Squaring the Circle: An Algorithm for Generating Polyhedral Invariant Sets from Ellipsoidal Ones

M. Lazar, *Student Member, IEEE* A. Alessio A. Bemporad, *Member, IEEE* W.P.M.H. Heemels

***Abstract*— This paper presents a new (geometrical) approach to the computation of *polyhedral* positively invariant sets for general (possibly discontinuous) nonlinear systems, possibly affected by disturbances. Given a β-contractive ellipsoidal set $\mathscr{E}$, the key idea is to construct a polyhedral set that lies between the ellipsoidal sets $\beta\mathscr{E}$ and $\mathscr{E}$. A proof that the resulting polyhedral set is positively invariant (and contractive under an additional assumption) is given, and a new algorithm is developed to construct the desired polyhedral set. An advantage of the proposed method is that the problem of computing polyhedral invariant sets is formulated as a number of Quadratic Programming (QP) problems. The number of QP problems is guaranteed to be finite and therefore, the algorithm has finite termination. An important application of the proposed algorithm is the computation of polyhedral terminal constraint sets for model predictive control based on quadratic costs.**

***Index Terms*— Positively invariant sets, Contractive sets, Model predictive control, Stability, Robust stability.**

I. Introduction

Positively invariant sets and contractive sets have been used in many control theoretical problems, such as synthesis of stabilizing controllers, computation of domains of attraction and robustness analysis, e.g. see [1] and [2] for a comprehensive overview. In particular, positively invariant sets play a very important role in the design of stabilizing Model Predictive Controllers (MPC). For example, the terminal cost and constraint set approach in MPC [3] requires that the terminal set is positively invariant under some appropriate local feedback.

The most utilized types of invariant sets are *ellipsoidal* invariant sets, which have a simple representation, but can be less flexible than *polyhedral* invariant sets, which, in turn, can be arbitrarily complex. Polyhedral invariant sets are preferred in various cases due to the fact that they are often derived from physical constraints on state and control variables, which makes them a better approximation of reachable sets and of domains of attraction for dynamical systems. Moreover, a polyhedral domain of attraction is more suitable for use in an optimization problem. Indeed, for *model predictive control based on quadratic costs* where, in order to guarantee stability, one has to constrain the terminal state to a *terminal set*, which can be naturally chosen as an ellipsoidal sublevel set of a constructed (local) quadratic Lyapunov function (which is needed as terminal cost). However, if an ellipsoidal set is used as the terminal set, then the MPC optimization problem becomes a *quadratically constrained* Quadratic Programming (QP) problem in case linear prediction models are used (or a mixed integer quadratically constrained QP problem, if piecewise affine prediction models are used), which is usually not tackled by standard solvers. If a *polyhedral* invariant set is employed instead, then the MPC optimization problem is a QP (or mixed integer QP) problem. Since most MPC algorithms with an a priori stability guarantee are based on quadratic costs, e.g. see the survey [3] for an overview, a lot of effort has been put in developing new approaches for computing *polyhedral* positively invariant sets. Significant advances in the computation of polyhedral invariant sets were recently obtained in [4], [5] for linear systems affected by additive disturbances and in [6] for linear systems affected by parametric uncertainties. Promising results were also reported for linear systems subject to input saturation [7] and piecewise affine systems [8], [9].

In this paper we consider the problem of constructing a polyhedral positively invariant set when an ellipsoidal one is already available, which is the case for MPC based on quadratic costs, as mentioned before. Given a β-contractive[1] ellipsoidal set $\mathscr{E}$, the key idea is to construct a polyhedral set that lies between the ellipsoidal sets $\beta\mathscr{E}$ and $\mathscr{E}$. We prove that the resulting polyhedral set is positively invariant (and contractive if an additional requirement is satisfied). The problem of fitting a polyhedral set between two ellipsoidal sets (with one ellipsoidal set contained in the interior of the other ellipsoidal set) is solved in the current paper by treating the ellipsoidal sets as sublevel sets of quadratic functions and constructing a PieceWise Affine (PWA) function that approximates the "outer" quadratic function well enough, i.e. so that its graph lies between the graphs of the two quadratic functions. A solution to the original problem is then obtained by retrieving a suitable sublevel set of the resulting PWA function.

One of the advantages of the proposed algorithm is that it requires the solution of a finite number of QP problems and its computational complexity is bounded. This bound guarantees that the algorithm has finite termination. Also, due to its unique geometrical approach, which is independent of the system dynamics, the method is applicable to a wide class

M. Lazar is with the Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, E-mail: m.lazar@tue.nl.

A. Alessio and A. Bemporad are with the Dipartimento di Ingegneria dell'Informazione, Universitá di Siena, Via Roma 56, 53100 Siena, Italy, E-mail: alessio@dii.unisi.it, bemporad@dii.unisi.it.

W.P.M.H. Heemels is with the Embedded Systems Institute, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, E-mail: maurice.heemels@esi.nl.

[1] A set $\mathscr{E}$ is a β-contractive set for an arbitrary discrete-time system, if for all initial conditions in $\mathscr{E}$, the state obtained after one discrete-time step lies in the set $\beta\mathscr{E}$.

1-4244-0210-7/06/$20.00 ©2006 IEEE

of systems, including linear systems affected by disturbances or subject to input saturation, switched linear systems under arbitrary switching and piecewise linear systems defined on conical regions in the state-space.

The paper is organized as follows. The problem statement and the proposed solution are presented in Section II. The algorithm for constructing the desired polyhedral set is given in Section III and computational complexity aspects are addressed in Section IV. Illustrative examples are presented in Section V and the conclusions are summarized in Section VI.

A. Notation and basic definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For a set $\mathscr{S} \subseteq \mathbb{R}^n$, we denote by $\partial\mathscr{S}$ the boundary of $\mathscr{S}$, by $\text{int}(\mathscr{S})$ its interior and by $\text{cl}(\mathscr{S})$ its closure. For some points of $\mathscr{S}$, i.e. $(\theta_0,\ldots,\theta_n)$, let $\text{Co}(\theta_0,\ldots,\theta_n)$ denote their convex hull. For any real $\lambda \geq 0$, the set $\lambda\mathscr{S}$ is defined as $\{x \in \mathbb{R}^n \mid x = \lambda y \text{ for some } y \in \mathscr{S}\}$. A polyhedron (or a polyhedral set) is a set obtained as the intersection of a finite number of open and/or closed half-spaces. A piecewise polyhedral set is the union of a finite number of polyhedra. Given $(n+1)$ *affinely independent* points $(\theta_0,\ldots,\theta_n)$ of $\mathbb{R}^n$, i.e. $(1\ \theta_0^\top)^\top,\ldots,(1\ \theta_n^\top)^\top$ are linearly independent in $\mathbb{R}^{n+1}$, we define a simplex S as

$$S \triangleq \text{Co}(\theta_0,\ldots,\theta_n) \triangleq \{x \in \mathbb{R}^n \mid x = \sum_{l=0}^{n}\mu_l\theta_l, \sum_{l=0}^{n}\mu_l = 1, \mu_l \geq 0 \text{ for } l = 0,1,\ldots,n\}.$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is a *quadratic function* if $f(x) := x^\top Px + Cx + \alpha$ for some $P \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{1\times n}$ and $\alpha \in \mathbb{R}$. A quadratic function f is *strictly convex* if and only if $P > 0$. An ellipsoid (or an ellipsoidal set) $\mathscr{E}$ is a sublevel set (corresponding to some constant level $f_0 \in \mathbb{R}_+$) of a strictly convex quadratic function, i.e. $\mathscr{E} := \{x \in \mathbb{R}^n \mid f(x) \leq f_0\}$. Let $\Omega_1,\ldots,\Omega_N$ denote a polyhedral partition of $\mathbb{R}^n$, i.e. Ω_i is a polyhedron (not necessarily closed) for all $i = 1,\ldots,N$, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $\cup_{i=1,\ldots,N}\Omega_i = \mathbb{R}^n$.

Definition I.1 A function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = x^\top P_i x + C_i x + \alpha_i$ when $x \in \Omega_i$, $i = 1,\ldots,N$ is called a *PieceWise Quadratic (PWQ) function*. A function $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ with $\bar{f}(x) = H_i x + a_i$ when $x \in \Omega_i$, for some $H_i \in \mathbb{R}^{1\times n}$, $a_i \in \mathbb{R}$, $i = 1,\ldots,N$ is called a *PieceWise Affine (PWA) function*.

A piecewise ellipsoidal set is a sublevel set of a piecewise quadratic function with matrices $P_i > 0$ for all $i = 1,\ldots,N$.

II. Problem Statement and Proposed Solution

Consider the discrete-time perturbed nonlinear system:

$$x_{k+1} = G(x_k, w_k, v_k), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{W} \subset \mathbb{R}^p$ and $v_k \in \mathbb{V} \subset \mathbb{R}^q$ are the state and an unknown *parametric uncertainty* and *disturbance input*, respectively, and $\mathbb{W}$ and $\mathbb{V}$ are known, bounded sets. $G : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ is an arbitrary, possibly discontinuous, nonlinear function. For simplicity, we assume that the origin is an equilibrium in (1) for zero disturbance input, meaning that $G(0,w,0) = 0$ for all $w \in \mathbb{W}$.

Definition II.1 For a given $0 \leq \lambda \leq 1$, a set $\mathscr{P} \subseteq \mathbb{R}^n$ is called a *(robust) λ-contractive set* for system (1) if for all $x \in \mathscr{P}$ it holds that $G(x,w,v) \in \lambda\mathscr{P}$ for all $w \in \mathbb{W}$ and all $v \in \mathbb{V}$. For $\lambda = 1$ a (robust) λ-contractive set is called a *(robust) positively invariant set*.

For a set $\mathscr{P} \subseteq \mathbb{R}^n$, let $\mathscr{Q}_1(\mathscr{P}) := \{x \in \mathbb{R}^n \mid G(x,w,v) \in \mathscr{P}, \forall w \in \mathbb{W}, \forall v \in \mathbb{V}\}$ denote the (robust) one-step controllable set for system (1), with respect to $\mathscr{P}$.

In this paper we address the problem of computing polyhedral Positively Invariant (PI) sets and polyhedral contractive sets for system (1).

Problem II.2 Suppose that a (piecewise) ellipsoidal β-contractive set with $\beta \in [0, 1)$ is known for system (1). (i) Construct a (piecewise) polyhedral PI set for system (1); (ii) Construct a (piecewise) polyhedral λ-contractive set with $\lambda \in [0,1)$ for system (1).

Note that systematic solutions to obtain β-contractive (piecewise) *ellipsoidal* sets are available in the literature for many relevant subclasses of (1), such as linear systems subject to input saturation [10], linear systems affected by parametric uncertainties [11] and/or additive disturbances [1], piecewise affine systems [12]. Typically, they are obtained as sublevel sets of quadratic (PWQ) Lyapunov functions, which can be calculated efficiently via semi-definite programming.

Most of the existent methods for solving Problem II.2 are based on recursive algorithms that compute one-step controllable or one-step reachable sets [13] and they are applicable to *perturbed linear systems*. For example, see the *forward procedure* presented in [13] (extensions of this method to piecewise affine systems were proposed in [8], [9]), the *backward procedure* introduced in [14] or the reachability based algorithm given in [1]. Although these algorithms do not require that an ellipsoidal contractive set is known, existence of a quadratic Lyapunov function (and thus, existence of an ellipsoidal contractive set) can be used to prove finite termination for the forward procedure, e.g. see [8].

In this paper we generalize results from [15] to obtain a novel solution to Problem II.2. In [15] (see Lemma 4.1 and Lemma 4.2), where perturbed linear systems are considered, it was shown that a polyhedral set contained in between two *convex* sublevel sets of a Lyapunov function is positively invariant and λ-contractive. The result of [15] is extended in the theorem presented below to a wide class of systems, which includes, for example, any PWQ stabilizable system.

Theorem II.3 *Consider system* (1) *and let $\mathscr{E} \subseteq \mathbb{R}^n$ be a β-contractive set for system* (1), *for some $\beta \in (0,1)$, that contains the origin in its interior.*

(i) *Suppose there exists a set* $\mathscr{P} \subseteq \mathbb{R}^n$ *that satisfies* $\beta\mathscr{E} \subset \mathscr{P} \subset \mathscr{E}$*. Then,* $\mathscr{P}$ *is a PI set for system* (1) *and* $0 \in \text{int}(\mathscr{P})$*;*

(ii) *Let* $\beta\mathscr{E} \subset \lambda\mathscr{P} \subset \mathscr{P} \subset \mathscr{E}$ *for some*[2] $\lambda \in (0,1)$*. Then,* $\mathscr{P}$ *is a* λ*-contractive set for system* (1) *and* $0 \in \text{int}(\mathscr{P})$*. Moreover,* $\mathscr{Q}_1(\lambda\mathscr{P})$ *is a* λ*-contractive set for system* (1) *and* $\mathscr{E} \subset \mathscr{Q}_1(\lambda\mathscr{P})$*.*

Proof: (i) For any $x \in \mathscr{P} \subset \mathscr{E}$ it follows that $G(x,w,v) \in \beta\mathscr{E} \subset \mathscr{P}$ for any $w \in \mathbb{W}$ and any $v \in \mathbb{V}$ due to the fact that $\mathscr{E}$ is a β-contractive set for system (1). Hence, $\mathscr{P}$ is a PI set for system (1). Since $\mathscr{E}$ contains the origin in its interior, $\beta\mathscr{E}$ contains the origin in its interior and thus, $0 \in \text{int}(\mathscr{P})$;

(ii) Applying the same reasoning as above we have that for any $x \in \mathscr{P} \subset \mathscr{E}$ it follows that $G(x,w,v) \in \beta\mathscr{E} \subset \lambda\mathscr{P}$ for any $w \in \mathbb{W}$ and any $v \in \mathbb{V}$ due to the fact that $\mathscr{E}$ is a β-contractive set for system (1). Hence, $\mathscr{P}$ is a λ-contractive set for system (1) and $0 \in \text{int}(\mathscr{P})$;

Moreover, from the fact that for any $x \in \mathscr{E}$ it holds that $G(x,w,v) \in \beta\mathscr{E} \subset \lambda\mathscr{P}$ for any $w \in \mathbb{W}$ and any $v \in \mathbb{V}$, it follows that $\mathscr{E} \subset \mathscr{Q}_1(\lambda\mathscr{P})$. Since $\mathscr{P} \subset \mathscr{E}$, we have that $\mathscr{P} \subset \mathscr{Q}_1(\lambda\mathscr{P})$ and thus, $\lambda\mathscr{P} \subset \lambda\mathscr{Q}_1(\lambda\mathscr{P})$. Then, for any $x \in \mathscr{Q}_1(\lambda\mathscr{P})$ we have that $G(x,w,v) \in \lambda\mathscr{P} \subset \lambda\mathscr{Q}_1(\lambda\mathscr{P})$ for any $w \in \mathbb{W}$ and any $v \in \mathbb{V}$. Hence, $\mathscr{Q}_1(\lambda\mathscr{P})$ is a λ-contractive set for system (1) and $\mathscr{E} \subset \mathscr{Q}_1(\lambda\mathscr{P})$. ■

Note that the results of Theorem II.3 also apply to certain types of *non-convex* sets $\mathscr{E}$ and $\mathscr{P}$, i.e. piecewise ellipsoidal and piecewise polyhedral sets, respectively (see [16] for an illustrative example). Also, a λ-contractive polyhedral set $\mathscr{P}$ can be obtained without the additional hypothesis of Theorem II.3-(ii). Indeed, if $\mathscr{E}$ is β-contractive with $\beta \in (0,1)$ we can solve the "tighter" inclusion $\sqrt{\beta}\mathscr{E} \subset \mathscr{P} \subset \mathscr{E}$. Then, we obtain

$$\beta\mathscr{E} \subset \sqrt{\beta}\mathscr{P} \subset \mathscr{P} \subset \mathscr{E},$$

which is the hypothesis of Theorem II.3-(ii) with $\lambda = \sqrt{\beta}$.

The case of interest in this paper is, as stated in Problem II.2, when $\mathscr{E}$ is a piecewise *ellipsoidal* set and $\mathscr{P}$ is a piecewise *polyhedral* set. By Theorem II.3, it is sufficient to construct a piecewise polyhedral set $\mathscr{P}$ that lies between the piecewise ellipsoidal sets $\beta\mathscr{E}$ and $\mathscr{E}$ to obtain a solution to Problem II.2. In the next section we present an algorithm for solving this problem of computational geometry.

Remark II.4 The result $\mathscr{E} \subset \mathscr{Q}_1(\lambda\mathscr{P})$ (of Theorem II.3-(ii)) is relevant when the state of system (1) is constrained in a compact polyhedral set $\mathbb{X} \subset \mathbb{R}^n$ with $0 \in \text{int}(\mathbb{X})$. Then, given the largest β-contractive piecewise *ellipsoidal* set contained in $\mathbb{X}$, a larger, piecewise *polyhedral*, λ-contractive set can be simply obtained by computing the set $\mathscr{Q}_1(\lambda\mathscr{P}) \cap \mathbb{X}$.

III. "Squaring the Circle"

In this section we present a solution to the problem of fitting a piecewise polyhedral set $\mathscr{P}$ between two piecewise ellipsoidal sets where one is contained in the interior of the other, i.e. $\beta\mathscr{E} \subsetneq \mathscr{E}$, with β a number[3] in $(0,1)$. In case $\mathscr{E}$ is an ellipsoid, the main idea is to treat the sets $\mathscr{E}$ and $\beta\mathscr{E}$ as sublevel sets of two *quadratic* functions $f_{\mathscr{E}}(x)$ and $f_{\beta\mathscr{E}}(x)$, respectively, that correspond to a certain constant (level) $f_0 \in \mathbb{R}_+$, i.e. $\mathscr{E} := \{x \in \mathbb{R}^n \mid f_{\mathscr{E}}(x) \leq f_0\}$ and $\beta\mathscr{E} := \{x \in \mathbb{R}^n \mid f_{\beta\mathscr{E}}(x) \leq f_0\}$. Then, we compute a PWA function $\bar{f}$ that satisfies $f_{\beta\mathscr{E}}(x) > \bar{f}(x) \geq f_{\mathscr{E}}(x)$ for all $x \in \mathbb{R}^n$. The desired polyhedral set is obtained as $\mathscr{P} := \{x \in \mathbb{R}^n \mid \bar{f}(x) \leq f_0\}$.

In the *piecewise quadratic* case we assume that the polyhedral partitioning $\{\Omega_j \mid j \in \mathscr{S}\}$ ($\mathscr{S}$ is a finite set of indexes) consists of cones, which ensures that $\beta\Omega_j \subseteq \Omega_j$. We write $\mathscr{E}$ as:

$$\mathscr{E} = \bigcup_{j\in\mathscr{S}} (\mathscr{E}_j \cap \Omega_j) \text{ with } \mathscr{E}_j \triangleq \{x \in \mathbb{R}^n \mid f_{\mathscr{E}_j}(x) \leq f_0\},$$

where $f_{\mathscr{E}_j} := x^\top P_j x + C_j x_+ \alpha_j$ is a strictly convex quadratic function for all $j \in \mathscr{S}$. Then, we construct a PWA function $\bar{f}_j(x)$, as in the quadratic case mentioned above, such that $f_{\beta\mathscr{E}_j}(x) > \bar{f}_j(x) \geq f_{\mathscr{E}_j}(x)$ for all $x \in \mathbb{R}^n$ and for all $j \in \mathscr{S}$. Then, a piecewise polyhedral set $\mathscr{P}$ that satisfies $\beta\mathscr{E} \subset \mathscr{P} \subset \mathscr{E}$ is simply obtained as

$$\mathscr{P} = \bigcup_{j\in\mathscr{S}} (\mathscr{P}_j \cap \Omega_j) \text{ with } \mathscr{P}_j \triangleq \{x \in \mathbb{R}^n \mid \bar{f}_j(x) \leq f_0\}.$$

Indeed, as $\mathscr{P}_j$ is a polyhedral set that satisfies $\beta\mathscr{E}_j \subset \mathscr{P}_j \subset \mathscr{E}_j$, $j \in \mathscr{S}$, we obtain

$$\mathscr{P} = \bigcup_{j\in\mathscr{S}} (\mathscr{P}_j \cap \Omega_j) \subset \bigcup_{j\in\mathscr{S}} (\mathscr{E}_j \cap \Omega_j) = \mathscr{E}.$$

Since $\beta\mathscr{E}_j \subset \mathscr{P}_j$ and $\beta\Omega_j \subseteq \Omega_j$ for all $j \in \mathscr{S}$, we have that:

$$\beta\mathscr{E} = \beta\left(\bigcup_{j\in\mathscr{S}} (\mathscr{E}_j \cap \Omega_j)\right) = \bigcup_{j\in\mathscr{S}} \beta(\mathscr{E}_j \cap \Omega_j) = \\ = \bigcup_{j\in\mathscr{S}} (\beta\mathscr{E}_j \cap \beta\Omega_j) \subseteq \bigcup_{j\in\mathscr{S}} (\mathscr{P}_j \cap \Omega_j) = \mathscr{P}.$$

As the PWQ case can be split into a finite number of quadratic instances of the problem, in the following we consider only the quadratic case, i.e. when the set $\mathscr{E}$ is a sublevel set of a strictly convex quadratic function $f_{\mathscr{E}}$.

Next, choose $P \in \mathbb{R}^{n\times n}$ (with $P > 0$) and $f_0, \alpha_{\mathscr{E}} \in \mathbb{R}$ (with $f_0 > \alpha_{\mathscr{E}}$) such that $\mathscr{E}$ is the sublevel set of $f_{\mathscr{E}}(x) := x^\top P x + \alpha_{\mathscr{E}}$, corresponding to the level f_0. Then, we have that $\beta\mathscr{E}$ is the sublevel set of $f_{\beta\mathscr{E}}(x) := x^\top P x + \alpha_{\beta\mathscr{E}}$, corresponding to the level f_0, where $\alpha_{\beta\mathscr{E}} := (1-\beta^2)f_0 + \beta^2\alpha_{\mathscr{E}} > \alpha_{\mathscr{E}}$. Consider now an initial polyhedron $\mathscr{P}_0 \subset \mathbb{R}^n$ that contains $\mathscr{E}$. Let $(\tilde{\theta}_0, .., \tilde{\theta}_m)$, with $m \geq n$, be the vertices of $\mathscr{P}_0$. An initial set of simpleces $S_1^0, \ldots, S_{l_0}^0$ that contains these points is determined by Delaunay triangulation [17]. Then, for every simplex $S_i^0 := \text{Co}(\theta_{0i}^0, \ldots, \theta_{ni}^0)$, $i = 1, \ldots, l_0$, the following operations are performed.

Algorithm III.1 1) Let $k = 0$.

2) For every simplex S_i^k, $i = 1, \ldots, l_k$, construct the matrix $M_i^k \triangleq \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \theta_{0i}^k & \theta_{1i}^k & \ldots & \theta_{ni}^k \end{bmatrix}$.

[2] Note that the result also holds when $\beta = 0$ and $\lambda = 0$ but in this case $\mathscr{P}$ does not necessarily contain the origin in its interior.

[3] The case $\beta = 0$ is trivial: any $\mathscr{P} \subset \mathscr{E}$ with $0 \in \text{int}(\mathscr{P})$ works.

3) Set $v_i^k \triangleq [\ f_{\mathscr{E}}(\theta_{0i}^k)\ f_{\mathscr{E}}(\theta_{1i}^k)\ \dots\ f_{\mathscr{E}}(\theta_{ni}^k)\ ]^\top$ and construct the function $\bar{f}_i^k(x) := (v_i^k)^\top (M_i^k)^{-1} \begin{bmatrix} 1 \\ x \end{bmatrix}$.

4) Solve the *QP* problem:

$$J_i^{k*} \triangleq \min_{x \in S_i^k} \left\{ J_i^k(x) \triangleq f_{\beta\mathscr{E}}(x) - \bar{f}_i^k(x) \right\}, \qquad (2)$$

and let $x_i^{k*} := \arg\min_{x \in S_i^k} J_i^k(x)$.

5) If $J_i^{k*} > 0$ for all $i = 1, \dots, l_k$, then Stop. Otherwise, for all S_i^k, $i = 1, \dots, l_k$, for which $J_i^{k*} \leq 0$ build new simpleces $\overline{S}_0^i, \overline{S}_1^i, \dots, \overline{S}_n^i$ defined by the vertices $(x_i^{k*}, \theta_{1i}^k, \dots, \theta_{ni}^k)$, $(\theta_{0i}^k, x_i^{k*}, \dots, \theta_{ni}^k)$, $\dots$ and $(\theta_{0i}^k, \dots, \theta_{ni}^k, x_i^{k*})$, respectively. Increment k by one, add the new simpleces $\overline{S}_0^i, \overline{S}_1^i, \dots, \overline{S}_n^i$ to the set of simpleces $\{S_i^k\}_{i=1,\dots,l_k}$ and repeat the algorithm recursively from Step 2.

Algorithm III.1 computes a simplicial partition of a given initial polyhedral set $\mathscr{P}_0$ that contains the ellipsoidal set $\mathscr{E}$, by splitting a single simplex S_i^k into $n+1$ simpleces. This is done by fixing a new vertex x_i^{k*} which is obtained by solving the QP problem (2), and by calculating a new PWA approximation over the new set of simpleces. The steps of Algorithm III.1 are repeated for all resulting simpleces, until $J_i^{k*} > 0$ for all simpleces. At every iteration k, a tighter PWA approximation of the quadratic function $f_{\mathscr{E}}$ is obtained. Algorithm III.1 proceeds in a typical branch & bound way, i.e. *branching* on a new vertex x_i^{k*}, and *bounding* whenever it finds a simplex S_i^k for which it holds that $J_i^{k*} > 0$.

Suppose Algorithm III.1 stops. At the $\bar{k}$-th iteration[4] for some $\bar{k} \in \mathbb{Z}_+$, the following PWA function is generated:

$$\begin{aligned} \bar{f}(x) &\triangleq \bar{f}_i^{\bar{k}}(x) \text{ when } x \in S_i^{\bar{k}}, i = 1, \dots, l_{\bar{k}} \\ &\triangleq H_i^{\bar{k}} x + a_i^{\bar{k}} \text{ when } x \in S_i^{\bar{k}}, i = 1, \dots, l_{\bar{k}}, \end{aligned} \qquad (3)$$

where $l_{\bar{k}}$ is the number of simpleces obtained at the end of Algorithm III.1 and $H_i^{\bar{k}} x + a_i^{\bar{k}} = (v_i^{\bar{k}})^\top (M_i^{\bar{k}})^{-1} \begin{bmatrix} 1 \\ x \end{bmatrix}$. The PWA function $\bar{f}$ constructed via Algorithm III.1 is a continuous function. Moreover, for $x = \sum_{j=0}^n \mu_j \theta_{ji}$ with $\sum_{j=0}^n \mu_j = 1$, the corresponding functions $\bar{f}_i^{\bar{k}}$ satisfy:

$$\bar{f}_i^{\bar{k}}(x) = \bar{f}_i^{\bar{k}} \left(\sum_{j=0}^n \mu_j \theta_{ji} \right) = \sum_{j=0}^n \mu_j f_{\mathscr{E}}(\theta_{ji}),$$

which, by strict convexity of $f_{\mathscr{E}}$, implies that $\bar{f}_i^{\bar{k}}(x) \geq f_{\mathscr{E}}(x)$ for all $x \in S_i^{\bar{k}}$ and all $i = 1, \dots, l_{\bar{k}}$. Hence, $\bar{f}(x) \geq f_{\mathscr{E}}(x)$ for all $x \in \mathscr{P}_0$. Since the stopping criterion defined in Step 4 of Algorithm III.1 assures that at the end of the entire procedure the optimal value $J_i^{\bar{k}*}$ of the QP problem (2) will be greater than zero in every simplex $S_i^{\bar{k}}$, $i = 1, \dots, l_{\bar{k}}$, it follows that

$$f_{\mathscr{E}}(x) \leq \bar{f}(x) < f_{\beta\mathscr{E}}(x), \quad \forall x \in \cup_{i=1,\dots,l_{\bar{k}}} S_i^{\bar{k}}.$$

Then, the sublevel set of $\bar{f}$ given by

$$\overline{\mathscr{P}} \triangleq \bigcup_{i=1,\dots,l_{\bar{k}}} \{x \in S_i^{\bar{k}} \mid H_i^{\bar{k}} x + a_i^{\bar{k}} \leq f_0\}$$

[4]The existence of a finite $\bar{k}$ will be proven in Section IV.

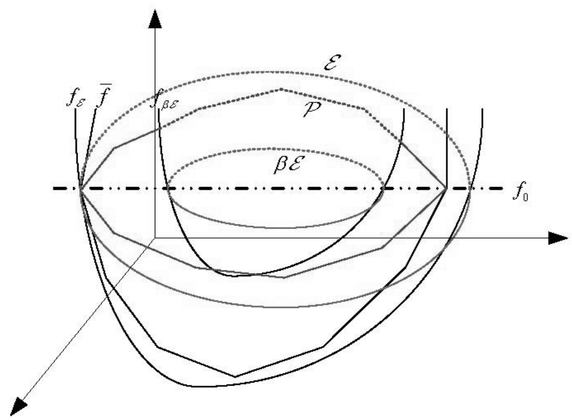

Fig. 1. Illustration of the proposed solution for constructing the polyhedral invariant set $\mathscr{P}$.

satisfies $\beta\mathscr{E} \subset \overline{\mathscr{P}} \subset \mathscr{E}$. Indeed, note that for $x \in \overline{\mathscr{P}}$ it holds that

$$\bar{f}(x) \leq f_0 \Rightarrow f_{\mathscr{E}}(x) \leq \bar{f}(x) \leq f_0 \Rightarrow x \in \mathscr{E},$$

and for $x \in \beta\mathscr{E}$ it holds that

$$f_{\beta\mathscr{E}}(x) \leq f_0 \Rightarrow \bar{f}(x) < f_{\beta\mathscr{E}}(x) \leq f_0 \Rightarrow x \in \overline{\mathscr{P}}.$$

The desired polyhedral set $\mathscr{P}$ (see Figure 1) satisfying $\beta\mathscr{E} \subset \mathscr{P} \subset \mathscr{E}$, is obtained as the convex hull of the vertices of $\overline{\mathscr{P}}$. Indeed,

$$\beta\mathscr{E} \subset \overline{\mathscr{P}} \subset \mathscr{P} \Rightarrow \beta\mathscr{E} \subset \mathscr{P}$$

and, by the convexity of $\mathscr{E}$, it holds that

$$\mathscr{P} \triangleq \text{Co}(\overline{\mathscr{P}}) \subseteq \text{Co}(\mathscr{E}) = \mathscr{E}.$$

Note that the computation of the vertices of $\overline{\mathscr{P}}$ and of their convex hull can be performed efficiently using, for instance, the Geometric Bounding Toolbox (GBT) [18]. Also, an ellipsoidal β-contractive set with $\beta \in (0,1)$ as small as possible is desirable, as this will result in a polyhedral positively invariant (λ-contractive) set of lower complexity.

Remark III.2 *The λ-contractive case*: the approximation error $\bar{\varepsilon} := \max_{x \in \mathscr{P}_0} [\bar{f}(x) - f_{\mathscr{E}}(x)]$ obtained at the end of Algorithm III.1 is upper bounded by the allowed maximum error defined as $\varepsilon_{\max} := \min_{x \in \mathscr{P}_0} [f_{\beta\mathscr{E}}(x) - f_{\mathscr{E}}(x)] > 0$. Thus, the Stop criterion of Algorithm III.1 can be set as $J_i^{k*} > \delta$ for some $\delta \in (0, \varepsilon_{\max})$, instead of just $J_i^{k*} > 0$, to create a gap between $\mathscr{P}$ and $\beta\mathscr{E}$. A larger δ will result in a smaller $\lambda \in (0,1)$ for which it holds that $\beta\mathscr{E} \subset \lambda\mathscr{P} \subset \mathscr{P} \subset \mathscr{E}$. The number of vertices of $\mathscr{P}$ tends to infinity, $\mathscr{P}$ recovers the ellipsoidal set $\mathscr{E}$ and λ tends to β when δ tends to $\varepsilon_{\max}$.

IV. An Estimate of the Computational Complexity

Algorithm III.1 computes at every iteration k a tighter PWA approximation $\bar{f}^k$ of the given strictly convex quadratic function $f_{\mathscr{E}}$. It stops when the approximation error obtained at the k-th iteration of the algorithm satisfies

$$\varepsilon_k \triangleq \max_{x \in \mathscr{P}_0} [\bar{f}^k(x) - f_{\mathscr{E}}(x)] \leq \varepsilon_{\max}, \quad k \in \mathbb{Z}_+.$$

Indeed, if the above inequality holds for some finite $\bar{k} \in \mathbb{Z}_+$, then for all $x \in \mathscr{P}_0$ it holds that $\bar{f}^{\bar{k}}(x) - f_{\mathscr{E}}(x) < \varepsilon_{\max}$, which implies that $\bar{f}^{\bar{k}}(x) < f_{\mathscr{E}}(x) + \varepsilon_{\max} \leq f_{\beta\mathscr{E}}(x)$. Consider now the following assumption.

Assumption IV.1 The optimum x_i^{k*} obtained in Step 4 of Algorithm III.1 for every simplex S_i^k, $i = 1, \ldots, l_k$, $k \in \mathbb{Z}_+$, satisfies $x_i^{k*} \in \text{int}(S_i^k)$.

In [19], the authors proved that under Assumption IV.1 the error ε_k committed at the k-th iteration of the algorithm is such that

$$\frac{\varepsilon_{k-1}}{4} \leq \varepsilon_k \leq \frac{\varepsilon_{k-1}}{2}, \quad \forall k \in \mathbb{Z}_+ \setminus \{0\}.$$

The algorithm builds recursively a tree where in each node it stores the vertices of the current simplex S_i^k and the pairs (H_i^k, a_i^k) such that $\bar{f}^k(x) = H_i^k x + a_i^k$, for all $x \in S_i^k$, $i \geq 1$, $k \in \mathbb{Z}_+$. If the value of J_i^{k*} for the current simplex is less than zero, then Algorithm III.1 splits S_i^k in $n+1$ simpleces and adds a new level to the tree. The height of the tree can be easily computed once the values of the initial error $\varepsilon_0 := \max_{x \in \mathscr{P}_0}[\bar{f}^0(x) - f_{\mathscr{E}}(x)]$ and of the allowed maximum error ε_{max} are known, which yields the following upper bound on the complexity of Algorithm III.1.

Theorem IV.2 *Suppose that the initial polyhedral set $\mathscr{P}_0$, the initial error ε_0 and the desired final approximation error ε_{max} are known. Furthermore, suppose Assumption IV.1 holds*[5]*. Then, the following bound holds on the height ξ_T of the tree generated by Algorithm III.1:*

$$\xi_T \leq \left\lceil \log_2 \frac{\varepsilon_0}{\varepsilon_{max}} \right\rceil,$$

where $\lceil a \rceil$ denotes the smallest integer larger than $a \in \mathbb{R}$.

Note that the height ξ_T of the tree and the number of nodes give the number of simpleces for which the steps of Algorithm III.1 have to be performed. This in turn yields the number of QP problems that have to be solved, which is of order $l_0(n+1)^{\left\lceil \log_2 \frac{\varepsilon_0}{\varepsilon_{\max}} \right\rceil}$, where l_0 is the initial number of simpleces and n is the dimension of the state-space. Hence, Algorithm III.1 always terminates in finite time.

V. Examples

In this section we present two examples to illustrate the effectiveness of the new algorithm for computing polyhedral positively invariant sets.

A. Linear systems subject to input saturation

Consider the following linear system subject to input saturation [10]:

$$x_{k+1} = Ax_k + B\,\text{sat}(u_k), \quad k \in \mathbb{Z}_+, \tag{4}$$

[5]If x_i^{k*} lies on a facet of S_i^k for some $i \geq 1$, $k \in \mathbb{Z}_+$, the same result holds with some minor modifications to the splitting strategy.

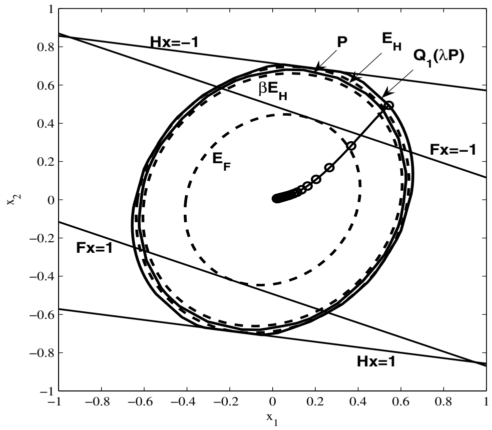

Fig. 2. Polyhedral (solid line) and ellipsoidal (dashed line) invariant sets.

where $A = \begin{bmatrix} 0.8876 & -0.5555 \\ 0.5555 & 1.5542 \end{bmatrix}$, $B = \begin{bmatrix} -0.1124 \\ 0.5555 \end{bmatrix}$ and $\text{sat}(u_k) := \text{sgn}(u_k)\min(1, |u_k|)$. In [10], a quadratically stabilizing state-feedback control law for system (4), i.e. $u_k = Fx_k$, with $F = [-0.7651 \; -2.0299]$, and a quadratic Lyapunov function $V(x) = x^\top Px$ with $P = \begin{bmatrix} 5.0127 & -0.6475 \\ -0.6475 & 4.2135 \end{bmatrix}$ were calculated. Since this control law does not take into account the input saturation, the maximal feasible domain of attraction for the closed-loop system is given by the ellipsoidal sublevel set $\mathscr{E}_F$ of V, corresponding to the level $f_0 = 0.8237$, see Figure 2. To obtain a larger ellipsoidal domain of attraction for the feedback F, we employed the LMI technique of [10], which yielded the new feedback matrix $H = [-0.2 \; -1.4]$ that takes into account the effect of saturation and establishes the enlarged ellipsoidal domain of attraction $\mathscr{E}_H$ (i.e. the sublevel set of V, corresponding to the level $f_0 = 2$) for system (4) in closed-loop with $u_k = Fx_k$.

Next, we employed the method developed in this paper in order to calculate a polyhedral set $\mathscr{P}$ such that $\beta\mathscr{E}_H \subset \mathscr{P} \subset \mathscr{E}_H$, where $\beta = 0.95$ is the contraction factor of $\mathscr{E}_H$. The resulting polyhedron is λ-contractive with $\lambda = 0.98$ and has 65 vertices. The set $\mathscr{P}$ and the enlarged polyhedral domain of attraction $\mathscr{Q}_1(\lambda\mathscr{P})$, which contains the (ellipsoidal) domain of attraction $\mathscr{E}_H$, are plotted in Figure 2 together with the closed-loop state trajectory for the initial state $x_0 = [0.5434 \; 0.4938]^\top$. The polyhedral set $\mathscr{Q}_1(\lambda\mathscr{P})$ can now be used as the terminal set in an MPC algorithm based on quadratic costs. The resulting MPC constrained optimization problem for system (4) is a QP problem, while if $\mathscr{E}_H$ would be used as the terminal set, the MPC optimization problem would be a quadratically constrained QP problem. Moreover, since $\mathscr{E}_H \subset \mathscr{Q}_1(\lambda_P)$, the MPC optimization problem with $\mathscr{Q}_1(\lambda\mathscr{P})$ as terminal set will have a larger feasible set in general, see, for example, the survey [3].

B. Linear systems subject to additive disturbances

Consider the following discrete-time triple integrator affected by additive disturbances:

$$x_{k+1} = Ax_k + Bu_k + v_k, \quad k \in \mathbb{Z}_+, \tag{5}$$

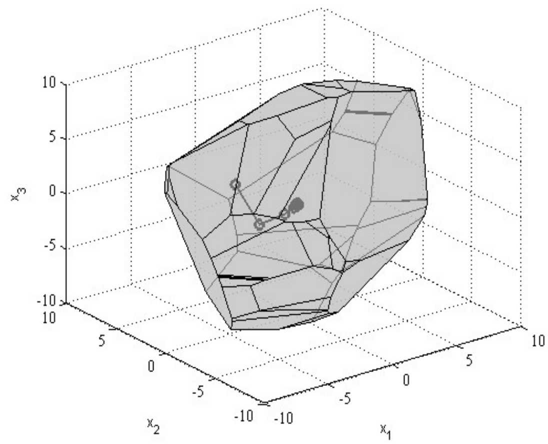

Fig. 3. Polyhedral invariant set and state trajectory for system (5) in closed-loop with $u = Kx$ and randomly generated disturbances v in $\mathbb{V}$.

where $A = \begin{bmatrix} 1 & T_s & \frac{T_s^2}{2} \\ 0 & 1 & T_s \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} \frac{T_s^3}{3!} \\ \frac{T_s^2}{2} \\ T_s \end{bmatrix}$, $T_s = 0.8$, $v_k \in \mathbb{V}$ is the additive disturbance input, and $\mathbb{V} = [-0.1,\, 0.1] \times [-0.1,\, 0.1] \times [-0.1,\, 0.1]$. We calculated a robust stabilizing state-feedback control law for system (5), i.e. $u_k = Kx_k$, with $K = [-1.1739\ -2.4071\ -2.0888]$, together with a robust quadratic Lyapunov function $V(x) = x^\top P x$ with $P = \begin{bmatrix} 14.4684 & 13.5850 & 4.0221 \\ 13.5850 & 17.4375 & 5.4581 \\ 4.0221 & 5.4581 & 2.5328 \end{bmatrix}$. The procedure presented in this paper was employed to calculate a polyhedral set $\mathscr{P}$ such that $\beta\mathscr{E} \subset \mathscr{P} \subset \mathscr{E}$, where $\mathscr{E}$ is the sublevel set of V, corresponding to the level $f_0 = 20$, and the contraction factor is $\beta = 0.8$. The resulting set $\mathscr{P}$ is λ-contractive with $\lambda = 0.9$ and has 56 vertices. A plot of $\mathscr{P}$ is given in Figure 3 together with a plot of the closed-loop system state trajectory obtained for $x_0 = [-3\ 2\ 2]^\top$ and randomly generated additive disturbance inputs.

VI. Conclusions

A new method for computing (piecewise) *polyhedral* positively invariant and contractive sets was developed based on a geometrical argument. The novelty of the proposed approach consists of formulating the problem of computing polyhedral invariant sets as solving a number of QP problems. This was achieved by observing that any polyhedral set that lies between two ellipsoidal sets $\beta\mathscr{E}$ and $\mathscr{E}$ with $\mathscr{E}$ β-contractive for some $\beta \in (0,1)$ is positively invariant (and contractive if an additional condition is satisfied). A new algorithm based on QP was developed in order to construct the desired polyhedral set. A guarantee that the number of QP problems that need to be solved is always finite was also given. This fact establishes finite termination for the algorithm.

The method works for a wide class of systems, including linear systems affected by parametric uncertainties and/or additive disturbances, linear systems subject to input saturation and certain relevant classes of hybrid systems. The algorithm presented in this paper has already been used in [16] to calculate a *polyhedral* terminal constraint set for a hybrid MPC scheme based on quadratic costs, which is an important application.

VII. Acknowledgements

The authors are grateful to the reviewers for their helpful comments. Research supported by the Dutch Science Foundation (STW), Grant "Model Predictive Control for Hybrid Systems" (DMR. 5675) and by the European Union through the Network of Excellence HYCON (contract number FP6-IST-511368).

References

[1] I. Kolmanovsky and E. Gilbert, "Theory and computation of disturbance invariant sets for discrete-time linear systems," *Mathematical Problems in Engineering*, vol. 4, pp. 317–367, 1998.

[2] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, pp. 1747–1767, 1999.

[3] D. Mayne, J. Rawlings, C. Rao, and P. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789–814, 2000.

[4] S. Raković, E. Kerrigan, K. Kouramas, and D. Mayne, "Invariant approximations of the minimal robust positively invariant set," *IEEE Transactions on Automatic Control*, vol. 50, pp. 406–410, 2005.

[5] S. Raković, D. Mayne, E. Kerrigan, and K. Kouramas, "Optimized robust control invariant sets for constrained linear discrete-time systems," in *16th IFAC World Congress*, Prague, 2005.

[6] B. Pluymers, J. Rossiter, J. Suykens, and B. De Moor, "The efficient computation of polyhedral invariant sets for linear systems with polytopic uncertainty," in *American Control Conference*, Portland, Oregon, 2005, pp. 804–809.

[7] A. Cepeda, D. Limon, T. Alamo, and E. Camacho, "Computation of polyhedral H-invariant sets for saturated systems," in *43nd IEEE Conference on Decision and Control*, Paradise Island, Bahamas, 2004, pp. 1176–1181.

[8] S. Raković, P. Grieder, M. Kvasnica, D. Mayne, and M. Morari, "Computation of invariant sets for piecewise affine discrete time systems subject to bounded disturbances," in *43nd IEEE Conference on Decision and Control*, Paradise Island, Bahamas, 2004, pp. 1418–1423.

[9] M. Lazar, W. Heemels, S. Weiland, and A. Bemporad, "Stabilization conditions for model predictive control of constrained PWA systems," in *43rd IEEE Conference on Decision and Control*, Paradise Island, Bahamas, 2004, pp. 4595–4600.

[10] T. Hu, Z. Lin, and M. Chen, "Analysis and design for discrete-time linear systems subject to actuator saturation," *Systems and Control Letters*, vol. 45, pp. 97–112, 2002.

[11] M. Kothare, V. Balakrishnan, and M. Morari, "Robust constrained model predictive control using linear matrix inequalities," *Automatica*, vol. 32, no. 10, pp. 1361–1379, 1996.

[12] G. Ferrari-Trecate, F. Cuzzola, D. Mignone, and M. Morari, "Analysis of discrete-time piecewise affine and hybrid systems," *Automatica*, vol. 38, no. 12, pp. 2139–2146, 2002.

[13] F. Blanchini, "Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 39, no. 2, pp. 428–433, 1994.

[14] F. Blanchini, F. Mesquine, and S. Miani, "Constrained stabilization with an assigned initial condition set," *International Journal of Control*, vol. 62, no. 3, pp. 601–617, 1995.

[15] F. Blanchini, "Nonquadratic Lyapunov functions for robust control," *Automatica*, vol. 31, no. 3, pp. 451–461, 1995.

[16] M. Lazar, W. Heemels, S. Weiland, and A. Bemporad, "On the stability of quadratic forms based Model Predictive Control of constrained PWA systems," in *24th American Control Conference*, Portland, Oregon, 2005, pp. 575–580.

[17] L. Yepremyan and J. Falk, "Delaunay partitions in $\mathbb{R}^n$ applied to non-convex programs and vertex/facet enumeration problems," *Computers & Operations Research*, vol. 32, pp. 793–812, 2005.

[18] S. Veres, *Geometric Bounding Toolbox (GBT) for Matlab*, 1995, official website: http://www.sysbrain.com.

[19] A. Alessio, A. Bemporad, B. Addis, and A. Pasini, "An algorithm for PWL approximations of nonlinear functions," Universita di Siena, Italy, Tech. Rep., 2005, web: http://control.dii.unisi.it/research/ABAP05.pdf.