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Stabilization of discrete-time switched linear systems: Lyapunov-Metzler inequalities versus S-procedure characterizations ⋆

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Abstract: In this paper we study connections between Lyapunov-Metzler inequalities and S-procedure characterizations in the context of stabilizing discrete-time switched linear systems using min-switching strategies. We propose two generalized versions of S-procedure characterization along the lines of the generalized versions of Lyapunov-Metzler inequalities recently proposed in the literature. It is shown that the existence of a solution to the generalized version(s) of Lyapunov-Metzler inequalities is equivalent to the existence of a solution to the generalized version(s) of S-procedure characterization with a restricted choice of the scalar quantities involved in the latter. This recovers some of our earlier works on the classical Lyapunov-Metzler inequalities as a special case. We also highlight and discuss an open question of whether the generalized versions of S-procedure characterization are strictly less conservative than the generalized versions of Lyapunov-Metzler inequalities, which in turn are equivalent to periodic stabilizability as was recently shown.

Keywords: Discrete-time switched linear systems, stabilizability, min-switching strategy, Lyapunov-Metzler inequalities, S-procedure characterizations, matrix inequalities

1. INTRODUCTION

A *switched system* consists of two components — a family of (sub)systems and a switching strategy. The *switching strategy* selects an *active subsystem* at every instant of time, i.e., the subsystem from the family that is currently determining the state evolution (Liberzon, 2003, §1.1.2). In this paper we study stabilization (Lee and Dullerud, 2006; Geromel and Colaneri, 2006) of discrete-time switched linear systems using so-called min-switching strategies. Two well-known tools in the literature for the design of min-switching strategies are the Lyapunov-Metzler inequalities and the S-procedure characterization.

The Lyapunov-Metzler inequalities proposed in (Geromel and Colaneri, 2006) comprise of a set of bilinear matrix inequalities (BMIs) involving the following to-be-designed parameters: a set of positive definite matrices, and a specific type of Metzler matrix. If for a given family of systems, the Lyapunov-Metzler inequalities are feasible, a stabilizing min-switching strategy has been constructed. The Lyapunov-Metzler inequalities are now standard in the literature for stabilization and performance analysis of switched systems. Recently, the Lyapunov-Metzler inequalities are extended to the case of constrained switched linear systems in (Jungers et al., 2016).

The S-procedure was first introduced in (Lur'e and Postnikov, 1944) with its theoretical justification in (Yakubovich, 1971). It is commonly used to ensure that certain "quadratic" functions only need to be negative if other "quadratic" functions are negative, see (Boyd et al., 1994; Feron, 1999; El Ghaoui and Niculescu, 2000) for a detailed discussion on this topic. The S-procedure characterizations are widely used for analyzing stability and performance of piecewise linear systems (the min-switching strategy results in a closed-loop system of this form), see e.g., (Ferrari-Trecate et al., 2002; Johansson and Rantzer, 1998). In the context of designing a min-switching strategy, the S-procedure characterization involves BMIs with the following to-be-designed parameters: a set of positive definite matrices, and sets of non-negative scalars.

Recently in (Heemels et al., 2017) we studied connections between Lyapunov-Metzler inequalities and S-procedure characterizations both in the continuous and discrete-time

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setting. Amongst others, we showed that in the context of *continuous-time* switched linear systems the S-procedure characterization is equivalent to the Lyapunov-Metzler inequalities. In this paper we are interested in the stabilization of discrete-time switched linear systems and compare *generalized* S-procedure characterizations and generalized Lyapunov-Metzler inequalities, which were recently proposed in the literature.

Indeed, in the recent work (Fiacchini et al., 2016, 2014) the authors proposed two generalized versions of the Lyapunov-Metzler inequalities, and showed that these are equivalent to periodic stabilizability (i.e., the existence of a time periodic switching strategy that results in a stable closed-loop switched linear system) and lead to stabilizing generalized min-switching strategies. Moreover, it was also established that the existence of a solution to the Lyapunov-Metzler inequalities is only a sufficient condition for the existence of the solution to the generalized Lyapunov-Metzler inequalities. Finally, in (Fiacchini et al., 2016) the authors showed that the existence of a stabilizing switching strategy does not imply periodic stabilizability.

In this paper we propose two generalized versions of the S-procedure characterization along the lines of the generalized versions of the Lyapunov-Metzler inequalities presented in (Fiacchini et al., 2016). It is shown that the existence of a solution to the generalized Lyapunov-Metzler inequalities is equivalent to the existence of a solution to the generalized S-procedure characterization with a restricted choice of the scalar quantities involved in the latter. Hence, generalized S-procedure characterizations are never more conservative than the generalized Lyapunov-Metzler inequalities. In fact, we recover some of the "discrete-time" results presented in (Heemels et al., 2017) that stated related relationships for the classical Lyapunov-Metzler inequalities and S-procedure characterizations as a special case.

An interesting question that arises from the above set of results is that whether the (generalized) S-procedure characterizations can "go beyond" (are less conservative) the (generalized) Lyapunov-Metzler inequalities (and hence periodic stabilizability). We investigate this question by considering a series of numerical examples, which seem to hint upon a "negative" answer to the above question. We provide a preliminary result that may aid in obtaining such an equivalence, although in the absence of analytical tools, it is unclear whether this equivalence is indeed true or not. Apart from presenting these new results towards generalizations and a comparative study between two well-known tools for stabilization of discrete-time switched linear systems available in the literature, another objective of this paper is to discuss the mentioned open question, thereby hopefully stimulating many researchers to consider this interesting problem.

2. PRELIMINARY RESULTS

In this section we formulate the problem under consideration and catalog the required preliminaries.

We consider a family of discrete-time linear systems

$$x(t+1) = A_j x(t),\ j \in \bar{\mathcal{N}},\ t \in \mathbb{N}_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the vector of states at time t, $\bar{\mathcal{N}} = \{1, 2, \ldots, N\}$ is an index set, and A_j, $j \in \bar{\mathcal{N}}$, are constant matrices in $\mathbb{R}^{n\times n}$. A *switching signal* $\sigma : \mathbb{N}_0 \to \bar{\mathcal{N}}$ selects an *active* system from the family (1) at every instant of time. The *switched system* generated by the family of systems (1) and a switching signal σ is given by

$$x(t+1) = A_{\sigma(t)} x(t),\ t \in \mathbb{N}. \tag{2}$$

Definition 1. (Fiacchini et al., 2016, Definition 2) The switched system (2) is (globally exponentially) stabilizable if there are scalars $c \geq 0$ and $\lambda \in [0, 1)$, and for all $x(0) \in \mathbb{R}^n$, there exists a switching strategy $\sigma : \mathbb{N} \to \bar{\mathcal{N}}$ such that the corresponding solution $x : \mathbb{N} \to \mathbb{R}^n$ satisfies:

$$\|x(t)\| \leq c\lambda^k \|x(0)\| \text{ for all } t \in \mathbb{N}. \tag{3}$$

We are interested in assessing this stabilizability property in this paper. In particular, we will focus on the stabilization of discrete-time switched linear systems (2) using so-called *min-switching strategies*. These switching strategies are widely used state-dependent signals defined as

$$\sigma(t) := \arg\min_{j\in\bar{\mathcal{N}}} x(t)^\top P_j x(t), \tag{4}$$

where $P_j \in \mathbb{R}^{n\times n}$ is a positive definite matrix associated to the j-th subsystem, $j \in \bar{\mathcal{N}}$.

In the context of stabilization of the switched system (2) using a min-switching strategy (4), two well-known analysis and design tools are Lyapunov-Metzler inequalities and S-procedure characterizations. We recall them briefly for self-containedness.

2.1 The Lyapunov-Metzler inequalities

A subclass of Metzler matrices plays a crucial role in Lyapunov-Metzler inequalities. Recall that a Metzler matrix is a square matrix in which all off-diagonal components are non-negative. The following subclass is of our interest:

Definition 2. For $N \in \mathbb{N} \setminus \{0\}$ the set $\mathcal{M}_d^N$ consists of all square matrices $\Pi \in \mathbb{R}^{N\times N}$ satisfying $\pi_{ij} \geq 0$ for all $i, j \in \bar{\mathcal{N}}$, and the column sums satisfying $\sum_{i=1}^{N} \pi_{ij} = 1$ for all $j \in \bar{\mathcal{N}}$.

Proposition 1. (Geromel and Colaneri, 2006, Theorem 3) Consider the discrete-time switched linear system (2). Suppose that there exist a set of positive definite matrices $\{P_j\}_{j\in\bar{\mathcal{N}}}$ and a matrix $\Pi \in \mathcal{M}_d^N$ such that the following set of inequalities is satisfied:

$$A_j^\top \left(\sum_{i\in\bar{\mathcal{N}}} \pi_{ij} P_i \right) A_j - P_j \prec 0 \text{ for all } j \in \bar{\mathcal{N}}. \tag{5}$$

Then the switched system (2) is stabilizable. In addition, the closed-loop switched linear system given by (2) and (4) is globally exponentially stable.

Condition (5) is known as the Lyapunov-Metzler inequalities for guaranteeing stabilizability of switched linear system (2) using strategies in (4). These inequalities are widely used in switched systems literature.

Recently in Fiacchini et al. (2016) the authors proposed two generalized versions of the Lyapunov-Metzler inequalities, and showed their equivalence to LMI-based conditions for stabilizability (Fiacchini et al., 2016, Theorem 15) and periodic stabilizability, which is defined as follows:

Definition 3. (Fiacchini et al., 2016, Definition 3) The discrete-time switched linear system (2) is called *periodically stabilizable* if there exists a *periodic* switching signal σ such that under the system (2) is globally exponentially stable. [1]

In the remainder of this section we recall the generalized versions of Lyapunov-Metzler inequalities from Fiacchini et al. (2016).

The following notations are employed:

- $\bar{\mathcal{N}}^k = \{(a_1, \ldots, a_k) \mid a_i \in \bar{\mathcal{N}}, i = 1, \ldots, k\}$
- $\bar{\mathcal{N}}^{[M:N]} = \cup_{k=M}^{N} \bar{\mathcal{N}}^k$ denotes all the possible sequences of numbers in $\bar{\mathcal{N}}$ of length from M to N,
- $\left|\bar{\mathcal{N}}^{[M:N]}\right|$ denotes the number of elements in $\bar{\mathcal{N}}^{[M:N]}$.
- Given $j = (j_1, \ldots, j_k)$ in $\bar{\mathcal{N}}^{[1:N]}$, we define $\mathbb{A}_j = \prod_{i=1}^{k} A_{j_i}$.

Proposition 2. (Fiacchini et al., 2016, Proposition 12) Consider the discrete-time switched linear system (2). Suppose that there exist $M \in \mathbb{N}$, a set of positive definite matrices $\{P_j\}_{j \in \bar{\mathcal{N}}^{[1:M]}}$, and a matrix $\Pi \in \mathcal{M}_d^{\left|\bar{\mathcal{N}}^{[1:M]}\right|}$ such that the following inequalities

$$\mathbb{A}_j^\top \left(\sum_{i \in \bar{\mathcal{N}}^{[1:M]}} \pi_{ij} P_i \right) \mathbb{A}_j - P_j \prec 0 \text{ for all } j \in \bar{\mathcal{N}}^{[1:M]} \tag{6}$$

are satisfied. Then the switched system (2) is stabilizable.

In (Fiacchini et al., 2016) the condition (6) are termed as the Lyapunov-Metzler inequalities Generalized I. It recovers condition (5) for $M = 1$. The condition (6) can be recast as the Lyapunov-Metzler condition (5) by considering the switched system (2) obtained by defining a fictitious subsystem for every matrix $\mathbb{A}_j$, $j \in \bar{\mathcal{N}}^{[1:M]}$, see (Fiacchini et al., 2016, Remark 13) for a detailed discussion.

Proposition 3. (Fiacchini et al., 2016, Proposition 14) If for every $j \in \bar{\mathcal{N}}$ there exist a set of indices $\mathcal{K}_j = \{1, 2, \ldots, h_j\}$, $h_j \in \mathbb{N}$, a set of positive definite matrices $\{P_k^{(j)}\}_{k \in \mathcal{K}_j}$, and there are $\pi_{m,k}^{(p,j)} \in [0, 1]$ satisfying

$$\sum_{p \in \bar{\mathcal{N}}} \sum_{m \in \mathcal{K}_p} \pi_{m,k}^{(p,j)} = 1, \tag{7}$$

for all $k \in \mathcal{K}_j$, such that the set of inequalities

$$A_j^\top \left(\sum_{p \in \bar{\mathcal{N}}} \sum_{m \in \mathcal{K}_p} \pi_{m,k}^{(p,j)} P_m^{(p)} \right) A_j - P_k^{(j)} \prec 0 \tag{8}$$

is satisfied for all $j \in \bar{\mathcal{N}}, k \in \mathcal{K}_j$. Then the switched system (2) is stabilizable.

Conditions (8) are termed Lyapunov-Metzler inequalities Generalized II. On the one hand, Lyapunov-Metzler inequalities Generalization I are based on increasing the length of a sequence. On the other hand, Lyapunov-Metzler inequalities Generalized II maintains the length of the sequence as 1, but increases the elements in the set of ellipsoids determined by $P_k^{(j)}$, $k \in \mathcal{K}_j$ associated to each subsystem $j \in \bar{\mathcal{N}}$, see (Fiacchini et al., 2016, p. 5) for a detailed discussion on this matter. Proposition 3 recovers Proposition 1 with $h_j = 1$ for all $j \in \bar{\mathcal{N}}$.

[1] We call the switching function σ periodic, if there exists a $T \in \mathbb{N}$ such that $\sigma(t + T) = \sigma(t)$ for all $t \in \mathbb{N}$.

2.2 The S-procedure characterization

The following proposition can be derived using the reasoning as in (Ferrari-Trecate et al., 2002) based on the S-procedure (Lur'e and Postnikov, 1944; Yakubovich, 1971), see also (Heemels et al., 2017, Theorem 9) for the details.

Proposition 4. Consider the discrete-time switched linear system (2). Suppose that there exist a set of positive definite matrices $\{P_j\}_{j \in \bar{\mathcal{N}}}$, and two sets of non-negative scalars $\{\alpha_k^{j \to i}\}_{i,j,k \in \bar{\mathcal{N}}, k \neq j}$ and $\{\beta_\ell^{j \to i}\}_{i,j,\ell \in \bar{\mathcal{N}}, \ell \neq i}$ such that the following set of inequalities is satisfied:

$$A_j^\top P_i A_j - P_j \prec \sum_{k \in \bar{\mathcal{N}}, k \neq j} \alpha_k^{j \to i} (P_j - P_k) + \sum_{\ell \in \bar{\mathcal{N}}, \ell \neq i} \beta_\ell^{j \to i} A_j^\top (P_i - P_\ell) A_j \text{ for all } i, j \in \bar{\mathcal{N}}. \tag{9}$$

Then the switched system (2) is stabilizable. In addition, the closed-loop switched linear system given by (2) and (4) is globally exponentially stable.

Condition (9) is known as the S-procedure characterization for guaranteeing stabilizability of switched linear system (2) using strategies in (4). This particular form for checking if the min-switching strategy (4) is indeed stabilizing for (2) was given in (Heemels et al., 2017). It is interesting to note that there are N positive definite matrices and $N(N-1)$ free scalar quantities (in the Metzler matrix) involved in a solution to the Lyapunov-Metzler inequalities (5), while a solution to the S-procedure characterization (9) contains N positive definite matrices and $2N^2(N-1)$ scalar quantities. This is due to the S-procedure characterization requiring $N^2(N-1)$ scalar quantities $\{\alpha_k^{j \to i}\}_{i,j,k \in \bar{\mathcal{N}}, k \neq j}$ corresponding to the regional conditions $x(t)^\top P_k x(t) \geq x(t)^\top P_j x(t)$ when $\sigma(t) = j$, and another $N^2(N-1)$ scalar quantities $\{\beta_\ell^{j \to i}\}_{i,j,\ell \in \bar{\mathcal{N}}, \ell \neq i}$ corresponding to regional conditions $x(t+1)^\top P_\ell x(t+1) \geq x(t+1)^\top P_i x(t+1)$ when $\sigma(t+1) = i$. Also, the S-procedure characterisation has N^2+N matrix inequalities (of size $n \times n$) and $2N^2(N-1)$ scalar inequalities, while the Lyapunov-Metzler inequalities have $N+1$ matrix inequalities (of size $n \times n$) and N^2 scalar inequalities.

We next propose two generalized versions of the S-procedure characterization, and establish their connections to the generalized versions to Lyapunov-Metzler inequalities described in §2.1.

3. MAIN RESULTS

3.1 S-procedure characterization: Generalization I

Proposition 5. Consider the discrete-time switched linear system (2). If there exist $M \in \mathbb{N}$, a set of positive definite matrices $\{P_j\}_{j \in \bar{\mathcal{N}}^{[1:M]}}$, and two sets of non-negative scalars $\{\alpha_k^{j \to i}\}_{i,j,k \in \bar{\mathcal{N}}^{[1:M]}, k \neq j}$ and $\{\beta_\ell^{j \to i}\}_{i,j,\ell \in \bar{\mathcal{N}}^{[1:M]}, \ell \neq i}$ such that the set of inequalities

$$\mathbb{A}_j^\top P_i \mathbb{A}_j - P_j \prec \sum_{k \in \bar{\mathcal{N}}^{[1:M]}, k \neq j} \alpha_k^{j \to i} (P_j - P_k) + \sum_{\ell \in \bar{\mathcal{N}}^{[1:M]}, \ell \neq i} \beta_\ell^{j \to i} \mathbb{A}_j^\top (P_i - P_\ell) \mathbb{A}_j \text{ for all } i, j \in \bar{\mathcal{N}}^{[1:M]} \tag{10}$$

is satisfied. Then the switched system (2) is stabilizable.

The proof of the above proposition follows under the same set of arguments as for S-procedure characterization (9) considering the switched system (2) with fictitious subsystems corresponding to each matrix $\mathbb{A}_j$, $j \in \bar{\mathcal{N}}^{[1:M]}$. We omit the proof here for brevity. It is evident that Proposition 5 recovers Proposition 4 for $M = 1$. A solution to condition (10) involves $\left|\bar{\mathcal{N}}^{[1:M]}\right|$ positive definite matrices and $2\left|\bar{\mathcal{N}}^{[1:M]}\right|^2\left(\left|\bar{\mathcal{N}}^{[1:M]}\right| - 1\right)$ scalars. In contrast, a solution to condition (6) involves $\left|\bar{\mathcal{N}}^{[1:M]}\right|$ positive definite matrices and $\left|\bar{\mathcal{N}}^{[1:M]}\right|\left(\left|\bar{\mathcal{N}}^{[1:M]}\right| - 1\right)$ free scalars.

We observe that S-procedure characterization Generalized I presented in Proposition 5 is related to Lyapunov-Metzler inequalities Generalized I discussed in Proposition 2 as formulated in the next theorem.

Theorem 1. Consider the discrete-time switched linear system (2). The following are equivalent:

(1) There is a solution to the Lyapunov-Metzler inequalities Generalized I (6).
(2) There is a solution to S-procedure characterization Generalized I (10) with for all $i, j \in \bar{\mathcal{N}}^{[1:M]}$ it holds that a) $\alpha_k^{j\to i} = 0$ for $k \neq j$, b) $\sum_{\ell\in\bar{\mathcal{N}}^{[1:M]},\ell\neq i} \beta_\ell^{j\to i} \le 1$, and c) $\beta_\ell^{j\to i} = \beta_\ell^{j\to r}$ for $\ell \in \bar{\mathcal{N}}^{[1:M]}\setminus\{i, r\}$, $r \in \bar{\mathcal{N}}^{[1:M]}$.

Proof. 1)⇒2): Pick an arbitrary $i \in \bar{\mathcal{N}}^{[1:M]}$. Given that there is a solution to (6), we have that

$$\mathbb{A}_j^\top\left(\sum_{\ell\in\bar{\mathcal{N}}^{[1:M]}} \pi_{\ell j} P_\ell\right)\mathbb{A}_j - P_j + \mathbb{A}_j^\top P_i \mathbb{A}_j - \mathbb{A}_j^\top P_i \mathbb{A}_j \prec 0 \tag{11}$$

By the properties of Π, the left-hand side of the above inequality is equal to $\mathbb{A}_j^\top\left(\sum_{\ell\in\bar{\mathcal{N}}^{[1:M]},\ell\neq i} \pi_{\ell j}(P_\ell - P_i)\right)\mathbb{A}_j - P_j$ $+\mathbb{A}_j^\top P_i \mathbb{A}_j$. Consequently, (11) can be written as

$$\mathbb{A}_j^\top P_i \mathbb{A}_j - P_j \prec \mathbb{A}_j^\top\left(\sum_{\ell\in\bar{\mathcal{N}}^{[1:M]},\ell\neq i} \pi_{\ell j}(P_i - P_\ell)\right)\mathbb{A}_j$$

for all $j \in \bar{\mathcal{N}}^{[1:M]}$.

Recall that $i \in \bar{\mathcal{N}}^{[1:M]}$ was chosen arbitrarily. We therefore conclude that (10) holds with $\alpha_k^{j\to i} = 0$ and $\beta_\ell^{j\to i} = \pi_{\ell j}$ for all $i, j, k, \ell \in \bar{\mathcal{N}}^{[1:M]}$, $k \neq j$, $\ell \neq i$. Clearly, $\sum_{\ell\in\bar{\mathcal{N}}^{[1:M]},\ell\neq i} \beta_\ell^{j\to i} = \sum_{\ell\in\bar{\mathcal{N}}^{[1:M]},\ell\neq i} \pi_{\ell j} = 1 - \pi_{ij} \le 1$, which gives property (b). Moreover, due to the choice of $\beta_\ell^{j\to i} = \pi_{\ell j}$ for all $i, j, \ell \in \bar{\mathcal{N}}^{[1:M]}$, $\ell \neq i$ also (c) holds.

2)⇒1): Given that (10) has a solution with (a), (b) and (c) satisfied for all $i, j \in \bar{\mathcal{N}}^{[1:M]}$, we have

$$P_j \succ \mathbb{A}_j^\top\left(P_i + \sum_{\ell\in\bar{\mathcal{N}}^{[1:M]},\ell\neq i} \beta_\ell^{j\to i}(P_\ell - P_i)\right)\mathbb{A}_j \tag{12}$$

for all $i, j \in \bar{\mathcal{N}}^{[1:M]}$.The right-hand side of the above inequality can be written as $\mathbb{A}_j^\top\left(\left(1 - \sum_{\ell\in\bar{\mathcal{N}}^{[1:M]}\ell\neq i} \beta_\ell^{j\to i}\right)P_i\right.$ $\left.+\sum_{\ell\in\bar{\mathcal{N}}^{[1:M]},\ell\neq i} \beta_\ell^{j\to i} P_\ell\right)\mathbb{A}_j$. By the hypothesis that $\sum_{\ell\in\bar{\mathcal{N}}^{[1:M]},\ell\neq i} \beta_\ell^{j\to i} \le 1$ and defining $\beta_i^{j\to i} := 1 - \sum_{\ell\in\bar{\mathcal{N}}^{[1:M]},\ell\neq i} \beta_\ell^{j\to i}$ ≥ 0, the above quantity is $\mathbb{A}_j^\top\left(\sum_{\ell\in\bar{\mathcal{N}}^{[1:M]}} \beta_\ell^{j\to i} P_\ell\right)\mathbb{A}_j$. Substituting this in (12), we obtain for all $i, j \in \bar{\mathcal{N}}^{[1:M]}$ that $\mathbb{A}_j^\top\left(\sum_{\ell\in\bar{\mathcal{N}}^{[1:M]}} \beta_\ell^{j\to i} P_\ell\right)\mathbb{A}_j - P_j \prec 0$. By taking now $i = 1$, we obtain (6) with $\pi_{\ell j} = \beta_\ell^{j\to 1}$ for all $j, \ell \in \bar{\mathcal{N}}^{[1:M]}$. Obviously, the resulting $\Pi \in \mathcal{M}_d^{\left|\bar{\mathcal{N}}^{[1:M]}\right|}$ thereby also satisfying (6). This completes the proof. □

Theorem 1 shows that the existence of a solution to Lyapunov-Metzler inequalities Generalized I is equivalent to the existence of a solution to S-procedure characterization Generalized I with the sets of scalars $\{\alpha_k^{j\to i}\}_{i,j,k\in\bar{\mathcal{N}}^{[1:M]},k\neq j}$ and $\{\beta_\ell^{j\to i}\}_{i,j,\ell\in\bar{\mathcal{N}}^{[1:M]},\ell\neq i}$ satisfying the conditions a), b), and c) in 2) above.

3.2 S-procedure characterization: Generalization II

We now proceed towards proposing a second generalized version of S-procedure characterization.

Proposition 6. Consider the discrete-time switched linear system (2). Suppose that for every $j \in \bar{\mathcal{N}}$ there exist $\mathcal{K}_j = \{1, 2, \ldots, h_j\}$, $h_j \in \mathbb{N}$, a set of positive definite matrices $\{P_k^{(j)}\}_{k\in\mathcal{K}_j}$, and two sets of non-negative scalars $\{\alpha_{(m,p)}^{(k,j)\to(\ell,i)}\}_{\substack{i,j,k,p\in\bar{\mathcal{N}},k\in\mathcal{K}_j,\ell\in\mathcal{K}_i,\\ m\in\mathcal{K}_p,(k,j)\neq(m,p)}}$ and $\{\beta_{(n,q)}^{(k,j)\to(\ell,i)}\}_{\substack{i,j,k,q\in\bar{\mathcal{N}},k\in\mathcal{K}_j,\ell\in\mathcal{K}_i,\\ n\in\mathcal{K}_q,(\ell,i)\neq(n,q)}}$ such that the following set of inequalities is satisfied:

$$\begin{aligned} A_j^\top P_\ell^{(i)} A_j - P_k^{(j)} \prec &\sum_{\substack{p\in\bar{\mathcal{N}},m\in\mathcal{K}_p,\\(k,j)\neq(m,p)}} \alpha_{(m,p)}^{(k,j)\to(\ell,i)}(P_k^{(j)} - P_m^{(p)}) \\ &+ \sum_{\substack{q\in\bar{\mathcal{N}},n\in\mathcal{K}_q,\\(\ell,i)\neq(n,q)}} \beta_{(n,q)}^{(k,j)\to(\ell,i)} A_j^\top(P_\ell^{(i)} - P_n^{(q)})A_j \end{aligned} \tag{13}$$

for all $i, j \in \bar{\mathcal{N}}$, $k \in \mathcal{K}_j$, $\ell \in \mathcal{K}_i$. Then the switched system (2) is stabilizable.

Proposition 6 utilizes a similar concept as in Proposition 3, i.e., while the length of the sequence is kept as 1, the number of elements in the set of ellipsoids determined by $P_k^{(j)}$, $j \in \bar{\mathcal{N}}$, is increased. We observe that under a similar set of restrictions on the scalar quantities $\alpha_{(m,p)}^{(k,j)\to(\ell,i)}$ and $\beta_{(n,q)}^{(k,j)\to(\ell,i)}$ as in Theorem 1, S-procedure characterization Generalized II and Lyapunov-Metzler inequalities Generalized II can be connected. This is formulated in the next theorem.

Theorem 2. Consider the discrete-time switched linear system (2). The following are equivalent:

(1) There is a solution to the Lyapunov-Metzler inequalities Generalized II (8).
(2) There is a solution to the S-procedure characterization Generalized II (13) with for all $i, j \in \bar{\mathcal{N}}$ it holds that a) $\alpha_{(m,p)}^{(k,j)\to(\ell,i)} = 0$ for all $p \in \bar{\mathcal{N}}$, $k \in \mathcal{K}_j$, $\ell \in \mathcal{K}_i$, $m \in \mathcal{K}_p$, $(k, j) \neq (m, p)$, b)

$$\sum_{q\in\bar{\mathcal{N}},n\in\mathcal{K}_q,(\ell,i)\neq(n,q)} \beta_{(n,q)}^{(k,j)\to(\ell,i)} \leq 1, \text{ and c) } \beta_{(n,q)}^{(k,j)\to(\ell,i)} = \beta_{(n,q)}^{(k,j)\to(\ell,r)} \text{ for } q \in \bar{\mathcal{N}}\backslash\{i,r\} \text{ and } r \in \bar{\mathcal{N}}.$$

The proof of the above theorem follows along similar lines as the proof of Theorem 1. We omit the proof for brevity. Recently, in Heemels et al. (2017) we showed that the existence of a solution to the Lyapunov-Metzler inequalities (5) is equivalent to the existence of a solution to the S-procedure characterization (9) with a restricted set of choice for the scalars $\{\alpha_k^{j\to i}\}_{i,j,k\in\bar{\mathcal{N}},k\neq j}$ and $\{\beta_\ell^{j\to i}\}_{i,j,\ell\in\bar{\mathcal{N}},\ell\neq i}$. We recall our result below.

Theorem 3. (Heemels et al., 2017, Theorem 10) Consider the discrete-time switched linear system (2). The following statements are equivalent:

(1) There is a solution to the Lyapunov-Metzler inequalities (5).
(2) There is a solution to the S-procedure characterization (9) with for all $i, j \in \bar{\mathcal{N}}$ it holds that a) $\alpha_k^{j\to i} = 0$ for $k \neq j$, b) $\sum_{\ell\in\bar{\mathcal{N}},\ell\neq i} \beta_\ell^{j\to i} \leq 1$, and c) $\beta_\ell^{j\to i} = \beta_\ell^{j\to r}$ for $\ell \in \bar{\mathcal{N}}\backslash\{i,r\}$ and $r \in \bar{\mathcal{N}}$.

Clearly, when $M = 1$, Theorem 1 recovers our earlier result Theorem 3. Also, Theorem 2 recovers Theorem 3 as a special case when $\mathcal{K}_j$ contains exactly one element for all $j \in \bar{\mathcal{N}}$.

Up to this point, we proposed two generalized versions of S-procedure characterizations, and discussed their connections to the generalized versions of Lyapunov-Metzler inequalities, which by the connections established in (Fiacchini et al., 2016, §V) are equivalent to periodic stabilizability.

3.3 Discussion

As observed in Theorems 1, 2 and 3, the S-procedure characterizations with a "restricted" choice of the scalar quantities is equivalent to the Lyapunov-Metzler inequalities. Indeed, right at the level of formulation, S-procedure characterizations involve "additional" regional conditions giving rise to the "additional" scalar quantities as already highlighted in §2. In addition, by definition of the Metzler matrix, the scalar quantities involved in Lyapunov-Metzler inequalities obey certain upper bounds unlike the scalar quantities in S-procedure characterization, which are just known to be non-negative. Figure 1 illustrates the connections among various existing stabilizability conditions obtained from Fiacchini et al. (2016) and Heemels et al. (2017) and Theorems 1, 2 and 3 in this paper.

4. OPEN QUESTIONS

Since the generalized L-M inequalities are equivalent to S-procedure characterizations with additional conditions on the scalars (some actually being 0), a natural question to pose is whether the S-procedure characterizations in the discrete-time setting are strictly less conservative than the corresponding Lyapunov-Metzler inequalities. In other words, are there families of (sub)systems that admit solutions to S-procedure characterizations (resp. generalized versions), but do not admit solutions to Lyapunov-Metzler inequalities (resp. generalized versions)? We formalize the questions as follows:

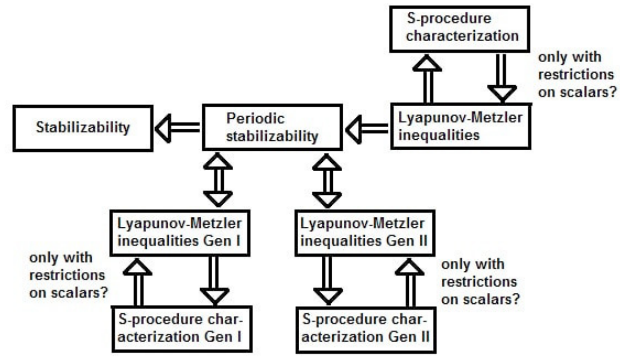

Fig. 1. Connections among stabilizability conditions based on Fiacchini et al. (2016); Heemels et al. (2017) and Theorems 1 and 2 in this paper

Question 1. (Classical version). Is the existence of a solution to the Lyapunov-Metzler inequalities (5) equivalent to the existence of a solution to the S-procedure characterization (9)?

Question 2. (Generalized version). Is the existence of a solution to the Lyapunov-Metzler inequalities Generalized I (6) (resp. Generalized II (8)) equivalent to the existence of a solution to the S-procedure characterization Generalized I (10) (resp. Generalized II (13))?

These questions are related to the question marks in Fig. 1. On the one hand, Question 1 is of interest because it pertains to the classical Lyapunov-Metzler inequalities. In addition, recently in Heemels et al. (2017) we showed that in the context of continuous-time switched linear systems the classical S-procedure characterization is equivalent to the classical Lyapunov-Metzler inequalities. It is therefore a natural question to study whether the same holds for discrete-time switched linear systems. On the other hand, Question 2 is of interest in the stablization of switched linear systems as the generalized Lyapunov-Metzler inequalities are equivalent to periodic stabilizability as was shown in Fiacchini et al. (2016) (while the classical Lyapunov-Metzler inequalities are not). It is of interest to know if the genearlized S-procedure characterizations I-II can actually go beyond the periodic stabilizabilition. In other words, are there switched systems for which the generalized S-procedure conditions can be true, but the system is not perioidcally stabilizable. We attempt to shed some light on these open questions using a set of numerical experiments.

5. NUMERICAL STUDY

Prior to presenting the observations from our numerical experiments, it is worth noting that the S-procedure characterizations involve BMIs, and identifying solutions (or their absence) to them is a difficult task. Thus, for the following set of results we rely on performance of standard BMI solving tools, which employ approximations. We stick to the simplest case of Lyapunov-Metzler inequalities (5) (which is also a sufficient condition for the existence of a solution to the generalized Lyapunov-Metzler inequalities (6) and (8)), and perform an "empirical study". While not finding a solution to (5) does not ensure that there is no solution to (6) (resp. (8)), it definitely provides some insights into the subtlety of the "gap" between Lyapunov-Metzler inequalities and S-procedure characterizations in general.

Example 4. We generate families of random 2×2 matrices with the entries of the matrices ranging in the interval $(-a, a)$. Here, a is picked from the interval $(0, k)$ uniformly at random. It is ensured that each matrix in the family is unstable by checking the maximum eigenvalue of the randomly generated matrices. The objective is to identify the cases (if any) where the Lyapunov-Metzler inequalities (5) do not have a solution, but the S-procedure characterization (9) admits a solution within a pre-specified duration of time. Our observations are listed in Table 1.

N	k	No. of samples samples	Solution to (5)	No solution to (5), but solution to (9)
2	2	100	11	0
	3	100	4	0
3	2	100	9	0
	3	100	1	0

Table 1. Data for Example 4

The above numerical study hints upon a situation similar to the case of continuous-time setting, see (Heemels et al., 2017) where the classical S-procedure characterization turned out to be equivalent to the classical Lyapunov-Metzler inequalities, thereby providing a "positive" answer to Question 1. However, a complete proof or a counterexample to assert the above, is currently missing. We state a first result in this direction, the proof of which is omitted for brevity.

Theorem 5. Consider the discrete-time switched linear system (2) with $\bar{\mathcal{N}} = \{1, 2\}$. Suppose that there exists a solution to the S-procedure characterization (9) with $\alpha_k^{j\to i} = 0$ for all $i, j, k \in \bar{\mathcal{N}}$, $k \neq j$. Then there exists a solution to the S-procedure characterization (9) with $\alpha_k^{j\to i} = 0$ for all $i, j, k \in \bar{\mathcal{N}}$, $k \neq j$ and $\beta_\ell^{j\to i} \leq 1$ for all $i, j, \ell \in \bar{\mathcal{N}}$, $\ell \neq i$. Moreover, the S-procedure characterization (9) reduces to

$$A_j^\top P_i A_j - P_j \prec \beta_\ell^{j\to i} A_j^\top (P_i - P_\ell) A_j \qquad (14)$$

for all $j \in \bar{\mathcal{N}}$ and any $i \in \bar{\mathcal{N}}$, $\ell \neq i$.

Observe that for $N = 2$, the Lyapunov-Metzler inequalities (5) involve two inequalities, while the S-procedure characterization (9) involves four inequalities. In Theorem 5 we show that for the S-procedure characterization (9) it suffices to solve two inequalities. Consequently, one of the restrictions on the parameters $\beta_\ell^{j\to i}$, $\ell \neq i$ in Theorem 3 is not restrictive at all for $N = 2$. However, even in the simplest case of S-procedure characterization (9) with $\alpha_k^{j\to i} = 0$ for all $i, j, k \in \bar{\mathcal{N}}$, $k \neq 0$ a general analytical proof for relaxing the conditions on $\beta_\ell^{j\to i}$, $i, j, \ell \in \bar{\mathcal{N}}$, $\ell \neq i$ does not seem to be obvious beyond $N = 2$. In particular, the following subquestions are unanswered: If there exists a solution to S-procedure characterization (9) with $\alpha_k^{j\to i} = 0$ for all $i, j, k \in \bar{\mathcal{N}}$, $k \neq j$, then does there exist a solution to S-procedure characterization (9) with $\alpha_k^{j\to i} = 0$ for all $i, j, k \in \bar{\mathcal{N}}$, $k \neq j$ and $\sum_{\ell\in\bar{\mathcal{N}},\ell\neq i} \beta_\ell^{j\to i} \leq 1$ for all $i, j \in \bar{\mathcal{N}}$? Moreover, if there exists a solution to S-procedure characterization (9), then does there exist a solution to S-procedure characterization (9) with $\alpha_k^{j\to i} = 0$ for all $i, j, k \in \bar{\mathcal{N}}$, $k \neq j$? If the answers to these are "yes", then it is evident that S-procedure characterization (9) is equivalent to Lyapunov-Metzler inequalities (5). Even though Lyapunov-Metzler inequalities (5) are just sufficient for their generalized versions to be satisfied, the proof techniques may carry over as is seen with the proof technique of Theorem 2.

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