Observer design for a class of piecewise linear systems

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SUMMARY

In this paper we present observer design procedures for a class of bi-modal piecewise linear systems in both continuous and discrete time. We propose Luenberger-type observers, and derive sufficient conditions for the observation error dynamics to be globally asymptotically stable, in the case when the system dynamics is continuous over the switching plane. When the dynamics is discontinuous, we derive conditions that guarantee that the relative estimation error with respect to the state of the observed system will be asymptotically small. The presented theory is illustrated with several examples. Copyright © 2007 John Wiley & Sons, Ltd.

1. INTRODUCTION

In this paper we present observer design procedures for a class of bimodal piecewise linear (PWL) systems in both continuous and discrete time. The systems of the considered class comprise two linear dynamics with the same input distribution matrix. The characteristic feature of our approach is that the state reconstruction is performed on the basis of input and measured output signals only, while the information on the active linear dynamics (or mode) is not required.

For the case when the mode is known, methods to construct observers in continuous and discrete time are presented in [1–3]. The proposed observers are of Luenberger type, and achieve

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global asymptotic stability of the observation error. The observer design procedure is based on constructing observer gains for each individual mode and a common quadratic Lyapunov function for the switched estimation error dynamics. The resulting conditions were formulated in terms of linear matrix inequalities.

A more difficult case, set in discrete time, when the discrete mode is not known, was considered in [4]. The proposed observers use discrete inputs and outputs of the hybrid plant, augmented with discrete signals derived from the continuous measurements when necessary, to obtain the estimate of the mode. Subsequently, the estimate of the continuous state can be obtained, for example, using the techniques of [1–3]. The designed observers aim to correctly identify the mode of the plant in a finite number of time steps, after which the continuous observation error exponentially converges to a bounded set. The class of systems considered in this paper does not have discrete inputs and outputs and therefore we propose a more direct approach for state estimation, which is more close to the classical Luenberger observers.

Another approach to state estimation for discrete time hybrid systems, based on moving horizon estimation, was considered in [5]. This approach is applicable to the general class of piecewise affine systems, but it is computationally demanding (mixed integer quadratic programming problems have to be solved online), which may be an obstacle for implementing it in applications with limited computational resources. The observers presented here exploit the structure of the considered class of PWL systems, and can be implemented in a numerically friendly way.

In this paper we focus directly on the observer design using Luenberger-type observers of which preliminary versions were presented in [6, 7]. In this case the state estimation error dynamics, defined by interconnecting the bimodal system with the bimodal observer has four modes. Contrary to the classical Luenberger observer for linear systems and to the case when the mode is known, the error dynamics is not autonomous, but depends on the state of the observed system, and hence, indirectly, on the control input. Global asymptotic stability of the estimation error may still be achieved, in particular when the bi-modal system is continuous over the switching plane. In the case of a discontinuous system, our approach guarantees that the norm of the error will asymptotically not exceed a certain bound, relative to the bound on the state of the observed system. The conditions that guarantee the existence of the observer are formulated in terms of linear matrix inequalities (LMI).

In the particular case when the system is continuous over the switching plane, the PWL systems under study can be reformulated as a Lur'e-type system with a $\max(0, \cdot)$ nonlinearity in the feedback path (see, for instance, [8]). Observer design for Lur'e-type systems, when the signal that enters the nonlinearity in the feedback path is not measured, is presented in [9]. We will establish a link between the result in [9] and the conditions obtained in this paper.

Of course, the design of an observer that reconstructs the state of a hybrid system is related to the *observability* or *detectability* properties of the underlying system. Loosely speaking, observability is the property that ensures that the state of the observed system can be recovered from the measurements of the system's output and input signals. Precise definitions of observability and results for some classes of piecewise affine and switched systems can be found in [10–15]. From a system theoretical point of view the investigation of observability conditions is an interesting problem and it might be a source of inspiration for the design of observers. However, in this paper we opted for the more direct approach of designing an observer. The feasibility conditions for the existence of the observer, of course, imply some observability (or detectability) properties of the system under study. Investigating the gap between the available

observability conditions in the literature and the conditions presented here for the existence of an (asymptotic) observer is of importance. However, at present this is mainly an open issue.

The observers that we consider can be used in situations when the state of the system (continuous state or discrete mode) is of interest (e.g. for diagnostics or discrete mode change detection). Research on control of piecewise affine systems has been mainly concerned with state feedback control designs [16–18]. The proposed observers enable the implementation of these controllers in situations when the state is not measured. Output feedback controller design (which implicitly consists of an observer part and a state feedback part) was presentedin [19, 20]. However, it is not straightforward to extract the observer from the dynamic feedback controller proposed by Rodrigues *et al.* [19] and Rodrigues and How [20].

The paper is organized as follows. In Section 2 we introduce some preliminary definitions and notation. In Section 3 we introduce the considered class of bi-modal PWL systems. In Section 4 we present the observer design procedures. Sliding modes, for the continuous time case, are analysed in Section 5. The discrete time case is analysed in Section 7 and examples are given in Section 8. Conclusions are presented in Section 9.

2. PRELIMINARIES AND NOTATION

Definition 2.1
A function $x : \mathbb{R}^+ \to \mathbb{R}^n$ is said to be *bounded by a constant* $x_{\max} > 0$, if $||x(t)|| \leqslant x_{\max}$ for all $t > 0$, i.e. $\sup_{t \in \mathbb{R}^+} ||x(t)|| \leqslant x_{\max}$. A function x is said to be *eventually bounded by* $x_{\max}$, if for all $\delta > 0$ there exists a $T_0 > 0$ such that $||x(t)|| \leqslant x_{\max} + \delta$, for all $t > T_0$, i.e. $\limsup_{t \to \infty} ||x \times (t)|| \leqslant x_{\max}$.

Definition 2.2
The sequence $(x(0), x(1), x(2), \ldots)$ is said to be *bounded by* $x_{\max}$ if $\forall k \geqslant 0$ $||x(k)|| \leqslant x_{\max}$. The sequence $(x(0), x(1), x(2), \ldots)$ is said to be *eventually bounded by* $x_{\max}$, if $\forall \delta > 0$ $\exists k_0 > 0$ $\forall k \geqslant k_0$ $||x(k)|| \leqslant x_{\max} + \delta$. i.e. $\limsup_{k \to \infty} ||x(k)|| \leqslant x_{\max}$.

$M^\top$ denotes the transpose of the matrix M. For a square matrix M, $M > 0$ means that M is symmetric i.e. $M = M^\top$ and positive definite. The operators $\ker M$ and $\operatorname{im} M$ denote the kernel and the image of the matrix M, respectively.

The operator $\operatorname{col}(\cdot, \cdot)$ stacks its operands into a column vector, i.e. for $v_1 \in \mathbb{R}^n$, $v_2 \in \mathbb{R}^m$

$$\operatorname{col}(v_1, v_2) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^{n+m}$$

In matrices we denote by $(*)$ at position (i, j) the transposed matrix element at position (j, i), e.g.

$$\begin{bmatrix} A & B \\ (*) & C \end{bmatrix} \quad \text{means} \quad \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$$

3. PROBLEM STATEMENT

We consider the following system:

$$\dot{x}(t) = \begin{cases} A_1 x(t) + Bu(t) & \text{if } H^\top x(t) < 0 \\ A_2 x(t) + Bu(t) & \text{if } H^\top x(t) > 0 \end{cases} \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$ are the state, output and the input of the system, respectively, in time $t \in \mathbb{R}^+$. The input $u : \mathbb{R}^+ \to \mathbb{R}^m$ is assumed to be an integrable function. We will consider Filippov solutions of the system (1) [21]. The matrices $A_1, A_2 \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ and $H \in \mathbb{R}^n$. The hyperplane defined by $\ker H^\top$ separates the state space $\mathbb{R}^n$ into the two half-spaces. The considered class of bimodal PWL systems has identical input distribution matrix B and output distribution matrix C for both modes.

Analogously, in the discrete time case we will consider the following system:

$$x(k+1) = \begin{cases} A_1 x(k) + Bu(k) & \text{if } H^\top x(k) < 0 \\ A_2 x(k) + Bu(k) & \text{if } H^\top x(k) > 0 \end{cases} \tag{2a}$$

$$y(k) = Cx(k) \tag{2b}$$

where all matrices have the same dimensions as before, and $k \in \mathbb{N}$ denotes the time index.

Remark 3.1
It is possible to consider a somewhat more general class of bimodal PWL systems, where the output distribution matrices would be different for each of the modes, and to derive observer design procedure following the procedure presented here. Stability properties of the designed observers remain the same as the class studied here. For reasons of ease of exposition, here we consider the classes (1) and (2).

Depending on the values of A_1 and A_2 we distinguish two situations:

(1) The vector field of the system is continuous over the switching plane, i.e. $A_1 x = A_2 x$, when $H^\top x = 0$. It is straightforward to show that in this case

$$A_2 = A_1 + GH^\top \tag{3}$$

for some vector G of appropriate dimensions. In this case Equation (1a) can be rewritten as

$$\dot{x} = A_1 x + G \max(0, H^\top x) + Bu$$

or, in the discrete time case (2a)

$$x(k+1) = A_1 x(k) + G \max(0, H^\top x(k)) + Bu(k)$$

Moreover, from (3) it follows that $\text{rank}(\Delta A) = 1$, where ΔA is defined as $\Delta A := A_1 - A_2$.

(2) The vector field of the system is not continuous over the switching plane, i.e. a parametrization as in (3) does not exist.

The problem at hand is to design a state estimation procedure, which, on the basis of the known system model, input u, and measured output y provides a state estimate $\hat{x}$, without directly measuring the mode of the system.

Remark 3.2
Information on the currently active linear dynamics may be available to the observer if $H^\top x$ can be reconstructed from the measured output, i.e. $H \in \operatorname{im} C^\top$. In this case the results from [1–3] apply.

4. MAIN RESULTS FOR THE CONTINUOUS TIME CASE

As an observer for the system (1), we propose a continuous time bimodal system with the following structure:

$$\dot{\hat{x}} = \begin{cases} A_1\hat{x} + Bu + L_1(y - \hat{y}) & \text{if } H^\top\hat{x} + K^\top(y - \hat{y}) < 0 \\ A_2\hat{x} + Bu + L_2(y - \hat{y}) & \text{if } H^\top\hat{x} + K^\top(y - \hat{y}) > 0 \end{cases} \tag{4a}$$

$$\hat{y} = C\hat{x} \tag{4b}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimated state at time t and $L_1, L_2 \in \mathbb{R}^{n \times p}$ and $K \in \mathbb{R}^p$.

The dynamics of the state estimation error, $e := x - \hat{x}$, is then described by

$$\dot{e} = \begin{cases} (A_1 - L_1C)e, & H^\top x < 0, \quad H^\top\hat{x} + K^\top(y - \hat{y}) < 0 \\ (A_2 - L_2C)e + \Delta Ax, & H^\top x < 0, \quad H^\top\hat{x} + K^\top(y - \hat{y}) > 0 \\ (A_1 - L_1C)e - \Delta Ax, & H^\top x > 0, \quad H^\top\hat{x} + K^\top(y - \hat{y}) < 0 \\ (A_2 - L_2C)e, & H^\top x > 0, \quad H^\top\hat{x} + K^\top(y - \hat{y}) > 0 \end{cases} \tag{5}$$

where x satisfies (1a) and $\hat{x}$ satisfies (4a). By substituting $\hat{x} = x - e$ in (5), we see that the right-hand side of the state estimation error dynamics is PWL in the variable $\operatorname{col}(e, x)$.

Note that the error dynamics in the first and the fourth mode of (5) is described by an n-dimensional autonomous state equation, while in the two other modes the external signal x is present, which, by (1a), depends on the input u. For given (open loop) input signals $u : \mathbb{R}^+ \to \mathbb{R}^m$ it is possible to consider the evolution of the error e in (5) as a time varying switched equation of the form

$$\frac{\mathrm{d}e}{\mathrm{d}t}(t) = f_i(t, e(t)), \quad i = 1, 2, 3, 4 \tag{6}$$

Hence, concepts and results of Lyapunov stability theory for hybrid systems (see, for instance, [8, 22, 23]) can now be applied to Equation (6).

The problems of observer design can now be formally stated as below.

Problem 4.1
Determine the observer gains L_1, L_2 and K in (4) such that global asymptotic stability of the estimation error dynamics (5) is achieved, for all functions $x : \mathbb{R}^+ \to \mathbb{R}^n$, satisfying (1) for some given locally integrable $u : \mathbb{R}^+ \to \mathbb{R}^m$.

Problem 4.2
Determine $\eta > 0$, and L_1, L_2 and K in (4) such that for all bounded trajectories $x : \mathbb{R}^+ \to \mathbb{R}^n$ it holds that

$$\limsup_{t \to \infty} ||e(t)|| \leqslant \eta \limsup_{t \to \infty} ||x(t)|| \tag{7}$$

which means that if $x(t)$ is (eventually) bounded by $x_{\max}$, then $e(t)$ should be eventually bounded by $\eta x_{\max}$.

The constant η can be seen as an asymptotic upper bound of the estimation error relative to the state, and it is desirable to design the observer so that η is as small as possible.

4.1. Continuous dynamics

Consider system (1) observer (4) and the error dynamics (5).

Theorem 4.3
The state estimation error dynamics (5) is globally asymptotically stable for all $x : \mathbb{R}^+ \to \mathbb{R}^n$ (in the sense of Lyapunov), if there exist matrices $P > 0$, L_1, L_2, K and constants $\lambda \geqslant 0$, $\mu > 0$ such that the following set of matrix inequalities is satisfied:

$$\begin{bmatrix} (A_2 - L_2 C)^\top P + P(A_2 - L_2 C) + \mu I & P \Delta A + \lambda \frac{1}{2}(H - C^\top K)H^\top \\ \Delta A^\top P + \lambda \frac{1}{2} H(H^\top - K^\top C) & -\lambda H H^\top \end{bmatrix} \leqslant 0 \tag{8a}$$

$$\begin{bmatrix} (A_1 - L_1 C)^\top P + P(A_1 - L_1 C) + \mu I & -P \Delta A + \lambda \frac{1}{2}(H - C^\top K)H^\top \\ -\Delta A^\top P + \lambda \frac{1}{2} H(H^\top - K^\top C) & -\lambda H H^\top \end{bmatrix} \leqslant 0 \tag{8b}$$

Remark 4.4
Inequalities (8a)–(8b) are nonlinear matrix inequalities in $\{P, L_1, L_2, K, \lambda, \mu\}$, but are linear in $\{P, L_1^\top P, L_2^\top P, \lambda, \lambda K, \mu\}$, and thus can be efficiently solved using available software packages (such as the free software LMItool).

Proof
In order to guarantee that system (5) is globally asymptotically stable it suffices to have a Lyapunov function $V(e)$ of the form

$$V(e) = e^\top P e \tag{9}$$

where $P > 0$ is such that

$$\dot{V} \leqslant -\mu e^\top e \tag{10}$$

for some $\mu > 0$.

Requirement (10) yields the following set of inequalities:

$$e^\top \{(A_1 - L_1 C)^\top P + P(A_1 - L_1 C) + \mu I\} e \leqslant 0 \tag{11a}$$

for $H^\top x < 0, H^\top (x - e) + K^\top C e < 0$

$$\begin{bmatrix} e \\ x \end{bmatrix}^\top \begin{bmatrix} (A_2 - L_2 C)^\top P + P(A_2 - L_2 C) + \mu I & P\Delta A \\ \Delta A^\top P & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leqslant 0 \tag{11b}$$

for $H^\top x < 0, H^\top (x - e) + K^\top C e > 0$

$$\begin{bmatrix} e \\ x \end{bmatrix}^\top \begin{bmatrix} (A_1 - L_1 C)^\top P + P(A_1 - L_1 C) + \mu I & -P\Delta A \\ -\Delta A^\top P & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leqslant 0 \tag{11c}$$

for $H^\top x > 0, H^\top (x - e) + K^\top C e < 0$

$$e^\top \{(A_2 - L_2 C)^\top P + P(A_2 - L_2 C) + \mu I\} e \leqslant 0 \tag{11d}$$

for $H^\top x > 0, H^\top (x - e) + K^\top C e > 0$.

Note that requirements (11a)–(11d) cannot be satisfied in the complete (e, x) space unless $\Delta A = 0$.

Regions of the (e, x) space where the second and the third linear dynamics of the error (5) is active can be covered with the quadratic constraint in the following way:

$$\begin{bmatrix} e \\ x \end{bmatrix}^\top \begin{bmatrix} 0 & -\frac{1}{2}(H - C^\top K)H^\top \\ -\frac{1}{2}H(H^\top - K^\top C) & HH^\top \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leqslant 0 \tag{12}$$

Inequality (12) is derived by multiplying the mode constraints $x^\top H(H^\top (x - e) + K^\top C e) \leqslant 0$. The quadratic constraint (12) is by construction negative in the region of interest, 0 at the boundaries, and non-negative elsewhere. Combining (10) with (12), using the S-procedure [23, 24], yields the inequalities (8a)–(8b). Note that inequalities (11a) and (11d) are implied by (8a) and (8b), respectively, and therefore can be omitted.

Note that the relaxed inequalities (8a)–(8b) can be only negative semidefinite by construction (because $-\lambda HH^\top$ is negative semidefinite), but that derivatives (11b)–(11c) are guaranteed to be negative whenever the appropriate dynamics is active and $e \neq 0$. Hence, the computed derivative of the candidate Lyapunov function (9) is negative definite in e, and the global asymptotic stability of the error dynamics (5) is guaranteed. □

Suppose that a feasible solution to (8a)–(8b) exists. Assume that $M > 0$ is a matrix. Since $M \leqslant 0$ and $z^\top M z = 0$ imply that $z \in \ker(M)$, it follows that

$$\mathrm{col}(0, h) \in \ker \begin{bmatrix} (A_2 - L_2 C)^\top P + P(A_2 - L_2 C) + \mu I & P\Delta A + \lambda \frac{1}{2}(H - C^\top K)H^\top \\ \Delta A^\top P + \lambda \frac{1}{2}H(H^\top - K^\top C) & -\lambda HH^\top \end{bmatrix}$$

(and analogously for the inequality (8b)), where $h \in \ker(HH^\top) = \ker(H^\top)$. Hence, we have that $\ker(H^\top) \subseteq \ker(P\Delta A) = \ker \Delta A$, since $P > 0$. From this inclusion it follows that the state

evolution matrices of the two modes are not independent, but are related *via* $A_2 = A_1 + GH^\top$ for some vector G of appropriate dimensions. This relation implies the continuity of the vector fields over the switching plane, as detailed in Section 3. Hence, an equivalent representation of the continuous bi-modal system (1) is:

$$\dot{x} = A_1 x + G \max(0, z) + Bu \tag{13a}$$

$$z = H^\top x \tag{13b}$$

$$y = Cx \tag{13c}$$

which is a Lur'e system [8], with $\max(0, \cdot) \in [0, 1]$ nonlinearity in the feedback path.

Observer design for this type of systems with slope restricted nonlinearities was presented in [9]. Here, we show that the observer design from [9] is a special case of our observer design. We will simplify our observer structure by assuming the same gain $L_1 = L_2 = L$ for both modes. Equation (8a) can then be transformed into Equation (8b), by pre-multiplying it with $Q^\top$ and post-multiplying it with Q, where

$$Q = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$$

Equation (8b) can be represented as

$$T^\top \begin{bmatrix} (A_1 - LC)^\top P + P(A_1 - LC) + \mu I & PG + \lambda(H - C^\top K) \\ G^\top P + \lambda(H^\top - K^\top C) & -2\lambda I \end{bmatrix} T \leqslant 0 \tag{14}$$

where

$$T = \begin{bmatrix} I & 0 \\ 0 & H^\top \end{bmatrix}$$

Pre- and post-multiplication with a matrix T introduces a kernel in the matrix inequality (14), but does not change feasibility conditions. Condition (14) is therefore equivalent to the LMI condition obtained in [9] (up to the scaling constant λ).

4.2. Discontinuous dynamics

Theorem 4.3 gives sufficient conditions for the solution of Problem 4.1. A drawback of the obtained result is that a necessary condition for the feasibility of (8a)–(8b) is the continuity of the bi-modal PWL system.

In order to give an approach for discontinuous systems as well we need to consider the relaxed Problem 4.2. The following theorem provides an answer to this problem.

Theorem 4.5
The state estimation error dynamics (5) is eventually bounded by a constant $e_{\max}$ (in the sense of Definition 2.1), under the assumption that x is eventually bounded by $x_{\max}$, if there exist matrices $P = P^\top > 0$, L_1, L_2 and constants $\lambda, \varepsilon \geqslant 0$ and $\mu, \alpha > 0$ such that the following set of matrix inequalities is satisfied:

$$\begin{bmatrix} \begin{array}{c} (A_2 - L_2 C)^\top P \\ + P(A_2 - L_2 C) + (\mu + \alpha)I \\ \Delta A^\top P + \dfrac{\lambda}{2} H(H^\top - K^\top C) \end{array} & \begin{array}{c} P\Delta A + \dfrac{\lambda}{2}(H - C^\top K)H^\top \\ \\ -\lambda HH^\top - \alpha \varepsilon^2 I \end{array} \end{bmatrix} < 0 \tag{15a}$$

$$\begin{bmatrix} \begin{array}{c} (A_1 - L_1 C)^\top P \\ + P(A_1 - L_1 C) + (\mu + \alpha)I \\ -\Delta A^\top P + \dfrac{\lambda}{2} H(H^\top - K^\top C) \end{array} & \begin{array}{c} -P\Delta A + \dfrac{\lambda}{2}(H - C^\top K)H^\top \\ \\ -\lambda HH^\top - \alpha \varepsilon^2 I \end{array} \end{bmatrix} < 0 \tag{15b}$$

Moreover, if

$$\gamma_1 I \leqslant P \leqslant \gamma_2 I \tag{16}$$

then

$$e_{\max} \leqslant \sqrt{\frac{\gamma_2}{\gamma_1}} \, \varepsilon x_{\max} \tag{17}$$

Proof
Note that the matrix inequalities (15) are relaxations of (8) by using the S-procedure with the quadratic constraint

$$||e||^2 \geqslant \varepsilon^2 ||x||^2 \tag{18}$$

and the multiplication constant α (see [23, 24]). Hence, they imply that the function $V(e) = e^\top P e$ satisfies

$$\dot{V}(e) \leqslant -\mu e^\top e \quad \text{when } ||e||^2 \geqslant \varepsilon^2 ||x||^2 \tag{19}$$

For an arbitrary $\delta > 0$, denote

$$V_{\max}^\delta := \max_{||e|| \leqslant \varepsilon x_{\max} + \delta} V(e)$$

Define the level set S_δ by

$$S_\delta = \{e \in \mathbb{R}^n | V(e) \leqslant V_{\max}^\delta\}$$

Since $\dot{V}(e) \leqslant -\mu e^\top e \leqslant -\beta V(e)$ for some $\beta > 0$ and $e \notin S_\delta$, it follows that S_δ is positively invariant i.e. if for $T_0 > 0y$

$$V(e(T_0)) < V_{\max}^\delta \Rightarrow V(e(t)) < V_{\max}^\delta \quad \forall_{t > T_0}$$

Moreover, S_δ satisfies a strong variant of attractivity in the sense that

$$\exists_{T_0 > 0} \quad V(e(T_0)) < V_{\max}^\delta$$

From (16) it follows that

$$V_{max}^{\delta} \leqslant \gamma_2 [\varepsilon x_{max} + \delta]^2$$

and consequently,

$$\forall_{\delta>0} \forall x_0 \forall e_0 \forall u \exists_{T_0>0} \forall_{t>T_0} \quad ||e(t)|| \leqslant \sqrt{\frac{\gamma_2}{\gamma_1}} [\varepsilon x_{max} + \delta]$$

This means that

$$e_{max} := \limsup_{t \to \infty} ||e(t)|| \leqslant \sqrt{\frac{\gamma_2}{\gamma_1}} \varepsilon x_{max} \qquad\qquad \square$$

Remark 4.6
If there exists a feasible solution for the system of inequalities

$$P > 0 \tag{20a}$$

$$(A_1 - L_1 C)^\top P + P(A_1 - L_1 C) + (\mu + \alpha)I < 0 \tag{20b}$$

$$(A_2 - L_2 C)^\top P + P(A_2 - L_2 C) + (\mu + \alpha)I < 0 \tag{20c}$$

(which implies that the pairs (A_1, C) and (A_2, C) are detectable) an ε can always be found such that (15a) and (15b) are feasible. Conditions (20) are exactly the conditions required for the observer design in the case when the current mode is known [1, 2].

Remark 4.7
Condition (18) can be stated in a more general form, when $|| \cdot ||$ is replaced by $|| \cdot ||_Q$. An interesting case is when $||e||$ is replaced by $||e||_P$. Then, given a certain $\eta_{spec} > 0$, existence of an observer that achieves bound $e_{max} \leqslant \eta_{spec} x_{max}$ follows from the feasibility of bilinear matrix inequalities (BMI) similar in form to (15) with $\varepsilon = \eta_{spec}$. The drawback is that BMIs cannot be solved in an efficient way.

Remark 4.8
Equations (15) are bi-linear in the variables $\{P, L_1, L_2, K, \varepsilon, \lambda, \mu, \alpha\}$. When ε is fixed, with the same change of variables as in Remark 4.4 we get a set of LMI.

Remark 4.9
The equation

$$P \geqslant I \tag{21}$$

can be added to (15a), (15b) without changing feasibility. Namely, if $\{P, L_1, L_2, K, \lambda, \mu_1, \mu_2, \varepsilon\}$ is the feasible solution of (15a), (15b), so is the scaled set $\{(1/\gamma_1)P, L_1, L_2, K, (1/\gamma_1)\lambda, (1/\gamma_1)\mu, \varepsilon\}$,

and $P^* = (1/\gamma_1)P \geqslant I$. The second part of the double inequality (16) follows from:

$$P - \gamma_2 I < 0 \qquad (22)$$

Any feasible solution of the Equations (15), (21) and (22) is a solution for the problem (2a), with $\eta = \sqrt{\gamma_2}\varepsilon$. An sub-optimal algorithm that aims to minimize the value of η follows from Theorem 4.5, and Remarks 4.9 and 4.8. Under the conditions of Remark 4.6 a minimal ε can be found when (15a), (15b) cease to be feasible. For a feasible value of ε close to the infeasible value an optimization problem:y

$$\min \gamma_2$$

under (15a), (15b), (21), (22) is solved. Then $e_{max} < \sqrt{\gamma_2}\varepsilon x_{max}$.

5. SLIDING MODE ANALYSIS

All derivations so far were done with the implicit assumption that sliding modes do not occur neither in the original system nor in the designed observer. In the discussion that follows we will show that the properties of the designed observers are retained, under the presence of sliding modes, in the system and in the designed observer. Note that even in the case when the system dynamics is continuous sliding modes may exist in the designed observer. This is the special case of the following analysis.

We will consider sliding modes along the switching surface ($H^\top x = 0$ for the system and $H^\top \hat{x} + K(\hat{y} - y) = 0$ for the observer) under the assumption that we have constructed an observer that satisfies Equations (15), and we are going to show that the estimation error remains eventually bounded. The mode of the system where $H^\top x < 0$ ($H^\top x > 0$) is referred to as the first mode (second mode), and in an analogous way for the observer.

First, consider the case where sliding occurs in the designed observer along the plane $H^\top \hat{x} + K(y - \hat{y}) = 0$. Then, the dynamics of the observer is given by a convex combination of the constituting linear dynamics (i.e. we use Filippov solutions as the formalization of the sliding dynamics [21]):

$$\dot{\hat{x}} = \xi\{A_1\hat{x} + Bu + L_1(y - \hat{y})\} + (1 - \xi)\{A_2\hat{x} + Bu + L_2(y - \hat{y})\}$$
$$\hat{y} = C\hat{x} \qquad (23)$$

where $\xi \in [0, 1]$. Consider next the situation where the system is in the first mode. Then the error dynamics is given byy

$$\dot{e} = \dot{x} - \dot{\hat{x}} = \xi\{(A_1 - L_1C)e\} + (1 - \xi)\{(A_2 - L_2C)e + \Delta Ax\}$$

which is a convex combination of the first and the second mode of the error dynamics (5). Since $\dot{V}(e)$ is negative when (18) holds, for both modes, it is also negative for their convex combinations under (18). Hence, the error is eventually bounded, as proven in Theorem 4.5. A similar argument holds when the system is in the second mode and the observer is in the sliding motion.

Consider now the case when a sliding mode exists on the switching plane of the system. Then, the system dynamics is given by a convex combination of the constituting linear dynamics

$$\dot{x} = \eta\{A_1 x + Bu\} + (1 - \eta)\{A_2 x + Bu\}$$

$$y = Cx \tag{24}$$

where $\eta \in [0, 1]$. If the observer is in the first mode, the error dynamics is given by

$$\dot{e} = \dot{x} - \dot{\hat{x}} = \eta\{(A_1 - L_1 C)e\} + (1 - \eta)\{(A_1 - L_1 C)e - \Delta Ax\}$$

which is a convex combination of the first and the third mode of the error dynamics. Hence, $\dot{V}(e)$ is negative when (18) holds. A similar argument holds for the case when the system is in the sliding mode, and the observer is in the second mode.

Consider now the situation where there are sliding modes in both the system and the observer. Then the dynamics of the system is given by (24) and the dynamics of the observer is given by (23). The error dynamics follows then as

$$\dot{e} = (\eta - \xi)\{(A_2 - L_2 C)e + \Delta Ax\} + (1 - \eta)\{(A_2 - L_2 C)e\} + \xi\{(A_1 - L_1 C)e\} \tag{25}$$

if $\eta - \xi \geqslant 0$, and

$$\dot{e} = (\xi - \eta)\{(A_1 - L_1 C)e - \Delta Ax\} + (1 - \xi)\{(A_2 - L_2 C)e\} + \eta\{(A_1 - L_1 C)e\} \tag{26}$$

if $\xi - \eta \geqslant 0$. We see that the error dynamics is again given as a convex combination of the modes of the error dynamics (5), and is, by a similar argument as in the previous cases, eventually bounded.

To summarize the above analysis, we conclude that the estimation error under sliding modes is eventually bounded.

6. CONTINUOUS TIME EXAMPLE

Example 6.1
Consider the bimodal system (1) with

$$A_1 = \begin{bmatrix} -1 & -0.2 \\ 0.2 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0.2 \\ -0.2 & 0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

We see that the switching is driven by the first state variable x_1, while x_2 is measured. Hence, the discrete mode cannot be reconstructed directly from the measurements (cf. Remark 3.2).

We will design an observer of the form (4). LMI were solved using the LMItool [25]. For the value of $\varepsilon = 0.1$, the following feasible solution of (15) was obtained:

$$L_1 = \begin{bmatrix} 4.7770 \\ 11.0158 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 4.9910 \\ 12.1719 \end{bmatrix}, \quad K = 0.1083$$

with $\gamma_2 = 1.2882$, and $\eta x_{\max} \approx 0.1022$.

An input signal that takes values randomly in $\{-1, 0, +1\}$, with a sampling period of 2 s was applied to the system. The initial conditions for the system were chosen as $x(0) = [1 \quad -1]^\top$ and for the observer as $\hat{x}(0) = [-1 \quad 1]^\top$. Note that the system and the observer start in different modes. The simulation results are shown in Figure 1. In Figure 2 left we see that sliding patches
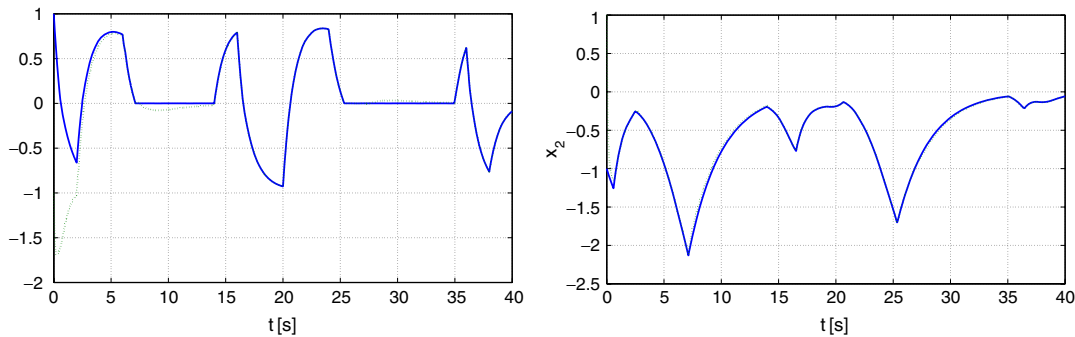
Figure 1. System (solid) and observer (dotted) response for the states x_1 (left figure) and x_2 (right figure).
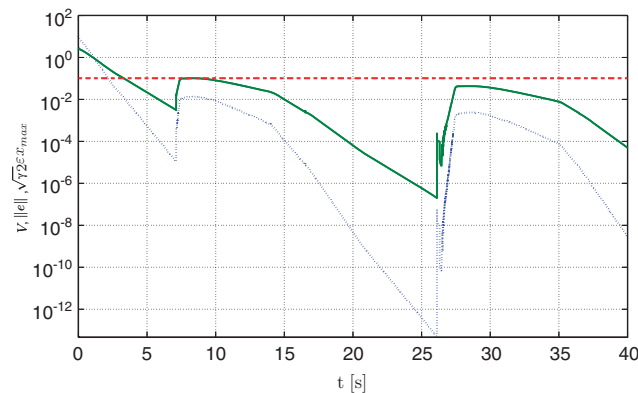

Figure 2. Norm of the error $||e||$ (solid); Lyapunov function of the error $e^\top P e$ (dotted) on log scale, the boundary $\sqrt{\gamma_2} \varepsilon x_{\max}$.

exist in the intervals from [6, 14] and [25, 35]. From Figure 2 we see that the observer error remains within the determined bounds, as predicted by the analysis.

7. MAIN RESULTS FOR THE DISCRETE TIME CASE

As an observer for the discrete time PWL system (2), we propose a bi-modal system with the following structure:

$$\hat{x}(k+1) = \begin{cases} A_1 \hat{x}(k) + Bu(k) + L_1(y(k) - \hat{y}(k)) & \text{if } H^\top \hat{x}(k) < 0 \\ A_2 \hat{x}(k) + Bu(k) + L_2(y(k) - \hat{y}(k)) & \text{if } H^\top \hat{x}(k) > 0 \end{cases} \tag{27a}$$

$$\hat{y}(k) = C\hat{x}(k) \tag{27b}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the estimated state at time k and L_1 and $L_2 \in \mathbb{R}^{n \times p}$ are matrices.

Remark 7.1
We may consider the switching surface of the form: $H^\top \hat{x} + K(\hat{y} - y) = 0$ for the observer (27). In order to simplify the exposition we chose to treat the case with $K = 0$. Derivation of results for $K \neq 0$ is straightforward.

The dynamics of the state estimation error $e = x - \hat{x}$ is then described by

$$e(k+1) = \begin{cases} (A_1 - L_1 C)e(k), & H^\top x(k) < 0, \quad H^\top \hat{x}(k) < 0 \\ (A_2 - L_2 C)e(k) + \Delta A x(k), & H^\top x(k) < 0, \quad H^\top \hat{x}(k) > 0 \\ (A_1 - L_1 C)e(k) - \Delta A x(k), & H^\top x(k) > 0, \quad H^\top \hat{x}(k) < 0 \\ (A_2 - L_2 C)e(k), & H^\top x(k) > 0, \quad H^\top \hat{x}(k) > 0 \end{cases} \tag{28}$$

where $x(k)$ satisfies (27a) and $\hat{x}(k)$ satisfies (27a). By substituting $\hat{x} = x - e$ in (28), we see that the right-hand side of the state estimation error dynamics is PWL in the variable $\mathrm{col}(e, x)$. The problems of observer design can be formally stated as follows.

Problem 7.2
Determine the observer gains L_1, L_2 in (27) such that global asymptotic stability of the estimation error dynamics (28) is achieved, for all sequences $x(1), x(2), \ldots$, satisfying (2) for some given input sequence $u(1), u(2), \ldots$.

Problem 7.3
Determine $\eta > 0$, and L_1, L_2 in (27) such that for all bounded sequences $x(0), x(1), \ldots$ satisfying (2) it holds that

$$\limsup_{k \to \infty} \|e(k)\| \leq \eta \limsup_{k \to \infty} \|x(k)\| \tag{29}$$

which means that if the sequence x is (eventually) bounded by $x_{\max}$, then e should be eventually bounded by $\eta x_{\max}$.

7.1. Continuous dynamics

In order to obtain stable error dynamics we search for a Lyapunov function of the form $V(x) = x^\top P x$, where $P = P^\top > 0$, such that

$$V(e(k+1)) - V(e(k)) \leq -\mu e(k)^\top e(k) \tag{30}$$

for $e(k) \neq 0$ and some $\mu > 0$.

Following the similar steps as in the continuous time case, after applying Schur complement (for details see [7]) we obtain the following result.

Theorem 7.4
The state estimation error dynamics (28) is globally asymptotically stable if there exist matrices $P > 0, L_1, L_2$ constant $\lambda \geq 0$ and $\mu > 0$ such that the following set of matrix

inequalities is satisfied:

$$\begin{bmatrix} P & 0 & (A_i - L_i C)^\top P & 0 \\ 0 & P & 0 & (*) \\ (*) & 0 & P - \mu I & (*) \\ 0 & \Delta A^\top P & \begin{array}{c} -\frac{1}{2}\lambda H H^\top \\ -(-1)^i \Delta A^\top P(A_i - L_i C) \end{array} & \lambda H H^\top \end{bmatrix} \geqslant 0 \qquad (31)$$

for $i = 1, 2$.

The previous result is applicable only to systems with continuous maps. Indeed, the term in the lower right corner is positive semidefinite by construction, and of rank at most 1. The following inclusion must hold (see discussion after Theorem 4.3 for details) $\ker H^\top \subseteq \ker \Delta A$ which implies: $A_2 = A_1 + GH^\top$ for some G of suitable dimensions.

7.2. Discontinuous dynamics

In order to obtain results applicable for discontinuous maps we search for another way to relax the requirement (30). Condition (30) will be required when the appropriate dynamics is active and

$$||e||^2 \geqslant \varepsilon^2 ||x||^2 \qquad (32)$$

Following the same steps as in the continuous time case we get the following theorem.

Theorem 7.5
Consider system (2), observer (27) and the estimation error dynamics (28). The state estimation error e is eventually bounded by $e_{\max}$, under the assumption that x is bounded by $x_{\max}$ if there exist matrices $P > 0$, L_1, L_2, and constant $\lambda \geqslant 0$ and $\mu, \alpha > 0$ such that the following set of matrix inequalities is satisfied:

$$\begin{bmatrix} P & 0 & (A_i - L_i C)^\top P & 0 \\ 0 & P & 0 & (*) \\ (*) & 0 & P - (\mu + \alpha)I & (*) \\ 0 & \Delta A^\top P & \begin{array}{c} -\frac{1}{2}\lambda H H^\top \\ -(-1)^i \Delta A^\top \ P(A_i - L_i C) \end{array} & \lambda H H^\top + \alpha \varepsilon I \end{bmatrix} \geqslant 0 \qquad (33)$$

for $i = 1, 2$. Moreover, if

$$\gamma_1 I \leqslant P \leqslant \gamma_2 I \qquad (34)$$

then

$$e_{\max} \leqslant \sqrt{\frac{\gamma_2}{\gamma_1}} \varepsilon x_{\max} \qquad (35)$$

The proof of the previous theorem is similar to the proof of continuous time case (Theorem 4.3).

Equation (35) explicitly gives an eventual upper bound of the estimation error. The observer gains L_1 and L_2 can be determined so as to minimize this upper bound, which amounts to minimizing γ_2/γ_1 and ε, under (33). If it is possible to design Luenberger observers for both constituting linear dynamics with a common Lyapunov function of the form (7.1), Equations (33) can always be made feasible for large enough ε (cf. Remark 4.6).

8. DISCRETE TIME EXAMPLE

Example 8.1
Consider system (2) with the following parameter values:

$$A_1 = \begin{bmatrix} 0.95 & 0.0475 \\ -0.0475 & 0.95 \end{bmatrix}, \quad A_2 = A_1^\top, \quad H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The system evolution depends only on the initial state $x(0)$. Both pairs (A_1, C), (A_2, C) are observable. Consider two initial state vectors $x^1(0) = [a \quad b]^\top$ and $x^2(0) = [-a \quad b]^\top$, where $a \geqslant 0$. The output sequences y^1 and y^2 generated from $x^1(0)$ and $x^2(0)$, respectively, are the same for any $k > 0$, while the state trajectories are not, when $a \neq 0$. In other words, the system is unobservable (in the sense that state cannot be uniquely determined from the measured output whenever the first component of the state differs from 0).

An observer of the form (27) is designed, using the methodology described in Theorem 7.5 The following observer gains were obtained:

$$L_1 = \begin{bmatrix} -0.0495 \\ 0.8387 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0.0455 \\ 0.7495 \end{bmatrix}$$

while the best found error bound that can be guaranteed is $e_{\max} \leqslant 24 x_{\max}$. The simulation is depicted in Figure 3, with initial states $x(0) = [0.2 \quad 4]^\top$ and $\hat{x}(0) = [-0.3 \quad 4]^\top$. We see that the
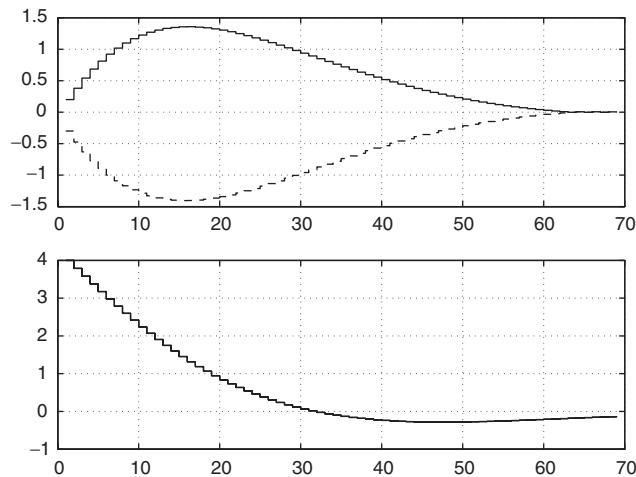

Figure 3. System (solid) and observer (dashed) response (upper: state x_1, lower: state x_2).

state estimate $\hat{x}$ converges towards the other possible state trajectory, starting in $[-0.2 \quad 4]^\top$, yielding the same output. The observer makes the output injection error zero, and hence recovers one corresponding state trajectory (not necessarily the real one).

For more examples of the observer design in the discrete time, we refer the reader to [7].

9. CONCLUSIONS

We have presented observer design procedures for a class of bimodal PWL systems both in continuous and discrete time. The proposed observers are of Luenberger type, but, unlike the classical Luenberger observer for the linear case, the estimation error dynamics is not autonomous. Sufficient conditions for global asymptotic stability of the estimation error dynamics were derived. It turned out that these conditions are only feasible in the case when the system dynamics is continuous over the switching plane. Moreover, in this situation we recovered the observer design presented in [9], as a special case of our observer design.

For the case when the system dynamics is discontinuous over the switching plane we derived conditions that guarantee that the estimation error is asymptotically bounded relative to the system state. The achievable relative upper bound of the estimation error can be optimized. It remains as an open problem whether it is possible to get global asymptotic stability of the estimation error with a Luenberger observer structure in the case of discontinuous bimodal PWL system. It was further shown that the desired properties are retained under the presence of sliding modes in the continuous time case. The theoretical results are illustrated by several examples. A practical application of the results presented in this paper was reported in [26]. These experimental results indicated that the robustness of the designed observers with respect to the model uncertainty is an important issue. This remains to be investigated in the near future.

The main line of reasoning can be applied to more general classes of piecewise affine systems, with the requirement that the input distribution matrix B is the same for all constituting affine dynamics. The major difficulty, however, is to understand the conditions implied by such relaxations (such as the continuity requirement implied by relaxation used in the proof of Theorem 4.3). Another possible extension is to use multiple Lyapunov functions in the observer design and state resets. Some results in this direction are presented in [27].

It can be shown that our observer design conditions imply that the observer is also input-to-state stable (ISS) [28] with the estimation error as a state and the original system state as the disturbance input. Future work will also focus on utilizing the obtained observers for output based controller design for the considered class of bimodal PWL systems, for instance, by using the ISS interconnection conditions from [29, 30].

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