A Consistent Threshold-Based Policy for Event-Triggered Control

B. Asadi Khashooei, D. J. Antunes, and W. P. M. H. Heemels

***Abstract*—A consistent event-triggered control (ETC) policy is defined as a policy that outperforms the performance of periodic control for the same average transmission rate and does not generate transmissions in the absence of disturbances. In this letter, we propose a threshold-based policy for periodic ETC that is consistent. Simulation results illustrate the strengths of the proposed method.**

***Index Terms*—Networked control systems, linear systems, stochastic optimal control.**

I. Introduction

For many years, periodic control has been prevalent in digital control systems, due to its simple implementation and the existence of powerful techniques to analyze such systems. However, in the context of networked control systems (NCSs), periodic sampling and control lacks the flexibility one needs to efficiently manage the computation, communication, and energy resources. This led to the advent of event-triggered control (ETC). The main idea behind ETC is to include state or output information for the determination of the times at which data-transmissions take place in a control loop. As such, in a networked control setting, the event-triggered controller aims at creating a balance between the performance of the system and the use of feedback (in terms of the average transmission rate).

Extensive research has been conducted on ETC over the past decade. Several works (e.g., [1]–[5]) have generalized the idea in [6] by proposing to trigger transmissions from sensors (with computational capacities) to a remote controller when the norm of the error between the current state and a state estimate (available both at the sensor's computational unit and at the controller) exceeds a certain (possibly state or output-dependent) threshold. Indeed, various triggering mechanisms have been proposed in the literature including relative triggering [1], [2], mixed triggering [2], [7] and, recently, dynamic triggering [8]–[10]. Furthermore, different criteria have been analyzed for ETC systems, including stability [1], $\mathcal{L}_p$-gain performance [2], [9] and quadratic cost criteria [4], [11]–[13]. When analyzing the performance based on a quadratic cost criteria, most of the works [11], [12], [14] formulate the event-triggered control problems in the framework of dynamic programming, and search for control policies that optimize the performance indices taking into account the state variables and the transmission rates. Despite their analytical importance, the obtained dynamic programming formulations in the aforementioned works suffer from the curse of dimensionality and, therefore, most of the optimal triggering policies proposed in the literature are hard to implement in practice, and lack the insight and simplicity of the basic policies described in the pioneering works [1], [6].

Prompted by these observations, in some works suboptimal ETC with guarantees on the closed-loop performance and/or on the network usage have been proposed [12], [15], [16]. One can trace back the suboptimal ETC approach to the early work of [6], where it was shown that a threshold-based sampling and control outperforms the periodic control in terms of the variance of the state at the same sampling rate for a scalar linear system.

Motivated by these developments, the concept of consistency was introduced in [17] and a consistent ETC policy taking the form of dynamic ETC was proposed. To recall, an ETC policy is consistent if it possesses the following two properties: (i) it outperforms the performance of periodic control for the same average transmission rate; (ii) it requires no sensor updates (i.e., operates in open-loop) in the absence of disturbances. Note that, the second consistency property typically only holds for ETC strategies in which a control-input-generator (CIG) based on a system model is implemented in the actuator side and the full state (not the output) of the system is transmitted via the network, e.g., [3]. In fact, in a control loop, where a zero-order hold (ZOH) is used as the CIG, the second property can not be guaranteed in general for an ETC.

In [17] an example was provided in which a policy, where transmissions are triggered if the Euclidean norm of the error is larger than a threshold, performs worse (in a quadratic cost sense) than a periodic control policy at the same (average) transmission rate. This revealed that ad-hoc error threshold policies may not necessarily result in a better trade-off between performance and transmission rate than periodic control for systems with higher order than 1 (as considered in [6]). Or stated differently, threshold-based policies are not necessarily consistent.

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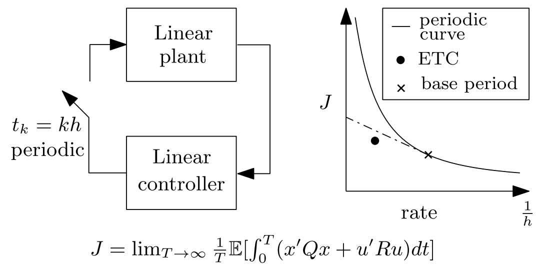

Fig. 1. Illustration of the key property to establish consistency in this letter. For periodic control, the trade-off curve between average quadratic performance and transmission rate is convex.

As a consequence, in this letter we are interested in the problem of designing a *threshold* policy, which is guaranteed to be consistent. Performance is measured by an average quadratic cost, as in the standard linear quadratic Gaussian (LQG) framework. While we consider continuous-time systems, the plant is only monitored periodically, at a fast rate, at which the transmission triggering condition is checked. Therefore, this can be seen as a periodic ETC (PETC) policy, see [2], where this term was introduced. The proposed solution builds upon a key result establishing the convexity of the trade-off curve between average quadratic cost performance and transmission rate for periodic control. In fact, we show that for the proposed ETC policy the pair (rate, performance) is below the tangent line of the periodic curve at the point corresponding to the base period of the PETC policy. This property is illustrated in Figure 1. The resulting threshold-based policy will use a weighted norm generally different from the Euclidean norm. Compared to the proposed consistent policy in [17], the new policy is simple to implement, because it does not require an integrator to realize the policy, and because the triggering condition is only evaluated at fixed-periodic sampling times making it suitable for digital implementation. In fact, since the triggering condition is only monitored at fixed-periodic sampling, a desired lower bound on minimum inter-event time can easily be tuned.

The remainder of this letter is organized as follows. Section II formulates the problem and Section III gives the cost of periodic control. Section IV provides the main result. Section V presents simulation results for a numerical example. Section VI provides concluding remarks.

II. Problem Formulation

Figure 2 shows the control structure considered in this letter. The plant model coincides with the jump linear model in [17] and is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \quad t \in \mathbb{R}_{\geq 0}\backslash\{s_\ell\}_{\ell\in\mathbb{N}_0}, \\ x(s_\ell) &= x(s_\ell^-) + \omega_\ell, \quad l \in \mathbb{N}_0, \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state and $u(t) \in \mathbb{R}^{n_u}$ is the control input at time $t \in \mathbb{R}_{\geq 0}$, and $\{\omega_\ell\}_{\ell\in\mathbb{N}_0}$, $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, is an independent and identically distributed (i.i.d.) sequence of random variables with zero mean and covariance $R_\omega = \mathbb{E}[\omega_\ell\omega_\ell^\intercal]$ for every $\ell \in \mathbb{N}_0$. Furthermore, we assume that (A, B) is controllable and $(A, R_\omega^{\frac{1}{2}})$ is observable. Moreover, $\{s_\ell\}_{\ell\in\mathbb{N}_0}$ denotes the sequence of times at which the disturbance impacts the state, where $s_0 = 0$ and $s_{\ell+1} - s_\ell$, $\ell \in \mathbb{N}_0$, are assumed to be independent and exponentially distributed with rate λ, i.e.,

$$\text{Prob}[s_{\ell+1} - s_\ell > a] = e^{-\lambda a} \quad \ell \in \mathbb{N}_0, \tag{2}$$

and $x(s_\ell^-) := \lim_{\epsilon\to 0,\epsilon>0} x(s_\ell - \epsilon)$.

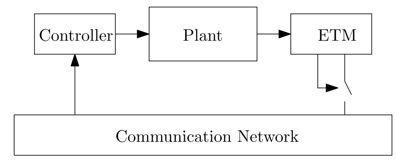

Fig. 2. Considered control structure. The ETC unit closes the loop when triggering conditions, checked in the event-triggering mechanism (ETM), are met. The controller computes the control actions based on the received data or the predictions. At both the controller and the ETM, there is a model-based control-input generator (CIG) based on estimates of the plant's state.

As explained in [17] and [18], this model can capture the behavior of the more common model used in literature where the process disturbances are modeled by a Wiener process when $\lambda \to \infty$, besides models with sporadic disturbances (small λ). One of the main motivations to consider it, is that it circumvents the technical difficulties associated with working directly with Wiener processes. However, note that our claims in the sequel hold only for fixed (possibly arbitrary large) λ, and considering the case $\lambda \to \infty$ warrants further research.

The event-triggering mechanism (ETM) samples the state of the plant at $t_k \in \mathbb{R}_{\geq 0}$ for $k \in \mathbb{N}_0$, and decides whether or not to transmit the value of the plant state $x(t_k)$ to the controller. Here, $k \in \mathbb{N}_0$ is the sample counter. In order to model the decision making process for the transmissions, we introduce $\sigma_k \in \{0, 1\} =: \mathcal{M}$, where $\sigma_k = 1$ means that at sample time t_k a transmission takes place and $\sigma_k = 0$ indicates that no transmission takes place at time t_k, $k \in \mathbb{N}_0$. At both the controller and the ETM, see Figure 2, there is a model-based control-input generator (CIG), which uses the linear estimator

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) \quad t \in \mathbb{R}_{\geq 0}\backslash\{t_k \mid \sigma_k = 1, \ k \in \mathbb{N}_0\}, \\ \hat{x}(t) &= x(t) \quad t \in \{t_k \mid \sigma_k = 1, \ k \in \mathbb{N}_0\}, \end{aligned} \tag{3}$$

with initial condition $\hat{x}_0 = x_0$, to generate the state estimate $\hat{x}(t)$ at time $t \in \mathbb{R}_{\geq 0}$. Moreover, the controller unit produces the control input $u(t)$, $t \in \mathbb{R}_{\geq 0}$, fed into the plant. Note that, as will be clear in the sequel, the control input $u(t)$ is also known by the ETM.

In this letter, we assume that the sampling times t_k, $k \in \mathbb{N}_0$, are equidistant, i.e., $t_k = k\tau$, where τ is a positive constant. Therefore, transmissions can only be triggered at integer multiples of the sampling period τ. In other words, if $\kappa_m \in \mathbb{R}_{\geq 0}$ denotes the m-th transmission time, where $m \in \mathbb{N}_0$ is a counter for the transmissions, then we have that

$$\{\kappa_m \mid m \in \mathbb{N}_0\} = \{k\tau \mid k \in N_0, \ \sigma_k = 1\}, \tag{4}$$

which is the set of transmission times as already used in (3). As already mentioned in the introduction, this class of ETC policies is known as periodic ETC [2].

We denote by $I^c(t)$ the available information at the controller at time $t \in R_{\geq 0}$, which is given by

$$I_m^c(t) = \{x(\kappa_m) \mid m \in \mathbb{N}_0, \kappa_m \leq t\}. \tag{5}$$

Furthermore, by I_k^s we denote the available information at the scheduler (ETM) at time $t_k, k \in \mathbb{N}_0$, given by

$$I_k^s = (I_{k-1}^s, x_k, \hat{x}_k) \quad \text{for } k \in \mathbb{N}_{\geq 1}, \tag{6}$$

where $x_k := x(t_k^-)$, $\hat{x}_k := \hat{x}(t_k^-)$ and $I_0^s = x_0$.

A control policy γ is a set of functions $\nu_m : (t,\ I^c_m(t)) \mapsto u(t)$ that map the available information at the controller at time t, i.e., $I^c_m(t)$, to the control actions $u(t)$, and hence, $u(t) = \nu_m(t,\ I^c_m(t))$. Similarly, the transmission policy θ is a set of functions $\mu_k : I^s_k \mapsto \mathcal{M}$ that map the available information at the ETM at time t_k, i.e., I^s_k, to the transmission decision σ_k in the set $\mathcal{M},\ k \in \mathbb{N}_0$. Finally, a policy π consists of a control policy and a transmission policy, i.e., $\pi := (\gamma, \theta)$.

In this letter, we are interested in designing control and transmission policies that outperform periodic time-triggered policies in terms of the trade-off between the performance criterion, V, and the average transmission rate, r, defined as

$$V := \limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}\,[\int_0^T x(t)^\mathsf{T} Qx(t) + u(t)^\mathsf{T} Ru(t)dt] \tag{7}$$

$$r := \frac{1}{\tau} \times \limsup_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{E}\,[\sigma_k], \tag{8}$$

where Q and R are positive-definite matrices. Moreover, we require that the policy is designed such that no new transmissions are triggered (i.e., the system operates in open-loop) in the absence of disturbances. These requirements were formally defined as the consistency properties in [17] and are restated next, in which we use the notation $V_p(r)$ to denote the optimal cost (7) of the optimal periodic control policy p with the transmission rate $r > 0$ that minimizes (7). This optimal periodic policy and corresponding cost will be computed in the next section (see (9)-(12)).

Definition 1 [17]: Let V_π and r_π denote the cost (7) and the average transmission rate (8) of a policy π, respectively. We say that the policy π is consistent if:

1) $V_\pi \le V_p(r_\pi)$, i.e., π achieves a better performance than that of the optimal periodic control policy for the same average transmission rate r_π.
2) Along any closed-loop trajectory given by (1)-(3) and policy π (and thus some realization given by $\{s_l\}_{l\in\mathbb{N}_0}$ and $\{w_l\}_{l\in\mathbb{N}_0}$ of the disturbance), and in case of full-state feedback, it holds that $\kappa_{m+1} \ge \bar{s}_m$ for every $m \in \mathbb{N}_0$, where

$$\bar{s}_m := \min\{s_\ell \mid s_\ell > \kappa_m, \ell \in \mathbb{N}_0\},$$

i.e., the scheduler generates no new transmissions after a given transmission time $\kappa_m,\ m \in \mathbb{N}_0$, as long as no disturbances act on the plant.

The second consistency property will follow easily from the use of a CIG (3) at the controller side, which in turn relies on full state feedback. However, this is an important property since there are full state based strategies in the literature that satisfy the first consistency property but not the second (see [12]). Clearly, some adjustments to the definition of this property are needed if only partial state information is available for feedback, since some transmissions might be triggered even in the absence of disturbances/noise while the state is being reconstructed. This warrants further research. Note that since (3) is a reset observer, the use of reset output feedback observers [19] can play an important role in this direction.

The problem we aim to tackle can be stated as: *Design an (easy-to-implement) consistent control and transmission policy for the system configuration discussed above.*

III. Periodic All-Time Transmission Policy

Before proposing our novel consistent PETC strategies, in this section we first present the optimal *periodic* policies and the corresponding performance and a few preliminary but instrumental observations leading to our main results. The optimal periodic policy p corresponds to the constant map $\mu_k(I^s_k) = 1,\ k \in \mathbb{N}_0$, for the transmission policy, i.e., $\kappa_m = t_m,\ m \in \mathbb{N}_0$, while applying the control policy

$$u(t) = L\hat{x}(t), \quad t \in \mathbb{R}_{\ge 0} \tag{9}$$

with

$$L := -R^{-1}B^\mathsf{T} P, \tag{10}$$

where P is the unique positive-definite solution of the continuous-time algebraic Ricatti equation (CARE)

$$A^\mathsf{T} P + PA - PBR^{-1}B^\mathsf{T} P + Q = 0. \tag{11}$$

This policy results in the average cost

$$V_p(r_p) = \lambda\,\mathrm{Tr}(PR_\omega) + r_p \int_0^{\frac{1}{r_p}} \mathrm{Tr}(L^\mathsf{T} RL\Sigma(t))dt, \tag{12}$$

where $r_p := \frac{1}{\tau}$ is the transmission rate, λ is the decay rate defined in (2), and

$$\Sigma(s) = \lambda \int_0^s e^{Ar} R_\omega e^{A^\mathsf{T} r} dr \tag{13}$$

corresponds to the covariance of the error

$$e(t) := x(t) - \hat{x}(t) \tag{14}$$

at time $t = t_k + s$, i.e., $\Sigma(s) = \mathbb{E}\,[e(t_k + s)e(t_k + s)^\mathsf{T}]$ for any $k \in \mathbb{N}_0$ (see [17]).

Before introducing our proposed PETC policy in the next section, we state the following key result upon which we build the PETC.

Theorem 1: The performance index (7) of the periodic policy p with respect to the transmission rate r_p is convex, i.e., the mapping V_p:$[0, \infty) \to [0, \infty)$ given by $r_p \mapsto V_p(r_p)$ is a convex function.

The proof of Theorem 1 follows from the following lemma.

Lemma 1: The function $g : \mathbb{R}_{>0} \to \mathbb{R}$, given by $g(r) = r\int_0^{\frac{1}{r}} f(s)ds$ is convex if f

- is continuously differentiable, and
- is non-decreasing, i.e., $\frac{d}{ds}f(s) \ge 0,\ s \ge 0$.

Proof: Since f is continuously differentiable, g is twice continuously differentiable and

$$\frac{d^2 g}{dr^2}(r) = \frac{1}{r^3}\frac{df}{dr}(\frac{1}{r}),$$

which is non-negative for $r \in \mathbb{R}_{>0}$ (as f is monotonically increasing), therefore, g is convex. ■

Proof of Theorem 1: The function $s \mapsto \mathrm{Tr}(L^\mathsf{T} RL\Sigma(s))$ satisfies both conditions of Lemma 1. Note that $L^\mathsf{T} RL$ and $\frac{d}{ds}\Sigma(s) = \lambda e^{As} R_\omega e^{A^\mathsf{T} s},\ s \in \mathbb{R}_{\ge 0}$, are positive semi-definite matrices, and hence,

$$\mathrm{Tr}(L^\mathsf{T} RL\frac{d}{ds}\Sigma(s)) \ge 0,\ s \in \mathbb{R}_{\ge 0}. \tag{15}$$

Therefore, the function $r_p \mapsto V_p(r_p)$ is convex. ■

Theorem 1 provides the corner stone for introducing our proposed consistent PETC (CPETC). This originates from

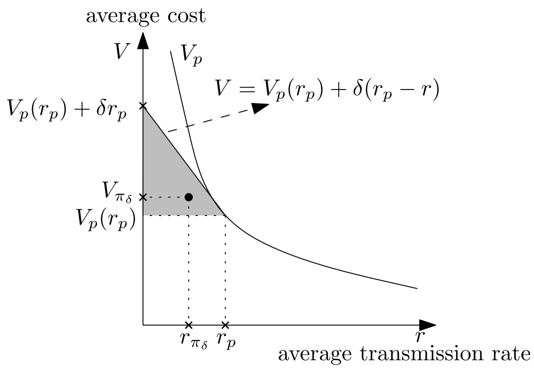

Fig. 3. Illustration of the trade-off between average cost and transmission rate for the proposed triggering policy, where δ is chosen as in (21). The gray area represents the performance region for the proposed triggering algorithm characterized by (18)-(22).

the fact that the graph of a convex function is above any tangent plane. Therefore, if we can provide a policy whose performance is bounded by the tangent of the optimal periodic policy, we ensure the first consistency property. This is illustrated in Figure 1.

IV. Consistent Periodic ETC Policy

The following theorem formally introduces the proposed consistent PETC (CPETC).

Theorem 2: Let $r_p > 0$ be given. Consider system (1) with the control policy $u(t) = L\hat{x}(t)$, $t \in \mathbb{R}_{\geq 0}$, as in (3) and (9). Then the ETC policy π_δ with the triggering policy θ_δ given by

$$\sigma_k = \mu_k(I_k^s) = \begin{cases} 1, & e_k^\intercal \Gamma(\frac{1}{r_p}) e_k > \delta \\ 0, & \text{otherwise}, \end{cases} \tag{16}$$

where $e_k := e(t_k^-)$, $t_k = k\frac{1}{r_p}$, $k \in \mathbb{N}_0$, and

$$\Gamma(\tau) = \int_0^\tau e^{A^\intercal t} L^\intercal R L e^{At} dt, \tag{17}$$

$\forall \tau > 0$, for any $\delta \in \mathbb{R}_{\geq 0}$ satisfies

$$V_{\pi_\delta} + \delta r_{\pi_\delta} \leq V_p(r_p) + \delta r_p \tag{18}$$

$$r_{\pi_\delta} \leq r_p, \tag{19}$$

where V_{π_δ} is the cost (7) of policy π_δ and r_{π_δ} is the average transmission rate of policy π_δ as defined in (8). Furthermore, system (1) with control policy (3) and triggering policy (16) is mean square stable, i.e.,

$$\mathbb{E}[x(T)^\intercal x(T)] \leq C, \quad \text{for every } T \in \mathbb{R}_{\geq 0}, \tag{20}$$

where $C < \infty$ is a positive scalar. Finally, for any $\delta \geq 0$ such that

$$\delta \leq -\frac{d}{dr_p} V_p(r_p) = \text{Tr}(L^\intercal R L(\frac{1}{r_p}\Sigma(1/r_p) - \int_0^{1/r_p} \Sigma(t)dt)), \tag{21}$$

the ETC policy π_δ is consistent and we have

$$V_p(r_p) \leq V_{\pi_\delta} \leq V_p(r_{\pi_\delta}). \tag{22}$$

Figure 3, shows the cost and the average transmission rate trade-off for the proposed policy. The gray area represents the guaranteed performance region characterized by (18)-(22) for the parameter δ chosen as in (21). Note that δ appears as a tuning knob which influences the slope of the upper bound defined by the line $\{(V, r) \mid V = V_p(r_p) + \delta(r_p - r),\ r \in (0,\ r_p]\}$. Furthermore, the higher the value of δ, the less stringent the triggering condition (16) leading to a lower transmission rate and a larger performance region.

Proof: To prove Theorem 2 we need some preliminaries. Consider the finite horizon performance index

$$\bar{V} := \frac{1}{T}\mathbb{E}\left[\int_0^T x(t)^\intercal Q x(t) + u(t)^\intercal R u(t) dt\right] \tag{23}$$

with $T = N\tau$, $N \in \mathbb{N}$. Since the transmission decisions can take place at discrete times $t_k = k\tau$ for $k \in \mathcal{F} = \{0, 1, \ldots, \text{N-1}\}$, we divide the integration interval $[0,\ T]$ of (23) into N equidistant subintervals of length τ. We assume that we apply the control policy $u = L\hat{x}$ as in (3) and (9). Based on similar arguments as provided in [17, eq. (37)], we then obtain

$$\begin{aligned} &\mathbb{E}\left[\int_{t_k}^{t_k+\tau} x(t)^\intercal Q x(t) + u(t)^\intercal R u(t) dt \mid x(t_k)\right] \\ &= x(t_k)^\intercal P x(t_k) + \tau\, \text{Tr}(\lambda P R_\omega) - \mathbb{E}[x(t_k+\tau)^\intercal P x(t_k+\tau) \mid x(t_k)] \\ &\quad + \mathbb{E}\left[\int_{t_k}^{t_k+\tau} (u(t) - Lx(t))^\intercal R (u(t) - Lx(t)) dt \mid x(t_k)\right]. \end{aligned}$$

Adding this equation from $k = 0$ until $k = N-1$ and dividing by $T = N\tau$, taking expected values and using the fact that $\mathbb{E}[. \mid x(t_k)] = \mathbb{E}[.]$, we obtain

$$\begin{aligned} \bar{V} = &\frac{1}{N\tau} x(0)^\intercal P x(0) + C_e - \frac{1}{N\tau}\mathbb{E}[x(N\tau)^\intercal P x(N\tau)] \\ &+ \text{Tr}(\lambda P R_\omega), \end{aligned}$$

where

$$C_e := \frac{1}{N\tau}\text{Tr}(L^\intercal R L \sum_{i=0}^{N-1}\int_{i\tau}^{(i+1)\tau} \mathbb{E}[e(s)e(s)^\intercal \mid e(t_i)]ds). \tag{24}$$

Using [17, eq. (45)] with initial condition $e(t_k)$, we obtain

$$\begin{aligned} &\mathbb{E}[e(t_k+r)e(t_k+r)^\intercal \mid e(t_k)] = e(t_k)e(t_k)^\intercal \\ &\quad + A\int_{t_k}^{t_k+r} \mathbb{E}[e(s)e(s)^\intercal \mid e(t_k)]ds \\ &\quad + \int_{t_k}^{t_k+r} \mathbb{E}[e(s)e(s)^\intercal \mid e(t_k)]ds A^\intercal + r\lambda R_\omega. \end{aligned} \tag{25}$$

Defining $W(r) := \mathbb{E}[e(t_k+r)e(t_k+r)^\intercal \mid e(t_k)]$, (24) can be rewritten as

$$\begin{aligned} W(r) = &e(t_k)e(t_k)^\intercal + A\int_0^r W(s)ds + \int_0^r W(s)ds A^\intercal \\ &+ r\lambda R_\omega, \end{aligned} \tag{26}$$

which is a Volterra equation of second type with unique solution given by

$$\begin{aligned} W(r) &= e^{Ar}e(t_k)e(t_k)^\intercal e^{A^\intercal r} + \Sigma(r) \\ \Sigma(r) &= \int_0^r e^{As}\lambda R_\omega e^{A^\intercal s} ds. \end{aligned}$$

Therefore, we can rewrite the term under summation in (24) as

$$\int_{i\tau}^{(i+1)\tau} \mathbb{E}\,[e(s)e(s)^{\mathsf{T}} \mid e(t_i)]ds = \int_0^{\tau} \Sigma(t)dt + \int_0^{\tau} e^{At}\,\mathbb{E}\,[e(i\tau)e(i\tau)^{\mathsf{T}}]e^{A^{\mathsf{T}}t}dt. \tag{27}$$

Consequently, replacing (27) in (24), we obtain

$$\sum_{i=0}^{N-1}\int_{i\tau}^{(i+1)\tau} \mathbb{E}\,[e(s)e(s)^{\mathsf{T}}]dt = \int_0^{\tau} e^{At}\sum_{i=0}^{N-1}\mathbb{E}\,[e(i\tau)e(i\tau)^{\mathsf{T}}]e^{A^{\mathsf{T}}t}dt + N\tau\int_0^{\tau}\Sigma(t)dt, \tag{28}$$

which leads to

$$C_e = \int_0^{\tau}\mathrm{Tr}(L^{\mathsf{T}}RL\Sigma(t))dt + \frac{1}{N\tau}\mathrm{Tr}(\Gamma(\tau)\sum_{i=0}^{N-1}\mathbb{E}\,[e(i\tau)e(i\tau)^{\mathsf{T}}]), \tag{29}$$

where $\Gamma(\tau)$ is as defined in (17). Due to the possibility of a transmission event at t_k for $k \in \mathcal{F}$, we can obtain $e(t_k) = 0$. This is key to the proposed policy, as the performance index (23) can be reformulated as

$$\bar{V} = \lambda\,\mathrm{Tr}(PR_{\omega}) + \int_0^{\tau}\mathrm{Tr}(L^{\mathsf{T}}RL\Sigma(t))dt + \frac{1}{N\tau}\sum_{i=0}^{N-1}\mathbb{E}\,[e(t_i)^{\mathsf{T}}\Gamma(\tau)e(t_i)] - \frac{1}{N\tau}\mathbb{E}\,[x(T)^{\mathsf{T}}Px(T)] + \frac{1}{N\tau}\mathbb{E}\,[x(0)^{\mathsf{T}}Px(0)]. \tag{30}$$

We focus on the third term and introduce the function $\bar{J}^N$ with an additional additive constant $\delta \in \mathbb{R}_{\geq 0}$ as the cost of transmission at time t_k, i.e.,

$$\bar{J}^N := \frac{1}{N\tau}\sum_{k=0}^{N-1}\mathbb{E}\,[e_k^{\mathsf{T}}\Gamma(\tau)e_k + \delta\sigma_k], \tag{31}$$

where $e_k = e(k\tau)$. Note that for the periodic all-time transmission policy, we have $\sigma_k = 1$ and $e_k = 0$ for every $k \in \mathcal{F}$. Therefore,

$$\bar{J}_p^N(r_p) = \delta\frac{1}{\tau} = \delta r_p, \tag{32}$$

where $\bar{J}_p^N(r_p)$ is (31) for $\sigma_k = 1,\ k \in \mathcal{F}$. Applying the proposed policy (16) leads to

$$\bar{J}_{\pi_\delta}^N \leq \delta r_p = \bar{J}_p \quad \text{for every } N \in \mathbb{N}, \tag{33}$$

which is due the fact that the policy (16) imposes

$$e_k^{\mathsf{T}}\Gamma(\tau)e_k + \delta\sigma_k = \min\{e_k^{\mathsf{T}}\Gamma(\tau)e_k, \delta\} \leq \delta. \tag{34}$$

Since the inequality (33) holds for all $N \in \mathbb{N}$, it also holds for the limit, i.e.,

$$J_{\pi_\delta} := \lim_{N\to\infty}\sup\bar{J}_{\pi_\delta}^N \leq \lim_{N\to\infty}\sup\bar{J}_p^N =: J_p = \delta r_p. \tag{35}$$

Applying the policy (16) and taking the limit as $T \to \infty$ (equivalently $N \to \infty$) from both sides of (30), on the left-hand side, we recover the performance index (7) for the policy π_δ. On the right-hand side, the summation term is bounded due to (35) and the last term converges to zero. Moreover, following a similar reasoning as in [17, eq. (47)] the fourth term vanishes as

$$\mathbb{E}\,[x(T)^{\mathsf{T}}x(T)] \leq C, \quad \text{for every } T \in \mathbb{R}_{\geq 0}, \tag{36}$$

which is also equivalent to mean square stability of the closed-loop system. Therefore,

$$V_{\pi_\delta} = \underbrace{\lambda\,\mathrm{Tr}(PR_{\omega}) + \frac{1}{\tau}\int_0^{\tau}\mathrm{Tr}(L^{\mathsf{T}}RL\Sigma(t))dt}_{V_p(r_p)} + \frac{1}{\tau}\lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1}\mathbb{E}\,[e_i^{\mathsf{T}}\Gamma(\tau)e_i]. \tag{37}$$

Furthermore, combining (35) and (37) leads to

$$V_{\pi_\delta} + \delta r_{\pi_\delta} \leq V_p(r_p) + \delta r_p. \tag{38}$$

Also note that by construction

$$r_{\pi_\delta} \leq r_p. \tag{39}$$

In order to prove the first consistency property of the proposed policy note that in $V - r$ coordinates $V(r) - V_p(r_p) = -\delta(r - r_p)$ characterizes a line passing through $(V_p(r_p),\ r_p)$ (see Figure 3). Since the curve of periodic performance vs rate is convex according to Theorem 1, we have

$$V_p(y) \geq V_p(x) + \frac{d}{dx}V_p(x)(y - x) \tag{40}$$

for any $x \in \mathbb{R}_{\geq 0}$ and for any $y \in \mathbb{R}_{\geq 0}$. If we set $x = r_p$ and $y = r_{\pi_\delta}$, we obtain

$$V_p(r_{\pi_\delta}) \geq V_p(r_p) + \frac{d}{dr_p}V(r_p)\underbrace{(r_{\pi_\delta} - r_p)}_{\leq 0}. \tag{41}$$

Choosing

$$\delta = -\frac{d}{dr_p}V_p(r_p) = Tr(L^{\mathsf{T}}RL(\frac{1}{r_p}\Sigma(1/r_p) - \int_0^{1/r_p}\Sigma(t)dt)),$$

then from (38) and (41), we conclude

$$V_p(r_{\pi_\delta}) \geq V_p(r_p) - \delta(r_{\pi_\delta} - r_p) \geq V_{\pi_\delta} \tag{42}$$

(the first consisitency criterion holds). Moreover, due to (39), (42) holds also for any $0 \leq \delta \leq -\frac{d}{dr_p}V_p(r_p)$. Also note that due to (37) the obtained cost of π_δ is at least equal to the cost of periodic policy with rate r_p. Therefore

$$V_p(r_p) \leq V_{\pi_\delta}. \tag{43}$$

Note that in the case of no disturbance after a given transmission time κ_m we have $e(t) = 0,\ t > \kappa_m$, and, therefore, there is no transmission for $t > \kappa_m$ (the second consistency criterion holds). ∎

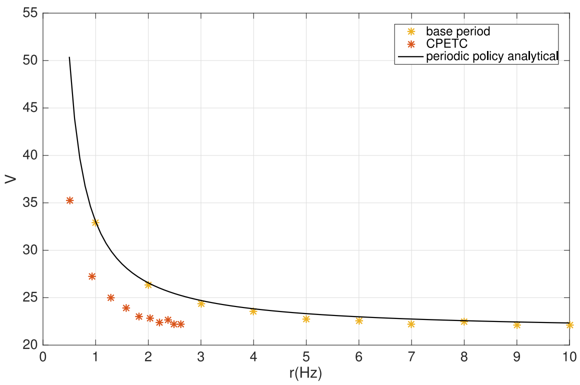

Fig. 4. Performance of the proposed CPETC for control of a double integrator system over a communication network. The solid line represents the analytical performance (45). The yellow stars represent the periodic control used as base policies via Montecarlo simulations and the red stars illustrate the performance trade-off of the corresponding consistent periodic event-triggered policy.

V. Simulation Results

In this section we illustrate the proposed ETC policy on a double integrator controlled over a communication network as depicted in Figure 2. The process and cost parameters are given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad Q = I; \quad R = 10. \tag{44}$$

This leads to $L = \begin{bmatrix} 0.3162 & 0.8558 \end{bmatrix}$ in (10). For the disturbances, we take $\lambda = 5$ and we consider that the ω_ℓ $\ell \in \mathbb{N}_0$, have a normal distribution with covariance $R_\omega = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0.5 \end{smallmatrix}\right]$, which influences the system through (1). Considering these numerical values, V_p, described by (12), can be computed analytically as

$$V_p(r_p) = 21.3959 + 9.1557\frac{1}{r_p} + 2.2553\frac{1}{r_p^2} + 0.2083\frac{1}{r_p^3}. \tag{45}$$

and the derivative of V_p follows easily. We consider the periodic policy with various transmission rates and the corresponding thresholds $\delta = -\frac{d}{dr_p}V_p(r_p)$. Note that the initial condition $x(0)$ (which we set to zero) is not relevant to compute the average cost performance and the average transmission rate, since these depend only on the asymptotic behavior of the system. Figure 4 illustrates the results of the simulation. The yellow stars illustrate the considered optimal periodic policies with cost computed through Monte-Carlo simulations. The black line corresponds to the analytical performance of the optimal periodic policies as in (45) and the red stars correspond to the applied CPETC for various value of sampling rate r_p computed via Monte-Carlo simulations. It is clearly visible that the triggering condition not only creates a better cost vs transmission ratio trade-off but also significantly reduces the transmission rate.

VI. Conclusion and Future Work

In this letter a *consistent* periodic ETC policy was introduced. The proposed policy is consistent in the sense that (i) it outperforms the performance of optimal periodic control policy for the same average transmission rate and (ii) it does not trigger sensor updates in the absence of disturbances. We showed that consistency can always be achieved considering linear systems, state feedback, and average quadratic costs as the performance measurement using the key observation that the average quadratic cost for periodic policies is a convex function of the transmission rate. This resulted in an error-based threshold policy using a weighted norm of the error.

An important open problem is the relaxation of the rather strong assumption that the full state, rather than a lower dimensional output, is available for feedback. Addressing the output-feedback case is challenging and warrants further research.

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