N-1]} \in \Pi_\delta(x)\}$. We will refer to δ by the term *optimality margin*. Note that $\delta = 0$ and $\Pi_\delta(x) \neq \emptyset$ correspond to the situation when the global optimum is attained in Problem 6. An optimality margin δ can be guaranteed a priori, for example, by using the sub-optimal mixed integer linear programming (MILP) solver proposed in Spjøtvold et al. (2007) or by specifying a tolerance with respect to achieving the optimum, which is a usual feature of most solvers.

Next, consider the following MPC closed-loop system corresponding to (4):

$$x(k+1) \in \Phi_\delta(x(k), w(k)) \\ := \{g(x(k), u, w(k)) \mid u \in \pi_\delta(x(k))\}, \quad k \in \mathbb{Z}_+. \tag{7}$$

To simplify the exposition we will make use of the following commonly adopted assumptions in tightened constraints MPC (Grimm et al., 2007; Limon et al., 2002). Let $h : \mathbb{R}^n \to \mathbb{R}^m$ denote a terminal control law and define $\mathbb{X}_\mathbb{U} := \{x \in \mathbb{X} \mid h(x) \in \mathbb{U}\}$.

Assumption 7. There exist $\mathcal{K}$-functions $\alpha_L(\cdot)$, $\alpha_F(s) := \tau s^\lambda$, $\alpha_1(s) := a s^\lambda$ and $\alpha_2(s) := b s^\lambda$, $\tau, a, b, \lambda \in \mathbb{R}_{>0}$, such that
(i) $L(x, u) \geq \alpha_1(\|x\|)$, $\forall x \in \mathbb{X}$, $\forall u \in \mathbb{U}$;
(ii) $|L(x, u) - L(y, u)| \leq \alpha_L(\|x - y\|)$, $\forall x, y \in \mathbb{X}$, $\forall u \in \mathbb{U}$;
(iii) $|F(x) - F(y)| \leq \alpha_F(\|x - y\|)$, $\forall x, y \in \Omega_{j_0} \cap \mathcal{L}_\mu^{N-1}$;
(iv) $V^*(x) \leq \alpha_2(\|x\|)$, $\forall x \in \mathbb{X}_f(N)$.

Assumption 8. There exist $N \in \mathbb{Z}_{\geq 1}$, $\theta > \theta_1 > 0$, $\mu > 0$ and a terminal control law $h(\cdot)$ such that
(i) $\alpha_F(\eta^{[N-1]}(\mu)) \leq \theta - \theta_1$;
(ii) $\mathbb{F}_\theta := \{x \in \mathbb{R}^n \mid F(x) \leq \theta\} \subseteq (\Omega_{j_0} \sim \mathcal{L}_\mu^{N-1}) \cap \mathbb{X}_\mathbb{U}$ and $g_{j_0}(x, h(x)) \in \mathbb{F}_{\theta_1}$ for all $x \in \mathbb{F}_\theta$;
(iii) $F\left(g_{j_0}(x, h(x))\right) - F(x) + L(x, h(x)) \leq 0, \ \forall x \in \mathbb{F}_\theta$.

Note that the hypotheses in Assumption 7-(i), (ii), (iii) usually hold by suitable definitions of $L(\cdot, \cdot)$ and $F(\cdot)$. Also, it can be shown that the hypothesis of Assumption 7-(iv) may hold, even for discontinuous value functions. For further details on how to satisfy Assumption 8-(i), (ii), (iii) we refer to Lazar et al. (2006) and Lazar, Heemels, Bemporad, and Weiland (2007).

4. Input-to-state stability results

The main result on ε-ISS of sub-optimal predictive control of hybrid systems is stated next.

Theorem 9. *Let $\delta \in \mathbb{R}_{>0}$ be given, suppose that Assumptions 5 and 7 and Assumption 8 hold for the nonlinear hybrid system (4) and Problem 6, and set $\mathbb{X}_T = \mathbb{F}_{\theta_1}$. Then:*

(i) *If Problem 6 is feasible at time $k \in \mathbb{Z}_+$ for state $x(k) \in \mathbb{X}$, then Problem 6 is feasible at time $k+1$ for any state $x(k+1) \in \Phi_\delta(x(k), w(k))$ and all $w(k) \in \mathcal{B}_\mu$. Moreover, $\mathbb{X}_T \subseteq \mathbb{X}_f(N)$;*
(ii) *The closed-loop system $x(k+1) \in \Phi_\delta(x(k), w(k))$ is $\varepsilon(\delta)$-ISS in $\mathbb{X}_f(N)$ for inputs in $\mathcal{B}_\mu$ with ISS margin $\varepsilon(\delta) := \left(\frac{3b}{a^2}\delta\right)^{\frac{1}{\lambda}}$.*

To prove Theorem 9 we will make use of the following technical lemmas (see the appendix for their proofs).

Lemma 10. *Let $x \in \Omega_j \sim \mathcal{L}_\mu^{i+1}$ for some $j \in \mathcal{S}$, $i \in \mathbb{Z}_+$, and let $y \in \mathbb{R}^n$. If $\|y - x\| \leq \eta^{[i]}(\mu)$, then $y \in \Omega_j \sim \mathcal{L}_\mu^i$.*

Lemma 11. *Let $(\phi(1), \ldots, \phi(N))$ be a state sequence of the unperturbed system (5), obtained from initial state $\phi(0) := x(k) \in \mathbb{X}$ and by applying an input sequence $\mathbf{u}_{[N-1]} = (u(0), \ldots, u(N-1)) \in \mathcal{U}_N(x(k))$. Let $(j_1, \ldots, j_{N-1}) \in \mathcal{S}^{N-1}$ be the corresponding mode sequence in the sense that $\phi(i) \in \Omega_{j_i} \sim \mathcal{L}_\mu^i \subset \Omega_{j_i}$, $i = 1, \ldots, N-1$. Let $(\bar{\phi}(1), \ldots, \bar{\phi}(N))$ be also a state sequence of the unperturbed system (5), obtained from the initial state $\bar{\phi}(0) := x(k+1) = \phi(1) + w(k)$ for some $w(k) \in \mathcal{B}_\mu$ and by applying the shifted input sequence $\bar{\mathbf{u}}_{[N-1]} := (u(1), \ldots, u(N-1), h(\bar{\phi}(N-1)))$. Then,*

$$(\bar{\phi}(i), \phi(i+1)) \in \Omega_{j_{i+1}} \times \Omega_{j_{i+1}}, \quad i = \overline{0, N-2}, \tag{8a}$$

$$\|\bar{\phi}(i) - \phi(i+1)\| \leq \eta^{[i]}(\|w(k)\|), \quad i = \overline{0, N-1}. \tag{8b}$$

Proof (*Proof of* Theorem 9). (i) We will show that $\bar{\mathbf{u}}_{[N-1]}$, as defined in Lemma 11, is feasible at time $k+1$. Let $(j_1, \ldots, j_{N-1}) \in \mathcal{S}^{N-1}$ be such that $\phi(i) \in \Omega_{j_i} \sim \mathcal{L}_\mu^i \subset \Omega_{j_i}$, $i = 1, \ldots, N-1$. Then, due to property (8b) and $\phi(i+1) \in \Omega_{j_{i+1}} \sim \mathcal{L}_\mu^{i+1}$, it follows from Lemma 10 that $\bar{\phi}(i) \in \Omega_{j_{i+1}} \sim \mathcal{L}_\mu^i \subset \mathbb{X}_i$ for $i = 1, \ldots, N-2$. From $\|\bar{\phi}(N-1) - \phi(N)\| \leq \eta^{[N-1]}(\|w(k)\|) \leq \eta^{[N-1]}(\mu)$ and Assumption 7-(iii) it follows that

$$F(\bar{\phi}(N-1)) - F(\phi(N)) \leq \alpha_F(\eta^{[N-1]}(\mu)),$$

which implies $F(\bar{\phi}(N-1)) \leq \theta_1 + \alpha_F(\eta^{[N-1]}(\mu)) \leq \theta$ due to $\phi(N) \in \mathbb{X}_T = \mathbb{F}_{\theta_1}$ and $\alpha_F(\eta^{[N-1]}(\mu)) \leq \theta - \theta_1$. Hence $\bar{\phi}(N-1) \in \mathbb{F}_\theta \subset \mathbb{X}_\mathbb{U} \cap (\Omega_{j_0} \sim \mathcal{L}_\mu^{N-1}) \subset \mathbb{X}_\mathbb{U} \cap \mathbb{X}_{N-1}$, so that $h(\bar{\phi}(N-1)) \in \mathbb{U}$ and $\bar{\phi}(N) \in \mathbb{F}_{\theta_1} = \mathbb{X}_T$. Thus, the sequence $\bar{\mathbf{u}}_{[N-1]}$ is feasible at time $k+1$, which proves the first part of (i). Moreover, since $g_{j_0}(x, h(x)) \in \mathbb{F}_{\theta_1}$ for all $x \in \mathbb{F}_\theta$ and

was defined in Assumption 5. However, we do not now require that the origin lies in the interior of one of the regions Ω_j in the state-space partition. The MPC problem set-up remains the same as the one described by Problem 6, with the only difference that the *set of admissible input sequences* for an initial condition $x \in \mathbb{X}$ is now defined as (without any tightening):

$$\mathcal{U}_N(x) := \{\mathbf{u}_{[N-1]} \in \mathbb{U}^N \mid \phi(i) \in \mathbb{X}, \quad i = 1, \dots, N-1, \phi(0) = x, \ \phi(N) \in \mathbb{X}_T\}. \quad (10)$$

All the definitions introduced in Sections 3 and 4 remain the same (e.g., $\mathbb{X}_f(N)$, $V^*(\cdot)$, $\Pi_\delta(\cdot)$, $\pi_\delta(\cdot)$, $\overline{\Pi}_\delta(\cdot)$, $\overline{\pi}_\delta(\cdot)$, etc.) with the observation that the set of admissible input sequences defined in (6) is replaced everywhere with the set defined in (10). We will use $\Xi_\delta(x(k)) := \{\xi(x(k), u) \mid u \in \pi_\delta(x(k))\}$ and $\overline{\Xi}_\delta(x(k)) := \{\xi(x(k), u) \mid u \in \overline{\pi}_\delta(x(k))\}$.

Theorem 16. *Let $\delta \in \mathbb{R}_{>0}$ be given and suppose that Assumption 7 holds for system (9) and Problem 6. Take $N \in \mathbb{Z}_{\geq 1}$, $\mathbb{X}_T$ with $0 \in \text{int}(\mathbb{X}_T)$ as a positively invariant set for system (9) in closed-loop with $u(k) = h(x(k))$, $k \in \mathbb{Z}_+$. Furthermore, suppose $F(\xi(x, h(x))) - F(x) + L(x, h(x)) \leq 0$ for all $x \in \mathbb{X}_T$.*

(i) *If Problem 6 is feasible at time $k \in \mathbb{Z}_+$ for state $x(k) \in \mathbb{X}$, then Problem 6 is feasible at time $k+1$ for any state $x(k+1) \in \Xi_\delta(x(k))$. Moreover, $\mathbb{X}_T \subseteq \mathbb{X}_f(N)$;*

(ii) *The closed-loop system $x(k+1) \in \Xi_\delta(x(k))$ is ε-AS in $\mathbb{X}_f(N)$ with $\varepsilon(\delta) := \left(\frac{2b}{a^2}\delta\right)^{\frac{1}{\lambda}}$;*

(iii) *Suppose that $\delta \in \mathbb{R}_{>0}$ satisfies $0 < \delta < a$, where $a \in \mathbb{R}_{>0}$ is the gain of the $\mathcal{K}$-function $\alpha_1(s) := as^\lambda$, introduced in Assumption 7. Then, the closed-loop system $x(k+1) \in \overline{\Xi}_\delta(x(k))$ is AS in $\mathbb{X}_f(N)$.*

The proof of the above theorem can be obtained *mutatis mutandis* by combining the reasoning used in the proof of Theorem III.2 in Lazar et al. (2006) (see also Lazar (2006)), and Theorem 4 for the case when $\sigma(s) \equiv 0$.

Remark 17. The result of Theorem 16, statement (ii), establishes that δ sub-optimal nonsmooth MPC is $\varepsilon(\delta)$-AS without requiring any additional assumption, other than the ones needed for AS of optimal smooth MPC (Mayne et al., 2000). Furthermore, the result of Theorem 16, statement (iii), introduces a slightly stronger condition, under which even AS can be guaranteed a priori for a specific class of sub-optimal predictive control laws. In contrast with the results in Scokaert et al. (1999) this is achieved without introducing additional stabilization constraints in the original MPC problem set-up.

6. Conclusion

In this paper we have considered discontinuous hybrid systems in closed-loop with predictive control laws. We presented conditions for ε-ISS and ε-AS of the resulting closed-loop systems. These conditions do not require continuity of the system dynamics nor optimality of the predictive control law. The latter is especially important as firstly, the infimum in an MPC optimization problem does not have to be attained and secondly, numerical solvers usually provide only sub-optimal solutions. An explicit relation was established between the deviation of the MPC control action from the optimum and the resulting deterioration of the ISS (AS) property of the closed-loop system. By imposing stronger conditions on the sub-optimal solutions, ISS can even be attained in this case.

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Appendix A. Proof of Lemma 10

Consider $y \in \mathbb{R}^n$ with $\|y - x\| \leq \eta^{[i]}(\mu)$. Let $\zeta \in \mathcal{L}^i_\mu$ and define $z := y - x + \zeta$. Then it holds that $\|z\| \leq \|y - x\| + \|\zeta\| \leq \eta^{[i]}(\mu) + \sum_{p=0}^{i-1} \eta^{[p]}(\mu) = \sum_{p=0}^{i} \eta^{[p]}(\mu)$ and thus, $z \in \mathcal{L}^{i+1}_\mu$. Together with $x \in \Omega_j \sim \mathcal{L}^{i+1}_\mu$ this yields $x + z \in \Omega_j$. Hence, $y + \zeta = x + z \in \Omega_j$. Since $\zeta \in \mathcal{L}^i_\mu$ was arbitrary, we have $y \in \Omega_j \sim \mathcal{L}^i_\mu$. □

Appendix B. Proof of Lemma 11

Property (8a) obviously holds for $i = 0$, since $\bar{\phi}(0) = \phi(1) + w(k)$, $w(k) \in \mathcal{B}_\mu = \mathcal{L}^1_\mu$ and $\phi(1) \in \Omega_{j_1} \sim \mathcal{L}^1_\mu$. Property (8b) holds for $i = 0$ as $\|\bar{\phi}(0) - \phi(1)\| = \|w(k)\| = \eta^{[0]}(\|w(k)\|)$. We proceed by induction. Suppose that both (8a) and (8b) hold for $0 \leq i-1 < N-2$. Then, since $\phi(i-1) \in \Omega_{j_i}$ and $\|\bar{\phi}(i-1) - \phi(i)\| \leq \eta^{[i-1]}(\|w(k)\|)$, it follows that:

$$\begin{aligned}\|\bar{\phi}(i) - \phi(i+1)\| &= \|g_{j_i}(\bar{\phi}(i-1), u(i)) - g_{j_i}(\phi(i), u(i))\| \\ &\leq \eta_{j_i}(\|\bar{\phi}(i-1) - \phi(i)\|) \leq \eta(\|\bar{\phi}(i-1) - \phi(i)\|) \\ &\leq \eta(\eta^{[i-1]}(\|w(k)\|)) = \eta^{[i]}(\|w(k)\|),\end{aligned} \quad (B.1)$$

and thus, (8b) holds for i. Next, as $\eta^{[i]}(\|w(k)\|) \leq \eta^{[i]}(\mu) \leq \sum_{p=0}^{i} \eta^{[p]}(\mu)$, it follows that $\bar{\phi}(i) - \phi(i+1) \in \mathcal{L}^{i+1}_\mu$. Then, since $\phi(i+1) \in \Omega_{j_{i+1}} \sim \mathcal{L}^{i+1}_\mu$, we have that $\phi(i+1) + (\bar{\phi}(i) - \phi(i+1)) = \bar{\phi}(i) \in \Omega_{j_{i+1}}$. Hence, (8a) holds for i. Thus, we proved that (8a) holds for $i = 0, \dots, N-2$ and (8b) holds for $i = 0, \dots, N-2$. Then, (8a) and (8b) for $i = N-2$ imply (8b) for $i = N-1$ via the reasoning used in (B.1). □

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M. Lazar (born in 1978) received his Ph.D. degree in Control Engineering from the Eindhoven University of Technology, Eindhoven, The Netherlands, September 2006. For his thesis he received the EECI (European Embedded Control Institute) Award for the best Ph.D. thesis in Europe on Networked and Embedded Control for the period 2006–2007. He received the Outstanding Reviewer distinction from the IEEE Transactions on Automatic Control journal for three years consecutively, starting with 2003. Since September 2006 he is an assistant professor in the Electrical Engineering Department of the Eindhoven University of Technology. He was invited as main speaker at the 3rd International Workshop on Assessment an Future Directions of Nonlinear Model Predictive Control, Pavia, Italy, 2008. His current research interests include nonlinear model predictive control, stability and robustness issues in hybrid systems, polyhedral invariant sets, networked control systems and real-time control of mechatronic systems.

W.P.M.H. Heemels (1972) received the M.Sc. degree (with honours) in mathematics and the Ph.D. degree (with highest distinction) from the department of Electrical Engineering (EE), Technische Universiteit Eindhoven (TU/e), Eindhoven, The Netherlands, in 1995 and 1999, respectively. From 2000 to 2004, he was with the Control Systems Group, EE, TU/e, as an Assistant Professor and from 2004 to 2006, with the Embedded Systems Institute (ESI) as a Research Fellow. Since 2006, he has been with the Control Systems Technology Group, department of Mechanical Engineering, TU/e, as an Associated Professor. He was a Visiting Professor at the Swiss Federal Institute of Technology (ETH), Zurich, Switzerland (2001) and at the University of California at Santa Barbara (2008). In 2004, he worked at the R & D laboratory, Oce, Venlo, The Netherlands, for four months. His current research interests include hybrid and nonsmooth dynamical systems, constrained (control) systems including model predictive control and networked control systems.