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Controllability of linear systems subject to packet losses ⋆

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Abstract: In this paper we study the controllability properties of discrete-time linear systems subject to packet losses. We tackle the problem from a switching systems perspective in which available information known on the packet loss signal, e.g., there cannot be more than a given maximum number of consecutive losses, is modelled through an automaton. For the resulting constrained switching system, we reformulate the controllability problem into an easier-to-study formulation through an algebraic characterization.
We show that the particular case where the packet loss signal does not contain more than N consecutive dropouts ($N \in \mathbb{N}$) boils down to a similar controllability problem with switching delays previously studied in the literature. For the general case, i.e., for an arbitrary automaton describing the lossy behaviour, we exploit the algebraic characterization and establish that our controllability problem of constrained switching systems is algorithmically solvable. This latter result is obtained by connecting it with the celebrated Skolem Theorem from linear algebra.

Keywords: Switching Systems, Packet Losses, Networked Control Systems, Controllability, Decidability, Constrained Switching, Automata.

1. INTRODUCTION

In several contexts a real-time control loop is intermittently disrupted by undesired events causing imperfect control updates. These events include packet drops in wireless communication, task deadline misses in shared embedded processors, and outliers in sensor data, which can typically be modeled as data losses. It is obvious that the loss of data can severely influence fundamental properties such as controllability, observability, stabilizability, detectability, etc. of control systems. Due to many important applications suffering from these imperfect control updates, packet losses (or similar phenomena) have attracted much attention recently, see, e.g., Sinopoli et al. (2004); Tabbara et al. (2007); Pajic et al. (2011); D'Innocenzo et al. (2013); Gommans et al. (2013) and the references therein. Most of these works focus on stability and observer/controller design, and only implicitly deal with the fundamental properties such as controllability, observability, stabilizability and detectability under packet losses. In fact, the methods proposed were only providing sufficient conditions for stability or stabilizing controller design, or were assuming that one could fix the switching signal (e.g. to a periodic one) in order to control the plant more easily. However, given that these fundamental properties form cornerstones of modern system theory and indicate the possibilities and impossibilities for controller design, it is important to be able to analyze these properties explicitly when packet losses are an intrinsic feature of the feedback loop.

Therefore, we are interested in this paper in deriving *necessary and sufficient conditions* for controllability of plants subject to data losses (although we envision that our results also apply to other fundamental properties such as observability). In particular, we are interested in effectively deciding whether or not the plant remains controllable despite the possibility of losing data, and despite our inability to control this loss of data. We study a *worst case situation* in the sense that we will investigate whether *there exists one particular* sequence of data losses, which can hamper controllability. Not only is this question relevant in various real-life applications, it also obviously provides sufficient conditions for controllability of a system with probabilistic data loss behavior (which is often considered in the literature), and, in our view, it allows to formally understand from an algebraic point of view the concept of controllability with packet losses.

Our problem setup will lead us to analyze controllability properties of discrete-time linear systems with data losses modelled through automata. The data loss automaton captures the information available on the packet loss behaviour, which typically depends on the networked and

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embedded control architectures used for implementing the control loop. This information could be, for instance, that there cannot be more than a given maximum number of consecutive losses. As we will show below, the overall system can be described as a *constrained switching system* (see Essick et al. (2014); Weiss and Alur (2007); Philippe et al. (2015)), being a switching system in which the admissible switching signals are generated by an automaton. As such, the problem studied in this paper boils down to studying controllability for this class of hybrid systems.

Controllability (and other fundamental system theoretic properties) have received quite some attention in the literature for different classes of hybrid systems, see, e.g., (Lin and Antsaklis, 2009; Blondel and Tsitsiklis, 1999; Bemporad et al., 2000; Ge and Sun, 2005; Babaali and Egerstedt, 2005; Sun and Zheng, 2001; Xie and Wang, 2003; Camlibel et al., 2008) and the references therein. However, it is known that general results for hybrid systems are hard to come by Blondel and Tsitsiklis (1999). This even holds in the context of switched linear systems (even without possible constraints on the switching) showing that analyzing controllability of switching systems is an extremely difficult task.

Although it is relatively easy to construct controllability conditions, which are generalizations of the classical tests for non-switching systems (see, for instance, (Ge and Sun, 2005, Theorem 4.31 p. 137) for such a general statement; our Proposition 1 below can be seen as a similar result for the systems at hand here), verifying them is another story. Indeed, as pointed out in the reference above,

> ...the conditions of (the) theorems are not verifiable in general. The proofs do not provide any information on how to find (...) controllability, reachability, etc.

In fact, these problems are typically *undecidable*, meaning that *there is no algorithm to solve these general problems* – see (Jungers, 2009, p. 29) and references therein for formal statements in this respect.

In this paper, we will exploit the particular algebraic structure of the systems at hand to show that in this case algorithms do exist that solve this problem. This particular algebraic structure comes from the fact that the switching is caused solely by packet losses. This causes that the underlying submodels of the switched systems share the same system (state) matrix and only the input matrix is different for each submodel.

It is worth mentioning that our work is close in philosophy to previous work initiated in Jungers et al. (2012), where controllability algorithms are constructed for other particular classes of switching systems, namely with a *switching delay* in the feedback loop. Below, we make one connection between these two settings: We show that the particular case of our problem mentioned above, where the constraint on the switching formalizes that there cannot be more than a given maximum number of consecutive losses, can be solved with techniques developed in Jungers et al. (2012) [1].

Another work close in nature to ours is Babaali and Egerstedt (2005). In their work, the authors provide sufficient conditions for observability (and controllability) of another class of switching systems which bears some similarities to ours. Similar to our framework, the system (state) matrix is the same for all the submodels, and only the input matrix switches in time. However, there are also differences with our work. For instance, the switching is arbitrary in their case, and there are several different input matrices. Moreover, the controllability conditions in Babaali and Egerstedt (2005) are only sufficient. They require constructing many pairs [2] of the type (A^l, b_i), where A is the plant matrix, l is an integer, and b_i is a possible input vector; and verifying that these pairs are controllable (in the classical sense). The authors prove that, if all these auxiliary pairs are controllable, then the switching system is also controllable.

Outline and contributions. In Section 2, we introduce our problem and start our running example. In Section 3, we provide a characterization of controllability for our systems. In Section 4, we prove our main theorem, leading to an algorithm for verifying controllability. Finally, in Section 5, we show that the controllability of linear systems with variable delays is a particular case of our problem.

2. PROBLEM FORMULATION

We study the controllability of the following discrete-time linear system

$$x(t+1) = \begin{cases} Ax(t) + bu(t), & \text{if } \sigma(t) = 1, \\ Ax(t), & \text{if } \sigma(t) = 0 \end{cases} \quad (1)$$

with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$ the state and control input, respectively, at time $t \in \mathbb{N}$. Moreover, $\sigma : \mathbb{N} \to \{0, 1\}$ is the *data loss signal*, which represents the packet dropouts in the sense that $\sigma(t) = 0$ corresponds to the case where the control packet is lost at time t, while $\sigma(t) = 1$ corresponds to the case where the control packet has arrived in good order at time t. Sometimes, we also call this the *actuation signal*. The trajectory generated by the system (1) with initial state $x(0) = x_0$, input sequence $u : \mathbb{N} \to \mathbb{R}$ and actuation signal $\sigma : \mathbb{N} \to \{0, 1\}$ is denoted by $x_{x_0,\sigma,u}$.

In this paper we assume that the data loss signals satisfy certain constraints representing the physics of the shared (wireless) communication network and/or the characteristics of the underlying embedded architecture. In our model, the admissible actuation signals are infinite words accepted by a certain automaton.

Definition 1. An *automaton* is a pair $\mathcal{A} = (M, s) \in \{0, 1\}^{N \times N} \times \{0, 1\}^N$ with N the *number of states*, the *transition matrix* $M \in \{0, 1\}^{N \times N}$, and the *vector of node labels* $s = [s_1 \; s_2 \; \ldots \; s_N]^\top \in \{0, 1\}^N$.

Automata define the set of data loss signals that are admissible according to the following definition.

Definition 2. A signal $\sigma : \mathbb{N} \to \{0, 1\}$ is said to be *admissible*, if there exists a sequence of states $v : \mathbb{N} \to \{1, \ldots, N\}$

[1] In fact, it can be proved that the *general* setting of Jungers et al. (2012) can be tackled with the tools developed here (though less efficiently, from a computational point of view). We defer this question for further work because of space constraints.

[2] The number of such pairs to test is bounded by making use of the Van Der Waerden Theorem, and is thus highly impractical. For instance, in dimension 3 and with 3 different input matrices, the bound on the number of pairs to test is larger than 10^{14610}.

such that for all $t \in \mathbb{N}$ it holds that $M_{v(t),v(t+1)} = 1$ and $\sigma(t) = s_{v(t)}$.

Example 1. Suppose that the system is such that no more than N dropouts can occur consecutively. We can easily model this constraint with an automaton with $N+1$ nodes, in which the node $i \in \{0, 1, 2, \ldots, N\}$ represents situations where the last i control packets were lost, and the one before arrived safely. Fig. 1 represents its transition graph, for $N = 3$ (where the bottom left node corresponds to node 0). Hence,

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } s = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

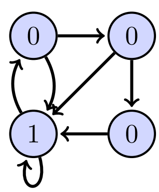

Fig. 1. Graph of an automaton forbidding more than 3 dropouts in a row. The labels s_i are directly indicated on the nodes.

Next we define controllability for the system given by the triple $(A, b, \mathcal{A})$. Note that the system is time-varying.

Definition 3. We say that the system given by the triple $(A, b, \mathcal{A})$ is *controllable*, if for all admissible actuation signals $\sigma : \mathbb{N} \to \{0, 1\}$, any initial state $x_0 \in \mathbb{R}^n$ and any final state $x_f \in \mathbb{R}^n$, there is an input signal $u : \mathbb{N} \to \mathbb{R}$ such that $x_{x_0,\sigma,u}(T) = x_f$ for some $T \in \mathbb{N}$. In case the system is not controllable it is said to be *uncontrollable.*

We consider the following general problem:

Problem 1. Determine if a system specified by the triple $(A, b, \mathcal{A})$ is controllable or uncontrollable.

Example 2. Let us consider the system (1) with

$$A = \begin{pmatrix} -2 & -13 & 9 \\ -5 & -10 & 9 \\ -10 & -11 & 12 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}. \tag{2}$$

Let us reconsider the situation where data loss signals cannot contain more than 3 dropouts in a row (Fig. 1). One can check that the pair (A, b) considered as a standard linear system is controllable. However, the data loss signal

$$\sigma = (0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, \ldots),$$

seems to hint upon the uncontrollability of the triple $(A, b, \mathcal{A})$. At least, 15 time steps are not enough to steer any initial state to any arbitrary final state. Indeed, for $x_0 = (2, 5, 8)^\top$, one can check that at no time $t \in \mathbb{N}_{\leq 15}$ it is possible to drive the state to, e.g., $(0, 0, 0)^\top$. Hence, it seems that there is a data loss signal with never more than 3 consecutive dropouts that prevents controllability of the system.

The question is now how to conclude that the system corresponding to the triple $(A, b, \mathcal{A})$ is *actually uncontrollable,* i.e., how to show the existence of an *infinite-length* switching signal under which we cannot control the state from any arbitrary initial state to any arbitrary final state? This is one of the questions we tackle in this paper.

In the remainder, we will make the following standing assumption.

Assumption 1. A is invertible and $b \in \mathbb{R}^n$ (single input).

This assumption is done for the sake of clarity and simplicity, but of course the questions raised in this paper can be posed for the more general setting where the plant matrix A and the input matrix B are arbitrary. In a similar manner also observability properties can be studied. For brevity we focus here on the controllability problem only and leave the other topics for future work.

3. PRELIMINARY RESULTS

Definition 4. Given $\sigma : \mathbb{N} \to \{0, 1\}$ we define the *controllability matrix* at time $t \in \mathbb{N}$ as $C_\sigma(t) \in \mathbb{R}^{n \times t}$, whose i-th column with $i \in \mathbb{N}_{[1,t]}$ is given[3] by $A^{(t-i)} b\sigma(i-1)$.

Hence, note that

$$C_\sigma(t) = [A^{(t-1)} b\sigma(0)\ A^{(t-2)} b\sigma(1)\ \ldots\ Ab\sigma(t-2)\ b\sigma(t-1)]$$

and thus, it is straightforward to check from (1) that for $t \in \mathbb{N}$,

$$x_{x_0,\sigma,u}(t) = A^t x_0 + C_\sigma(t) u^{t-1} \tag{3}$$

in which $u^{t-1} = [u(0)\ u(1)\ \ldots\ u(t-1)]^\top$.

For later purposes we also introduce the *backward reachability* matrix $R_\sigma(t)$ at time t for a given actuation signal σ as the matrix $A^{-(t-1)} C_\sigma(t)$, i.e.

$$R_\sigma(t) = [b\sigma(0)\ A^{-1} b\sigma(1)\ \ldots\ A^{-(t-1)} b\sigma(t-1)]. \tag{4}$$

The sequence of matrices $(R_\sigma(1), R_\sigma(2), \ldots)$ has the advantage that the image of the matrices is growing in the sense that $\mathrm{Im} R_\sigma(t) \subset \mathrm{Im} R_\sigma(t+1)$ for all $t \in \mathbb{N}$. This property does not hold for the controllability matrices.

Proposition 1. The system $(A, b, \mathcal{A})$ is controllable if and only if there exists no admissible signal $\sigma :\ \mathbb{N} \to \{0, 1\}$ such that for all $t \in \mathbb{N}$ the controllability matrix $C_\sigma(t)$ is of rank smaller than n. Equivalently, the system $(A, b, \mathcal{A})$ is controllable if and only if there exists no admissible signal $\sigma :\ \mathbb{N} \to \{0, 1\}$ such that for all $t \in \mathbb{N}$ $\mathrm{rank} R_\sigma(t) < n$.

Proof. [sketch] $\Rightarrow$ Proceed by contradiction. Suppose there is a σ with, for all $t \in \mathbb{N}$, the controllability matrix $C_\sigma(t)$ being of rank smaller than n. Take $x_0 = 0$. From (3), we have that

$$x(t) = C_\sigma(t) u^{t-1},$$

where $x = x_{x_0,\sigma,u}$. Hence, the set of reachable vectors at time t is a strict linear subspace of $\mathbb{R}^n$. Thus, the set of reachable points, which is a countable union of these strict linear subspaces, cannot be equal to $\mathbb{R}^n$.

$\Leftarrow$ Conversely, if for each σ there is a t such that $C_\sigma(t)$ has full rank, then the system is clearly controllable based on (3).

Since A^{-t} is invertible for all $t \in \mathbb{N}$, the statement directly translates to the backward reachability matrices as well. □

Proposition 2. The system $(A, b, \mathcal{A})$ is uncontrollable if and only if there exists a linear subspace R^* of dimension smaller than n together with an admissible actuation signal $\sigma : \mathbb{N} \to \{0, 1\}$ such that $\mathrm{Im} R_\sigma(t) \subseteq R^*$.

[3] A^0 is the identity by convention.

Proof. $\Rightarrow$ If the system is uncontrollable, according to Proposition 1 there is an admissible actuation signal σ such that for all $t \in \mathbb{N}$, $\text{rank}R_\sigma(t) < n$. Since $\text{Im}R_\sigma(t) \subset \text{Im}R_\sigma(t+1)$ for all $t \in \mathbb{N}$ the result is immediate.

$\Leftarrow$ Clearly, if there exists a linear subspace R^* of dimension smaller than n together with an admissible actuation signal $\sigma : \mathbb{N} \to \{0,1\}$ such that $\text{Im}R_\sigma(t) \subseteq R^*$, it follows that for all $t \in \mathbb{N}$ $\text{rank}R_\sigma(t) < n$. Using again Proposition 1, it is clear that the system is uncontrollable. □

We call a signal σ uncontrollable (for the system $(A, b, \mathcal{A})$) when it has the property as indicated in the above proposition for some linear subspace R^* of dimension smaller than n.

4. MAIN RESULTS

In this section we tackle Problem 1 and show it is decidable. For this we will make use of the following celebrated result from linear algebra, sometimes called the Skolem-Mahler-Lech theorem.

Theorem 1. (Skolem (1934)). Consider a matrix $A \in \mathbb{R}^{n\times n}$ and two vectors $b, c \in \mathbb{R}^n$. The set of values of n such that $c^\top A^n b = 0$ is eventually periodic in the sense that there exist two natural numbers $P, T \in \mathbb{N}$ such that

$$\forall t \in \mathbb{N}_{>T}, \quad c^\top A^t b = 0 \Leftrightarrow c^\top A^{t+P} b = 0. \tag{5}$$

Theorem 2. The general controllability problem is decidable in the sense that there is an algorithm that given a system $(A, b, \mathcal{A})$ decides in a finite amount of time whether the system is controllable or uncontrollable.

Proof. [sketch] Our algorithm consists in running two sub-algorithms in parallel. The first sub-algorithm simply generates all the admissible signals σ for increasing length, until they make the corresponding controllability matrix full-rank, which would imply controllability of $(A, b, \mathcal{A})$. Simultaneously, we run a second sub-algorithm that generates all *cycles* of increasing length in the automaton. This second algorithm analyses the periodic signals corresponding to repeating these cycles, and we show how to decide whether such an infinite periodic signal allows for controllability or not.

In the long version of this proof, we show that on the one hand, if the system is controllable, there exists a time T such that for all admissible σ the corresponding controllability matrix at time T is of full rank; if, on the other hand, the system is uncontrollable, then there exists a *periodic* signal allowing to establish uncontrollability of $(A, b, \mathcal{A})$. Thus, one of our sub-algorithms will stop in finite time, allowing us to conclude.

Example 3. Reconsider Example 2 with the admissible actuation signals described by the automaton of Fig. 1. It can be verified that the signal

$$\sigma = (1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots)$$

also makes the system uncontrollable. This signal obviously corresponds to a cyclic path in Fig. 1.

Remark 1. In the long version of the proof, we use Skolem's theorem (Theorem 1), which allows us to prove the finite time halting of the algorithm described in the proof of Theorem 2. However, it does not allow us to compute a bound on the running time of the algorithm. In fact, somewhat surprisingly, even though Theorem 1 has a simple statement, no simple proof of it is known, and it is a well-known open problem to provide a proof which would allow to find an explicit bound on the quantities T and P in its statement (see Tao (2007); Bell et al. (2010)). Thus, even though we know that the algorithm will stop, we cannot deduce (at least from our proof) an explicit bound on the time needed.

We leave open the question of whether our proof can be simplified, so as to derive an explicit upper bound on the length of the cycle. Recall that by 'cycle' we mean a cyclic path with repetitions of vertices allowed. If one can prove the stronger property that this cycle has to be simple (i.e. with no repetitions), it would have strong implications in terms of algorithmic complexity. This is an open problem for future work.

Open Question 1. Is it true that if System 1 is uncontrollable, there is always an uncontrollable actuation signal corresponding to a simple cycle (i.e. a cycle made of only different nodes, except the first and the last) in the automaton?

5. THE MAXIMALLY CONSECUTIVE DROPOUTS CONSTRAINT

In this section we study the particular case where the automaton represents the constraint that the actuation signal cannot contain more than N dropouts in a row (like in Example 1). We show that in this particular case the controllability problem can be restated as a controllability problem with switching delays. These problems have been studied recently in Jungers et al. (2014). Hence, controllability with a maximum number of dropouts in a row can be solved with the methodology developed there, for which a bound on the running time of the algorithm is available.

We briefly recall the setting of Jungers et al. (2014). A *system with varying delays* is defined by a triple (A, b, D) with $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $D = \{d_1, d_2, \ldots, d_N\} \subset \mathbb{N}$ is the set of *admissible delays*. Its dynamics is determined by a *delay signal*

$$d : \mathbb{N} \to D$$

and described by the equations

$$x(t+1) = Ax(t) + \sum_{t' \leq t, t'+d(t')=t} bu(t'). \tag{6}$$

The meaning of Equation (6) is the following: At each time $t' = 1, 2, \ldots,$ the control packet is subject to a certain delay $d(t')$, which is determined by an exogeneous switching signal d. Then, the control packet $u(t')$ will impact the system not at time $t' + 1$, as in a classical LTI system, but at time $t' + d(t') + 1$. If several packets arrive at the actuator at the same time, they are simply added.

The trajectory generated by the system (6) with initial state $x(0) = x_0$, input sequence $u : \mathbb{N} \to \mathbb{R}$ and delay signal $d : \mathbb{N} \to D$ is denoted by $x_{x_0,d,u}$.

In the spirit of Definition 3 the following definition was introduced in Jungers et al. (2014):

Definition 5. We say that a system with varying delays described by the triple (A, b, D) is *controllable* if for any delay signal $d : \mathbb{N} \to D$, any initial state $x_0 \in \mathbb{R}^n$ and any final state $x_f \in \mathbb{R}^n$, there is an input signal $u : \mathbb{N} \to \mathbb{R}$ such that $x_{x_0,d,u}(T) = x_f$ for some $T \in \mathbb{N}$. In case the system is not controllable it is said to be *uncontrollable.*

In this setting too, given a delay signal $d : \mathbb{N} \to \mathbb{N}$ one can introduce the controllability matrix $\bar{C}_d(t)$.

Definition 6. Given $d : \mathbb{N} \to \mathbb{N}$ we define the *controllability matrix* at time $t \in \mathbb{N}$ as $\bar{C}_d(t) \in \mathbb{R}^{n \times t}$, whose i-th column (for all $i \in \mathbb{N}_{[1,t]}$) is given by

- $A^{(t-i-d(i-1))}b$, if $i + d(i-1) \leq t$;
- the zero column, if $i + d(i-1) > t$.

The state of a system with varying delays as in (6) can be expressed thanks to its controllability matrix as

$$x_{x_0,\sigma,u}(t) = A^t x_0 + \bar{C}_d(t) u^{t-1}. \quad (7)$$

Again, controllability can be related to the matrices $\bar{C}_d(t)$ as formalized next.

Proposition 3. [Adapted from Proposition 4 in Jungers et al. (2014)] The delay system given by (A, b, D) is uncontrollable if and only if there is a delay signal

$$d : \mathbb{N} \to D$$

such that for all $t \in \mathbb{N}$ it holds that $\mathrm{Im}(\bar{C}_d(t)) \neq \mathbb{R}^n$.

The above proposition gives us a characterization of controllable systems, but again not an algorithmic one. Indeed it is not clear how to generate the signal σ like in the proposition, and even less clear how to prove the existence of such an infinite-length signal. However, it is shown in Jungers et al. (2014) that this problem is decidable and even more, that one can bound the time needed by the algorithm to answer the question. In the next theorem we show that for the particular problem described in Example 1 (i.e. when the constraint is expressed in terms of a maximal number of consecutive dropouts), one can in fact apply the methods for varying delays in order to answer the controllability problem. This leads to a better method than the one presented in the previous section, because we can bound the time needed for the algorithm to return the answer.

However, we attract the attention of the reader on the fact that the techniques developed in Jungers et al. (2014) can only be applied for the very particular case where the automaton is of the shape in Fig. 1 (i.e., when the constraint on the data loss signal is a bound on the number of consecutive losses). It is relatively obvious that the case with an arbitrary automaton is much more general and powerful (in terms of modeling of a data losses signal) than a mere constraint on a possible set of delays. We leave for further work a formal proof of this statement.

Theorem 3. Let us consider a system $(A, b, \mathcal{A})$ with at most N consecutive dropouts (like defined in Example 1) for some $N \in \mathbb{N}$. The system $(A, b, \mathcal{A})$ is controllable if and only if the delay system (A, b, D) with $D = \{0, 1, \ldots, N\}$, is controllable.

Proof. First, observe that in both Definitions 4 and 6, the columns appearing in the controllability matrix are uniquely defined by the actuation signal, which characterizes the set of times where a new column b enters the controllability matrix. To formalize this statement, recall that in the data loss case $\sigma : \mathbb{N} \to \{0, 1\}$ represents the actuation signal. In the delay case, given the delay signal $d : \mathbb{N} \to \mathbb{N}$, we define the *actuation signal* $\tau_d : \mathbb{N} \to \{0, 1\}$ for $t \in \mathbb{N}$ by

$$\tau_d(t) = \begin{cases} 1, & \text{if there is a } t' \in \mathbb{N} \text{ such that } t' + d(t') = t, \\ 0, & \text{otherwise.} \end{cases}$$

With this definition in place, we have that for all $d : \mathbb{N} \to \mathbb{N}$ and all $t \in \mathbb{N}$ it holds that $\mathrm{Im}C_{\tau_d}(t) = \mathrm{Im}\bar{C}_d(t)$. Hence, if we can show that the language of actuation signals σ in the max-N-consecutive-dropouts case, and the language of actuation signals τ_d induced by delay signals $d : \mathbb{N} \to D$ in the varying delay case with $D = \{0, 1, \ldots, N\}$ are the same, we have established the proof of the theorem. Indeed, if we can show this then there exists an actuation signal σ which is uncontrollable in the max-N-consecutive-dropouts case if and only if there exists a delay signal $d : \mathbb{N} \to D$ (inducing a τ_d) in the switching delays case being uncontrollable.

To show that the languages are indeed equal, take any signal $\sigma : \mathbb{N} \to \{0, 1\}$ satisfying the max-N-consecutive-dropouts constraint, that is

$$\forall t \in \mathbb{N}, \exists t' \in \mathbb{N}_{[t,t+N]} \text{ such that } \sigma(t') = 1. \quad (8)$$

The signal σ is the same as τ_d corresponding to the delay signal $d : \mathbb{N} \to \mathbb{N}$ given for $t \in \mathbb{N}$ by

$$d(t) = \min_{t' \geq t, \sigma(t')=1} (t' - t).$$

By Equation (8), the signal d is admissible in the sense that $d : \mathbb{N} \to D$ with $D = \{0, 1, \ldots, N\}$, as the righthand-side in the equation above is always equal to or smaller than N.

Conversely, a signal $d : \mathbb{N} \to \{0, 1, \ldots, N\}$ obviously incurs an actuation signal τ_d with at most N zeros in a row, which is clearly contained in the set of admissible signals σ for the max-N-consecutive-dropout case. □

6. CONCLUSION AND FURTHER WORK

In this paper we studied the controllability properties of discrete-time linear systems with data losses modelled through automata. The data loss automaton captured the information available on the packet loss behaviour, as could be derived by formal methods from the underlying embedded or networked control architecture. For the resulting class of constrained switching systems, we have given a characterization of (un)controllable systems, which allowed to derive a general decidability procedure to recognize (un)controllable systems. We then analysed the perhaps most natural situation, where the automaton imposes a hard bound on the maximum number of consecutive data losses. We showed that this situation is solvable more efficiently with previously proposed techniques for the control of systems with arbitrary switching delays. In fact, more connections can be made with the latter problem. Indeed, the general problem of switching delays (with an arbitrary set of possible delays $D \subset \mathbb{N}$) can actually be shown to be a particular case of the general framework presented here, although we leave this for future work. In fact, let us emphasize that the results presented here cover more general cases that cannot be cast in a switching delay setting.

Many more questions remain to be solved. Next to the open problems we already formulated, in the future, we plan to generalize our proof to include also singular matrices, and an arbitrary number of inputs. Also, we will study whether it is possible to derive an upper bound on the length of the periodic uncontrollable signal in Theorem 2. Also we believe that results presented here do not only apply for controllability, but also for other fundamental system-theoretic notions including observability. Finally, we are working at leveraging our analysis towards a concept taking into account the probability of packet losses, closer to more classical models for wireless control networks.

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