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On observability in networked control systems with packet losses

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Abstract— **This paper deals with observability properties of networked control systems subject to packet losses. We employ a switching systems perspective in which available information on the packet loss signal, e.g., there can be at most a pre-specified number of consecutive losses, is modelled through an automaton. Based on this perspective we address several natural extensions envisioned in [7]. Our specific contributions are as follows. Firstly, we show that the method introduced in [7] in the context of controllability of linear systems subject to packet losses extends to the question of observability. The proposed characterization is necessary and sufficient as well as algorithmically verifiable. For the observability problem, our proof is valid also for non-invertible matrices, thereby improving upon the previous results in [7]. Secondly, we show that the model employed for our analysis encompasses the model of wireless control networks with switching delays introduced in [6] (though at a cost of exponential encoding). We raise several open questions related to the algebraic nature of the problem under consideration.**

I. Introduction

In many modern control systems, it often happens that a real-time feedback loop is disrupted intermittently by undesired events such as packet losses in wireless communication, task deadline misses in shared embedded processors, outliers in sensor data, etc. These events are typically modelled as data losses and they lead to imperfect control updates. Clearly, fundamental system theoretic properties such as controllability, observability, stabilizability, detectability, etc. can be affected in the presence of data losses, see [7, Section I] for an extended discussion. This calls for a careful re-assessment of the characterizations of these fundamental properties as they form cornerstones of modern control theory.

Systems with data losses have attracted considerable research attention in the recent past, see e.g., [11], [13], [10], [4], [5], [3] and the references therein. However, most of the above works studied the fundamental properties only implicitly in the sense that they provided sufficient conditions for stability or stabilizing controller design, or assumed that the data loss signal could be abstracted (e.g., replaced with a periodic downsampling) in order to control the plant easily. Recently in [7] we derived a *necessary and sufficient characterization* of controllability of discrete-time linear time-invariant plants subject to data losses. This characterization was algorithmically verifiable.

Towards obtaining the above characterization, we employed a constrained switching systems perspective for modelling the system under consideration.[1] The constraints arise from our consideration that the packet loss signal has some known properties that can be captured through an automaton. For instance, the automaton could model the constraint that there can be at most a pre-specified number of consecutive losses. Exact characterization of observability and controllability were studied earlier for some other classes of switching systems [1], [6]. In [1] the authors provided sufficient conditions for observability (and controllability) of switching systems with a constant system matrix for all subsystems but the input matrix is selected from a given set at every instant of time. The proposed condition involved constructing a finite number of *auxiliary pairs* involving the system matrix and input matrices, and verifying that these pairs are controllable (in the classical sense). It was shown that if all the auxiliary pairs are controllable, then the switching system is also controllable. Controllability algorithms for a class of systems containing switching delays in the feedback loop were studied in [6].

Our setting here is a little different as we consider a plant subject to packet losses under the constraint that the losses respect the behaviour described by a given automaton. In [7] we proposed an algorithm that decides the controllability of such a plant in finite time. The results were derived in the particular setting of single input, and when the system matrix is invertible. In addition, it was shown that the particular case of the controllability problem, where the constraint on the switching formalizes that there cannot be more than a given maximum number of consecutive losses, can be restated as a controllability problem with switching delays. Consequently, this particular case (i.e. with a given maximum number of consecutive losses) can also be tackled with the techniques developed in [6].

In this paper our contributions are as follows:

- ◦ We provide a necessary and sufficient as well as algorithmically verifiable characterization of observability in networked control systems subject to packet losses. This

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[1] Recall that a *constrained switching system* is a switching system in which the admissible switches are *constrained*, e.g., it is allowed to switch from subsystem a to subsystem b, but it might not be allowed to switch from subsystem b to subsystem a, etc.

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characterization is obtained following a similar line of analysis as in [7], but is not entirely straightforward.

- Our results are stronger than in our previous work as we relax our previous assumption [7, Assumption 1] on invertibility of the system matrix. We also provide an ad hoc analysis for the case of non-invertible matrices, which is more efficient from a computational point of view, and allows us to relax [7, Assumption 1] in the context of controllability as well.
- We show a deeper connection between the model of wireless control networks with switching delays introduced in [6] and the systems with dropouts studied here via the construction of an auxiliary automaton. In particular, we show that the model in the present paper encompasses the model in [6] (though at the cost of an exponential-size encoding associated to this construction).

We employ the celebrated Skolem Theorem [12] from Linear Algebra and arguments from automata theory as the main apparatuses for our analysis.

The remainder of this paper is organized as follows: In §II we formulate the observability problem under consideration. In §III, we reformulate our problem as a purely algebraic one, in the Kalman-criterion fashion. We then provide an algorithmic solution to this problem in §IV, where we also provide an ad hoc analysis for the case of non-invertible system matrices. In §V we show how the previously introduced model of wireless networks with switching delays from [6] is actually a particular case of our model with dropouts. We conclude in §VI with a brief discussion on future directions. Throughout the paper we raise several open questions on the algebraic and algorithmic nature of the problem. Standard notations are employed unless otherwise stated. Whenever the proposed results are proven based on similar arguments as used in the proofs presented in [7], we highlight the parts that are different from [7] with precise references.

II. Problem Formulation

We consider networked control systems with packet losses, which can be modelled as the following discrete-time linear system

$$x(t+1) = Ax(t),$$
$$y(t) = \begin{cases} Cx(t), & \sigma(t) = 1 \\ \emptyset, & \sigma(t) = 0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, and $y(t) \in \mathbb{R}^p$ are the vectors of states and outputs, respectively, at time $t \in \mathbb{N}$, $\emptyset$ indicates that no output information is available, $\sigma : \mathbb{N} \to \{0, 1\}$ is the *data loss signal*, which represents the packet losses in the sense that $\sigma(t) = 0$ corresponds to the case where the output packet is lost at time t, and $\sigma(t) = 1$ corresponds to the case where the output packet has arrived in good order at time t.

We assume that the data loss signals σ satisfy certain constraints capturing the physics of the shared (wireless) communication network and/or the characteristics of the underlying embedded architecture. An example could be that there can be at most $l \in \mathbb{N}$ consecutive packet losses, etc. We assume that the data loss behaviour can be modeled with a switching system whose switching signal is constrained by an automaton. The admissible data loss signals correspond to the infinite words accepted by the automaton abstracting the structure of the communication network.

Definition 1: An *automaton* is a pair $\mathcal{A} = (M, s) \in \{0, 1\}^{N\times N} \times \{0, 1\}^N$, where N is the *number of states*, $M \in \{0, 1\}^{N\times N}$ is the *transition matrix*, and $s = (s_1\ s_2\ \dots\ s_N)^\top \in \{0, 1\}^N$ is the *vector of node labels*.

Definition 2: A data loss signal $\sigma : \mathbb{N} \to \{0, 1\}$ is said to be *admissible*, if there exists a sequence of states $v : \mathbb{N} \to \{1, 2, \dots, N\}$ such that for all $t \in \mathbb{N}$ it holds that $M_{v(t),v(t+1)} = 1$ and $\sigma(t) = s_{v(t)}$.

Example 1 ([7, Example 1]): Suppose that there can be at most $l \in \mathbb{N}$ consecutive data losses (l is a fixed parameter). This can be captured by an automaton containing $l+1$ nodes, in which the node $i \in \{1, 2, \cdots, l\}$ represents situations where the last i packets were lost, and the one before arrived safely. The node $l+1$ represents the situation where the last packet arrived safely. In Fig. 1 we consider this automaton for $l = 3$. In this case we have

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \text{ and } s = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

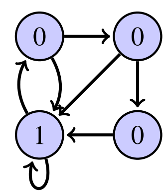

Fig. 1: Automaton representing our model with $l = 3$.

Definition 3: The system specified by the triple $(A, C, \mathcal{A})$ is said to be *observable* if for all admissible data loss signals σ and any initial state $x(0) = x_0$, there exists a time $t_f > 0$ such that the knowledge of the outputs $y(t)$ suffices to determine the state x_0. The system $(A, C, \mathcal{A})$ is said to be *unobservable* otherwise.

In the sequel we are interested in observability of a system specified by the triple $(A, C, \mathcal{A})$. In particular, we solve the following general problem:

Problem 1: Determine if a system specified by the triple $(A, C, \mathcal{A})$ is (un)observable.

III. Preliminary Results

Definition 4: Given a data loss signal σ, we define the *observability matrix* $O_\sigma(t) \in \mathbb{R}^{(t+1)p\times n}$ of the system $(A, C, \mathcal{A})$ at time $t \in \mathbb{N}$ as follows:

$$O_\sigma(t) = \begin{pmatrix} \sigma(0)C \\ \sigma(1)CA \\ \vdots \\ \sigma(t)CA^t \end{pmatrix}. \tag{2}$$

Proposition 1: The system $(A, C, \mathcal{A})$ is observable if and only if there is no admissible data loss signal σ such that for all $t \in \mathbb{N}$ the observability matrix $O_\sigma(t)$ is of rank smaller than n.

Proof: [Sketch] **If**: Given that for all admissible σ, there is a t such that $O_\sigma(t)$ is full rank, $x(0)$ can be obtained from $\begin{pmatrix} y(0) \\ \vdots \\ y(t) \end{pmatrix} = O_\sigma(t)x(0)$. The observability of the system $(A, C, \mathcal{A})$ follows at once.

Only if: Assume that there exists an admissible σ with no $t \in \mathbb{N}$ that allows n linearly independent columns of $O_\sigma(t)$. Then there exists a nonzero $\alpha \in \mathbb{R}^n$ such that for all t, $O_\sigma(t)\alpha = 0$. Pick $x(0) = x_0 = \alpha$. Then $y(t) = 0$ for all $t \in \mathbb{N}$. Hence, the initial state $x(0)$ cannot be distinguished from the zero state, and the system $(A, C, \mathcal{A})$ is unobservable. ■

Proposition 1 provides a necessary and sufficient characterization of observability of the system $(A, C, \mathcal{A})$. It is similar in spirit to [7, Proposition 1] stated for the controllability property.

IV. Main Results

We are therefore interested in solving the following purely algebraic problem: *Given a system specified by the triple* $(A, C, \mathcal{A})$*, is there an admissible data loss signal* σ *for which the rank of the observability matrix* $O_\sigma(t)$ *is smaller than* n *for all* $t \in \mathbb{N}$?

Our main result is derived employing the Skolem Theorem [12] from Linear Algebra. We first recall the theorem:

Theorem 1 ([12]): Consider a matrix $A \in \mathbb{R}^{n \times n}$ and two vectors $b, c \in \mathbb{R}^n$. The set of values of n such that $c^\top A^n b = 0$ is eventually periodic in the sense that there exist two natural numbers $P, T \in \mathbb{N}$ such that

$$\forall t \in \mathbb{N}_{>T}, \quad c^\top A^t b = 0 \Leftrightarrow c^\top A^{t+P} b = 0. \qquad (3)$$

In the following, we call a signal σ *unobservable*, if $O_\sigma(t)$ has rank strictly smaller than n for all $t \in \mathbb{N}$. The first main result of this paper is:

Theorem 2: Problem 1 is decidable in the sense that given a system specified by $(A, C, \mathcal{A})$, there is an algorithm that decides (un)observability of the system in finite time.

Proof: [Sketch] Our algorithm involves simultaneous execution of two subalgorithms:

- A first subalgorithm generates all admissible σ of increasing length until the corresponding observability matrix $O_\sigma(t)$ is full rank. Following Proposition 1, if this property is satisfied for all admissible signals of a given length, then the system $(A, C, \mathcal{A})$ is observable.
- A second subalgorithm verifies admissible *periodic* data loss signals corresponding to cycles of increasing length and continues until an unobservable periodic signal is found, which establishes unobservability of the system $(A, C, \mathcal{A})$.

The main fact we prove below is that restricting ourself to *periodic* switching signals does not hamper our ability to detect unobservability. This shows that the algorithm in discussion terminates in finite time if the system $(A, C, \mathcal{A})$ is unobservable.

Case I: The system $(A, C, \mathcal{A})$ is observable. The uniform bound on the maximal unobservable signal follows under simple compactness arguments, similar as presented in the proof of [7, Theorem 2]. This implies that the algorithm halts in finite time.

Case II: The system $(A, C, \mathcal{A})$ is unobservable.

Claim 1: There exists an admissible periodic data loss signal σ with the period corresponding to a cycle in the automaton representing the constraints, which is unobservable.[2]

In this case our proof differs slightly from the case of controllability presented in [7, Theorem 2] in terms of the application of Skolem's Theorem. We now highlight this modification.

If the system $(A, C, \mathcal{A})$ is unobservable, by definition, there is an admissible data loss signal σ for which the rank of the observability matrix $O_\sigma(t)$ is strictly less than n for all $t \in \mathbb{N}$. Hence the image of all the rows of $O_\sigma(t)$ remains in a subspace of dimension less than n.

Fix non-zero independent vectors $v_1, \ldots, v_k$ in the orthogonal complement of the subspace under consideration. We now apply Skolem's Theorem with $A^\top$ to every pair v_i, c_j, where c_j are rows of the matrix $C^\top$. We obtain $T_{ij}, P_{ij} \in \mathbb{N}$ (where i ranges over the vectors v_i and j ranges over the columns of $C^\top$) satisfying (3) with the corresponding vectors v_i. From this, defining

$$T = \max_{i,j} T_{i,j},$$

$$P = \prod P_{i,j},$$

we have that (3) is satisfied for all rows c_j of the Matrix C and for all vectors v_i.

Consider now the unobservable signal σ, and the corresponding path generating it in the automaton. There are two times $t_1, t_2 \in \mathbb{N}_{>T}$ such that

$$t_1 = t_2 \bmod P, \quad \text{and} \quad w_\sigma(t_1) = w_\sigma(t_2), \qquad (4)$$

where $w_\sigma(t)$ is the node of the infinite path corresponding to σ in the automaton $\mathcal{A}$ at time $t \in \mathbb{N}$. By Equation (4) the infinite periodic signal consisting of the repetition of $(\sigma(t_1), \ldots, \sigma(t_2))$ corresponds to an admissible data loss signal σ. Moreover, Equation (3) shows that this signal is unobservable, and the claim is proved.

Now, it is easy to check whether any periodic signal is unobservable. Thus, if one checks unobservability of each periodic signal corresponding to cycles (for growing length of these cycles), the algorithm will come across the uncontrollable cycle after a finite time, and the algorithm will terminate. ■

Thus, the proof of Theorem 2 provides an algorithmically verifiable necessary and sufficient characterization of observability in networked control systems subject to packet losses. The proof is similar in spirit to [7, Theorem 2] proposed in the context of controllability. It is however important to note

[2] For us a *cycle* is a path $v_1, v_2, \ldots, v_\ell$ of length ℓ such that $M_{v_j, v_{j+1}} = 1$ for all $j = 1, 2, \ldots, \ell - 1$ and $v_1 = v_\ell$ (with repetitions of vertices allowed).

that in [7] our results were derived under the assumption that the system matrix A is invertible and that the input is a single vector $b \in \mathbb{R}^n$. The sketch of proof presented above for the case of observability translates to the case of controllability with A invertible and $B \in \mathbb{R}^{n\times m}$, (i.e., $u \in \mathbb{R}^m$), and consequently, addresses the open question of relaxing [7, Assumption 1] partially.

Corollary 1: [7, Problem 1] is decidable in the sense that given a system specified by $(A, B, \mathcal{A})$ with A invertible and $B \in \mathbb{R}^{n\times m}$, there is an algorithm that decides (un)controllability of the system in finite time.

Example 2: Consider the system $(A, C, \mathcal{A})$ with $A = \begin{pmatrix} -2 & -13 & 9 \\ -5 & -10 & 9 \\ -10 & -11 & 12 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$. Let $\mathcal{A}$ be as considered in Example 1. We observe that even though (A, C) is an observable pair (in the standard sense), under the following admissible data loss signal σ the system $(A, C, \mathcal{A})$ is unobservable:

$$\sigma = 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots.$$

The above data loss signal is periodic and the period corresponds to a cycle in the automaton representing the constraints. We note that here we have a simple cycle generating the unobservable signal, but this might not be the case in more involved examples.[3]

In terms of algorithmic complexity, the above comment raises the following natural question:

Open Question 1: Can we provide an upper bound on the length of the unobservable cycle?

A solution to the above open question would provide an explicit upper bound on the complexity of our algorithm. However, we believe that this could be very hard to obtain. Indeed, it is a well-known open problem to provide a proof of Theorem 1 including explicit upper bounds on the quantities T and P, see e.g., [2], [14].

Observe that unlike in the case of [7, Theorem 2], our result on observability does not rely on a backward matrix and consequently does not require the invertibility assumption on the system matrix A. This is somehow natural and corresponds to the intuition that for observability, we do not have to predict the future, since we can observe the system by making observations a posteriori.

We now present an ad hoc analysis towards relaxing the invertibility assumption also in the context of controllability. We analyse the particular case where the matrix A has only zero eigenvalues. The reason for this is twofold: first, in this case we provide an efficient (polynomial time) algorithm to decide observability. Second, it is sufficient to consider nilpotent matrices in order to drop the invertibility assumption for the controllability question (see [7]): indeed, by applying a change of basis it is clear that the problem can be split between two subproblems: one on the (generalized) eigenspace corresponding to the zero eigenvalue, and one corresponding to the other eigenvalues.

[3]A *simple cycle* is a cycle $v_1, v_2, \ldots, v_\ell$ in which no vertex is repeated except that $v_1 = v_\ell$.

The case where A is a nilpotent matrix is in fact very simple: Obviously, if the pair (A, C) is not observable in the classical sense, the system with dropouts will not be either. Now, if the pair is observable, it is clear that one needs to have the first signal $Cx(0)$ in the observability matrix in order to be full rank. Thus, any signal starting with a zero is not observable, and hence, the only observable systems are the ones with an automaton where all nodes have label one.

V. Connection between Networked Systems with Delays [6] and Networked Systems with Dropouts

In this section we show how our technique can be applied to the previously studied problem of controllability with *varying delays.* This setting is also inspired from control over wireless networks, but focuses on another well known typical problem of these systems: when transmitted accross a wireless network, data can arrive with some delay to the plant, and, moreover, due to random communication failures and consequent rerouting, these delays are not fixed, but may vary depending on the encountered failures.

In [9], the varying delays incurred on the control packets are modeled by a disturbance signal $\delta(t) \in D : t \geq 0$, where $D \subseteq \{0, 1, \ldots, d_{max}\}$ is the set of possible delays introduced by all routing paths and d_{max} is the maximum delay. If delays are varying, it is obvious that several packets may arrive at the same time to the plant. In the model discussed here, it is assumed that in this case, the input at a particular time is defined as the sum of the numerical values of the control packets arriving at this time at the plant; see Equation (5) below (note that other assumptions can be made, under which the results presented here remain valid). This leads to the following dynamical equations of an LTI system with varying delays:

Definition 5: An *LTI system with varying delays,* described by a plant matrix $A \in \mathbb{R}^{n\times n}$, an input matrix $B \in \mathbb{R}^{n\times m}$, and a set of possible delays $D \subseteq \{0, 1, \ldots, d_{max}\}$ satisfies the following equations:

$$x(t+1) = Ax(t) + B \sum_{\substack{t - d_{max} \leq t' \leq t : \\ t' + \delta(t') = t}} u(t'), \quad (5)$$

where $u(\cdot) \in \mathbb{R}^m$ represents the output of the controller and the switching signal $\delta(\cdot) \in D$ represents the successive delays incurred on the control packets. We define the signal of *actuation times* $\tau : \mathbb{N} \to \{0, 1\}$ such that $\tau(t) = 1$ if there exists $t' \leq t$ with $t = t' + \delta(t')$, $\tau(t) = 0$ otherwise.

The system is said to be *controllable* if for any initial state x_0 and final state x_f, and for any delay signal $\delta : \mathbb{N} \to D$, there exist a control sequence $u : \mathbb{N} \to \mathbb{R}^m$ and a time T such that $x(T) = x_f$ in (5).

We suppose here that the plant matrix A is invertible. It is shown in [8] how to reduce the problem to the study of a problem with an invertible matrix in case A is not invertible. Our main result is summarized in the following theorem:

Theorem 3: Let (A, B, D) represent a system as in (5) with A an invertible matrix. One can build an automaton

$\mathcal{A}$ such that the system $(\tilde{A}, B^\top, \mathcal{A})$, with $\tilde{A} := A^{-\top}$, is observable with dropout signals as described by $\mathcal{A}$ if and only if the initial system (A, B, D) with varying delays is controllable.

Proof: We first recall from [9] how a controllability matrix can be built, analogous to the observability matrix introduced in this paper, in order to describe the controllability of a system with varying delays. Given a system (5) and a switching signal δ we call the *controllability matrix at time* t, denoted $C_t(A, B, D, \delta)$, whose columns are given (by increasing order of t') by

$$\{A^{t-t'-\delta(t')}B : \, t' \geq 0, \; t - t' - \delta(t') \geq 0\}. \qquad (6)$$

It is not difficult to see [9, Proposition 2] that the System (5) is uncontrollable if and only if there exists a switching signal δ such that, for all $t \in \mathbb{N}$, the rank of C_t is smaller than n.

Thus, the controllability of System (5) also boils down to the question whether there exists a switching signal such that some matrix determined by this switching signal has its rank smaller than n. Since A is assumed invertible, this is equivalent to ask whether for all t, the matrix $C'_t(A, B, D, \delta)$ with columns in

$$\{A^{-t'-\delta(t')}B : \, t' \geq 0, \; t' + \delta(t') \leq t\} \qquad (7)$$

is of rank smaller than n. This latter equation suggests to build a dropout system $(\tilde{A}, B^\top, \mathcal{A})$, with $\tilde{A} = A^{-\top}$ such that the possible observability matrices are as in (6).

We now build an automaton simulating the possible delay sequences $\delta(t)$. It corresponds to the so-called *De Bruijn graph* of the sequences on the alphabet D. The construction is as follows: In this graph, vertices correspond to the vectors in $D^{d_{max}}$, and for each vertex $v = (d_0, \dots, d_{d_{max}})$, and each delay $d \in D$, there is an edge labeled with d pointing from v towards $(d, d_0, \dots, d_{d_{max}-1})$. Each vertex in the graph represents the last values of the delay signal at time t, and from this information one can easily compute if the actuation signal at time t is equal to zero or one. More precisely, σ is equal to one if and only if there is an $i \in \mathbb{N}_{[0,d_{max}]}$ such that $v(i) = i$ (where the indices of v range from 0 to d_{max}).

It remains to prove that there is an infinite path in the automaton $\mathcal{A}$ such that the corresponding observability matrix is of rank smaller than n for all t if and only if there is a delay signal $\delta : \, \mathbb{N} \to D$ such that $C'_t(A, B, D, \delta)$ in (6) is of rank smaller than n for all t.

If: Let us take such a signal $\delta : \, \mathbb{N} \to D$. We start from the vertex corresponding to the vector $(\delta(d_{max}), \delta(d_{max} - 1), \dots, \delta(0))$, and then follow the path with edges labeled $\delta(d_{max}+1), \delta(d_{max}+2), \dots$. It is clear from the definition of the automaton that the actuation signal σ corresponding to this path satisfies

$$\sigma(t) = 1 \Leftrightarrow \exists t' \in [t, d_{max} + t] : \, \delta(t') = t + d_{max} - t'.$$

Thus, from (2), the rows of the observability matrix at an arbitrary time t'' are included in

$$\{B^\top \tilde{A}^t : \, \exists t' \in [t, t + d_{max}] : \, \delta(t') = t + d_{max} - t'\}. \qquad (8)$$

This set is exactly the set of rows $(A^{d_{max}} C'_t(A, B, D, \delta))^\top$, where C'_t is the matrix describing controllability of our system with varying delays (see (6)), which is never of rank n by hypothesis.

Only if: Conversely, if there is a path such that the observability matrix is never of rank n, the corresponding delay signal δ obtained by the successive labels $d_1, d_2, \dots, d_t$ on the edges of this path leads to a controllability matrix (see (6))

$$C_t(A, B, D, \delta) = \{A^{t-t'-\delta(t')}B : \, t' \geq 0, \; t - t' - \delta(t') \geq 0\}$$

which, premultiplying by A^{-t} and transposing, gives rows of the shape

$$B^\top \tilde{A}^{t'+\delta(t')}.$$

All these rows are rows of the observability matrix (2) corresponding to the path, and so the matrix C_t cannot be of rank n. ■

We conclude this section by noting that this result gives an algorithm for deciding controllability of systems with varying delays: Simply construct the automaton $\mathcal{A}$ described in the above proof, and decide observability of the system $(A^{-\top}, B^\top, \mathcal{A})$ with the techniques presented in this paper. We note, however, that the alternative technique presented in [9] seems more advantageous, since: 1. It does not require to build an auxiliary automaton, whose size can be exponential in the size of the initial problem and 2. in [9], the total computational time of the algorithm is bounded w.r.t. the size of the initial problem. Nevertheless, we believe that it is worth to remark, at least for theoretical purposes, that the problem of controllability with varying delays is actually a particular case of the general problem treated in this paper.

VI. CONCLUSION

In this paper we studied a fundamental systems theoretic property of networked control systems subject to packet losses, namely observability. This was a natural follow up on our previous work where we had provided a necessary and sufficient as well as algorithmically verifiable characterization of *controllability* of discrete-time linear systems subject to data losses. In the current paper we dealt with several extensions envisioned in [7]. We employed a constrained switching systems perspective in which available information on the packet loss signal is modelled through an automaton. Our analysis is based on Skolem's Theorem from linear algebra and tools from Automata theory.

Our first contribution is towards providing an algorithmically verifiable necessary and sufficient characterization for observability in networked control systems subject to packet losses. Following a similar line of analysis as presented in [7] we arrived at this characterization. However, unlike in the case of controllability, our analysis for observability does not rely on the invertibility assumption on the system matrix A.

We addressed the problem of relaxing [7, Assumption 1] in the following fashion: we show how our analysis for observability with $C \in \mathbb{R}^{p \times n}$ follows naturally for the case of controllability with A invertible and $B \in \mathbb{R}^{n \times m}$, i.e., the multi-input case; and also present an ad hoc analysis for non-invertible A.

We then established a connection between the networked control systems model with varying delays dealt with in [6] and the model with dropouts considered in this paper via the construction of an auxiliary automaton. It was shown that at the cost of an exponential-size encoding involved with this construction, the problem of controllability on a networked system with varying delays is a special case of the setup with dropouts employed in this paper.

Finally, we raised several open questions related to the algebraic nature and the algorithmic complexity of the problems under consideration.

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