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Observer Design for a Class of Piece-wise Affine Systems [1]

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Abstract

In this paper we propose an observer design procedure for a class of bi-modal piece-wise affine systems. The designed observers have the characteristic feature that they do not require information on the currently active dynamics of the piecewise linear system. A design procedure which guarantees global asymptotic stability of the estimation error is presented. It is shown that the applicability of the presented procedure is limited to continuous piece-wise affine systems. Therefore, we present an observer design procedure, applicable also to discontinuous systems, which guarantees that the estimation error is bounded, with respect to the state bounds, asymptotically. Sliding motions in the observed system, and the observer are discussed. The presented theory is illustrated with an example.

1 Introduction

In this paper we consider the problem to design an observer for a dynamical system in a class of bi-modal piece-wise affine systems. Systems in this class have the characteristic feature that they switch between two different (linear) state evolution maps, depending on whether the state belongs to either one of the half-spaces defined by a separating hyperplane in the state space.

For systems of this type we consider observers of Luenberger type which are also bi-modal in the sense that the state evolution map of the observer may also switch depending on the half-space in which the observer state resides. It is a distinguishing feature of this observer structure that the observer does not need information about the currently active linear dynamics (mode) of the system, in contrast to structures proposed in [1, 2, 3].

We will be interested in the state estimation error dynamics defined by interconnecting the bi-modal system with the bi-modal observer. Contrary to the classical Luenberger observer for linear systems and to the case when the mode is known, in this case the error dynamics is not autonomous, but depends on the state of the observed system, and hence, indirectly, on the control input. Global asymptotic stability of the estimation error may still be achieved, in particular when the bi-modal system is continuous over the switching plane. In the case of a discontinuous system, our approach guarantees that the norm of the error will asymptotically not exceed a certain bound, relative to the bound on the state of the observed system. Observers for switched systems presented in [1, 2, 3] achieve global asymptotic stability of the estimation error by exploiting information on the exactly known discrete mode.

The observers which we consider here are designed for situations where the state of the system (continuous or discrete mode) is of independent interest (e.g. for diagnostics, or discrete mode change detection). Output feedback controller design (which implicitly consists of an observer part and a state feedback part) was presented in [6, 7]. It is not straightforward to extract observer design in the proposed methodology.

Observer design for Lur'e type systems (see, for instance [8]), when the signal that enters the nonlinearity in the feedback path is not measured, is presented in [10]. An approach, related to [10], but far less general, has been presented in [9]. A link to this results will be established in the paper.

This paper is organized as follows. In section 2 we introduce the class of bi-modal piece-wise linear systems together with the observer structure. The design problems are defined, as well. In section 3 we derive sufficient conditions for global asymptotic stability of the observation error. We show that the proposed technique is applicable only when the system is continuous over the switching plane. Motivated by this, we propose an extension, to the more general case, and discuss properties of the obtained solutions. In section 4 sliding modes and their consequences are discussed. The presented theory is illustrated on an example, in section 5. Conclusions and future work are discussed in section 6.

2 Problem statement

Consider the following system

$$\dot{x} = \begin{cases} A_1 x + Bu, & \text{if } H^\top x \leq 0 \\ A_2 x + Bu, & \text{if } H^\top x > 0 \end{cases} \tag{1a}$$

$$y = Cx, \tag{1b}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$, $A_1, A_2 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $H \in \mathbb{R}^n$. The hyperplane defined by $\ker H^\top$ therefore separates the two half-spaces in which the state of the system resides. The considered class of bi-modal piece-wise affine systems has identical input distribution matrix B for both modes. The output distribution matrix C is taken to

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be the same for both modes as well, but this feature is not essential for the derivation of the results.

As an observer for the system (1), we propose a bimodal system with the following structure:

$$\dot{\hat{x}} = \begin{cases} A_1\hat{x} + Bu + L_1(y-\hat{y}), & \text{if } H^\top \hat{x} \le 0 \\ A_2\hat{x} + Bu + L_2(y-\hat{y}), & \text{if } H^\top \hat{x} > 0 \end{cases} \tag{2a}$$

$$\hat{y} = C\hat{x}, \tag{2b}$$

where $\hat{x} \in \mathbb{R}^n$ and L_1 and $L_2 \in \mathbb{R}^{n\times p}$.

The dynamics of the state estimation error, $e := x - \hat{x}$, is then described by

$$\dot{e} = \begin{cases} (A_1 - L_1C)e, & H^\top x \le 0, \quad H^\top \hat{x} \le 0 \\ (A_2 - L_2C)e + \Delta A x, & H^\top x \le 0, \quad H^\top \hat{x} > 0 \\ (A_1 - L_1C)e - \Delta A x, & H^\top x > 0, \quad H^\top \hat{x} \le 0 \\ (A_2 - L_2C)e, & H^\top x > 0, \quad H^\top \hat{x} > 0, \end{cases} \tag{3}$$

where x satisfies (1a), $\hat{x}$ satisfies (2a), and $\Delta A := A_1 - A_2$. By substituting $\hat{x} = x - e$ in (3), we see that the right-hand side of the state estimation error dynamics is piece-wise linear in the variable $v := \mathrm{col}(e, x)$. (Here, col stacks subsequent entries of its argument in a column matrix).

Note that the error dynamics in the two modes of (3) is described by an n-dimensional autonomous state equation, while in the two other modes the external signal $x(t)$ is present, which, by (1a), depends on the input u. For given (open loop) input signals $u : \mathbb{R}^+ \longrightarrow \mathbb{R}^m$ it is possible to consider the evolution of the error e in (3) as a time varying equation of the form

$$\frac{de}{dt}(t) = f(t, e(t)). \tag{4}$$

Standard concepts and results of Lyapunov stability theory (see for instance [8]) can now be applied to equation (4).

Remark 2.1 Information on the currently active linear dynamics may be available to the observer if $H^\top x$ is one of the components in the measured output, i.e. $H \in \mathrm{im}\, C$. In this case the results from [1, 2, 3] apply.

To present the problem formulations, we need the following definition:

Definition 2.2 A function $x : \mathbb{R}^+ \to \mathbb{R}^n$ is said to be *bounded by* $x_{max} > 0$, if

$$\|x(t)\| \le x_{max}$$

for all $t > 0$, i.e. $\sup_{t\in\mathbb{R}^+} \|x(t)\| \le x_{max}$. A function x is said to be *eventually bounded by* x_{max}, if for all $\delta > 0$ there exists a $T_0 > 0$ such that

$$\|x(t)\| \le x_{max} + \delta,$$

for all $t > T_0$, i.e. $\limsup_{t\longrightarrow\infty} \|x(t)\| \le x_{max}$.

Problem 1. Determine the observer gains L_1, L_2 in (2) such that global asymptotic stability of the estimation error (3) is achieved, for all functions $x : \mathbb{R}^+ \longrightarrow \mathbb{R}^n$, satisfying (1) for some given $u : \mathbb{R}^+ \longrightarrow \mathbb{R}^m$.

Problem 2a. Determine $\eta > 0$, and L_1, L_2 in (2), if they exist, such that for all bounded trajectories $x : \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ it holds that

$$\limsup_{t\to\infty} \|e(t)\| \le \eta \limsup_{t\to\infty} \|x(t)\|,$$

which means that if $x(t)$ is (eventually) bounded by x_{max}, then $e(t)$ should be eventually bounded by ηx_{max}.

Problem 2b. Find a (sub)optimal η for which problem 2 is solvable.

3 Main results

3.1 Continuous case

Consider system (1), observer (2), and the error dynamics (3).

Theorem 3.1 *The state estimation error dynamics (3) is globally asymptotically stable for all $x : \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ (in the sense of Lyapunov), if there exist matrices $P = P^\top > 0$, L_1, L_2 and constants $\lambda_1, \lambda_2 \ge 0$ such that the following set of matrix inequalities is satisfied:*

$$(A_1 - L_1C)^\top P + P(A_1 - L_1C) < 0 \tag{5a}$$

$$\begin{bmatrix} (A_2 - L_2C)^\top P + P(A_2 - L_2C) & P\Delta A + \lambda_1 \frac{1}{2} HH^\top \\ \Delta A^\top P + \lambda_1 \frac{1}{2} HH^\top & -\lambda_1 HH^\top \end{bmatrix} \le 0 \tag{5b}$$

$$\begin{bmatrix} (A_1 - L_1C)^\top P + P(A_1 - L_1C) & -P\Delta A + \lambda_2 \frac{1}{2} HH^\top \\ -\Delta A^\top P + \lambda_2 \frac{1}{2} HH^\top & -\lambda_2 HH^\top \end{bmatrix} \le 0 \tag{5c}$$

$$(A_2 - L_2C)^\top P + P(A_2 - L_2C) < 0. \tag{5d}$$

Remark 3.2 The inequalities (5a)-(5d) are not linear in $\{P, L_1, L_2, \lambda_1, \lambda_2\}$, but are linear in $\{P, L_1^\top P, L_2^\top P, \lambda_1, \lambda_2\}$, and thus can be efficiently solved using the available software packages.

Proof: In order to stabilize the system (3) we search for a Lyapunov function $V(e)$ of the form

$$V(e) = e^\top P e, \tag{6}$$

where $P = P^\top > 0$ is of appropriate dimensions.

Calculation shows that:

$$\dot{V}(e) = e^\top \{(A_1 - L_1C)^\top P + P(A_1 - L_1C)\} e, \tag{7a}$$

for $H^\top x \leq 0, H^\top(x-e) \leq 0$,

$$\dot{V}(e) = \begin{bmatrix} e \\ x \end{bmatrix}^\top \begin{bmatrix} (A_2 - L_2C)^\top P \\ +P(A_2 - L_2C) & P\Delta A \\ \Delta A^\top P & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix}, \tag{7b}$$

for $H^\top x \leq 0, H^\top(x-e) > 0$,

$$\dot{V}(e) = \begin{bmatrix} e \\ x \end{bmatrix}^\top \begin{bmatrix} (A_1 - L_1C)^\top P \\ +P(A_1 - L_1C) & -P\Delta A \\ -\Delta A^\top P & 0 \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix}, \tag{7c}$$

for $H^\top x > 0, H^\top(x-e) \leq 0$,

$$\dot{V}(e) = e^\top\{(A_2 - L_2C)^\top P + P(A_2 - L_2C)\}e, \tag{7d}$$

for $H^\top x > 0, H^\top(x-e) > 0$.

The condition that the matrices on the right hand side of equations (7a)-(7d) should be negative definite, together with the condition $P = P^\top > 0$, yields a system of matrix inequalities in $\{P, L_1, L_2\}$ which guarantee global asymptotic stability of the error dynamics. Note that a feasible solution of this set of equations does not exist unless $\Delta A = 0$.

In order to include the information on the switching structure of the system (1a),(1b) (and to relax inequalities (7b),(7c); see also [5]) we can use the S-procedure [4]. Relaxed in this way, the quadratic forms (7b),(7c) can become non-definite, but are guaranteed to be negative in the region of interest.

Regions of the (e, x) space where second and the third linear dynamics of the error (3) is active can be covered with the quadratic constraint in the following way:

$$\begin{bmatrix} e \\ x \end{bmatrix}^\top \begin{bmatrix} 0 & -\frac{1}{2}HH^\top \\ -\frac{1}{2}HH^\top & HH^\top \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0 \tag{8}$$

Equation (8) is derived by multiplying the mode constraints: $x^\top HH^\top(x-e) \leq 0$. The quadratic constraint (8) is by construction negative in the region of interest, 0 at the boundaries, and nonnegative elsewhere. Combining (7b),(7c) with (8), using S-procedure, yields the inequalities (5a)-(5d).

Note that the relaxed inequalities (5b),(5c) can be only negative semidefinite by construction (because $-\lambda_i HH^\top$ is negative semidefinite), but that derivatives (7b),(7c) are guaranteed to be negative whenever the appropriate dynamics is active and $e \neq 0$ (cf. discussion after the theorem). Hence, the computed derivative of the candidate Lyapunov function (6) is negative definite in e, and the global asymptotic stability of the error dynamics (3) is guaranteed. ∎

Suppose that the feasible solution to (5a)-(5d) exists. Since $M \leq 0$ and $z^\top Mz = 0$ imply that $z \in \ker(M)$ (where M is a matrix) it follows that $\mathrm{col}(0, h) \in \ker(5b)$ ($\ker(5c)$, respectively), whenever $h \in \ker(HH^\top) = \ker(H^\top)$ (in fact $\ker(5b) = \ker(5c) = \{\mathrm{col}(0, h) | h \in \ker H^\top\}$ by dimension argument). Hence, for any $h \in \ker(H^\top)$ we have that

$$h \in \ker(P\Delta A).$$

Since $P > 0$, we conclude that $h \in \ker \Delta A$, or

$$\ker H^\top \subseteq \ker \Delta A.$$

From the last inclusion it follows that the state evolution matrices of the two modes are not independent, but are related via:

$$A_2 = A_1 + GH^\top$$

for some vector G of appropriate dimensions. This relation implies the continuity of the vector fields over the switching plane. Note that an equivalent representation of the continuous bi-modal system (1) is:

$$\begin{aligned} \dot{x} &= A_1x + G\max(0, z) + Bu & \text{(9a)} \\ z &= H^\top x & \text{(9b)} \\ y &= Cx, & \text{(9c)} \end{aligned}$$

which is a Lur'e type system, with $max(0, \cdot) \in [0, 1]$ nonlinearity in the feedback path.

Observer design for systems with slope restricted nonlinearities was presented in [10]. Here we show the equivalence between (5a)-(5d), and the approach presented in [10], under certain simplifications. We will simplify our observer structure by assuming the same gain $L_1 = L_2 = L$ for both modes, and the same multiplicative constant $\lambda_1 = \lambda_2 = \lambda$. Equation (5b) can be transformed into equation (5c), by pre-multiplying it with $Q^\top$ and post- multiplying it with Q, where

$$Q = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}.$$

Equation (5c) can be represented as:

$$T^\top \begin{bmatrix} (A_1 - LC)^\top P & PG \\ +P(A_1 - LC) & +\lambda H \\ G^\top P & \\ +\lambda H^\top & -2\lambda \end{bmatrix} T \leq 0 \tag{10}$$

where

$$T = \begin{bmatrix} I & 0 \\ 0 & H^\top \end{bmatrix}.$$

Pre- and post- multiplication with a matrix T introduces a kernel in the matrix inequality (10), but does not change feasibility conditions. The matrix in the middle is therefore equivalent to the LMI condition obtained in [10] (up to the scaling constant λ).

Theorem 3.1 gives sufficient conditions for the solution of problem 1. A drawback of the obtained result is that the necessary condition for the feasibility of (5a)-(5d) is the continuity of the bi-modal piece-wise affine system.

3.2 Discontinuous case

In order to tackle discontinuous systems we need to consider the relaxed problems 2a and 2b. The following theorem directly states an answer to the problem 2a.

Theorem 3.3 *The state estimation error dynamics (3) is eventually bounded by e_{max} (in the sense of definition 2.2), under assumption that x is eventually bounded by x_{max}, if there exist matrices $P = P^\top > 0$, L_1, L_2 and constants $\lambda_1, \lambda_2, \varepsilon \geq 0, \mu_1, \mu_2 > 0$ such that the following set of matrix inequalities is satisfied:*

$$\begin{bmatrix} (A_2 - L_2 C)^\top P + P(A_2 - L_2 C) + \mu_1 I & P\Delta A + \lambda_1 \frac{1}{2} H H^\top \\ \Delta A^\top P + \lambda_1 \frac{1}{2} H H^\top & -\lambda_1 H H^\top - \mu_1 \varepsilon^2 I \end{bmatrix} < 0 \tag{11a}$$

$$\begin{bmatrix} (A_1 - L_1 C)^\top P + P(A_1 - L_1 C) + \mu_2 I & -P\Delta A + \lambda_2 \frac{1}{2} H H^\top \\ -\Delta A^\top P + \lambda_2 \frac{1}{2} H H^\top & -\lambda_2 H H^\top - \mu_2 \varepsilon^2 I \end{bmatrix} < 0 \tag{11b}$$

Moreover, if

$$\gamma_1 I \leq P \leq \gamma_2 I \tag{12}$$

then

$$e_{max} \leq \sqrt{\frac{\gamma_2}{\gamma_1}} \varepsilon x_{max}. \tag{13}$$

Proof: We will further relax equations (5). Consider the quadratic constraint:

$$\|e\|^2 \geq \varepsilon^2 \|x\|^2, \tag{14}$$

for some $\varepsilon > 0$, and suppose that $\dot{V}(e) < 0$ when (14) holds.

For an arbitrary $\delta > 0$, denote

$$V_{max}^\delta = \sup_{\|e\| \leq \varepsilon x_{max} + \delta} V(e).$$

Define the bounded set S_δ by:

$$S_\delta = \{e | V(e) < V_{max}^\delta\}.$$

Since $\dot{V}(e) < 0$ for $e \notin S_\delta$, it follows that S_δ positively invariant i.e. if for $T_0 > 0$

$$V(e(T_0)) < V_{max}^\delta \implies V(e(t)) < V_{max}^\delta \quad \forall_{t > T_0}$$

and satisfies the strong variant of attractivity, in the sense that

$$\exists_{T_0 > 0} \quad V(T_0) < V_{max}^\delta.$$

From (12) it follows that:

$$V_{max}^\delta \leq \gamma_2 [\varepsilon x_{max} + \delta]^2$$

and consequently

$$\forall_{\delta > 0} \exists_{T_0 > 0} \forall_{t > T_0} \quad \|e(t)\| \leq \sqrt{\frac{\gamma_1}{\gamma_2}} [\varepsilon x_{max} + \delta].$$

i.e.

$$e_{max} := \limsup_{t \to \infty} \|e(t)\| \leq \sqrt{\frac{\gamma_2}{\gamma_1}} \varepsilon x_{max}.$$

Combining the quadratic constraint (14) with the equations (5b),(5c), using the S-procedure, we obtain the equations (11a),(11b). A feasible solution of the equations ensures the property $\dot{V}(e) < 0$ when (14) holds. Note that equations (5a),(5d) are implied by equations (11a),(11b), and therefore redundant. ∎

Remark 3.4 The conditions of the theorem guarantee that the norm of the error will not exceed a certain bound, relative to the state bound. Note that $e = 0$ is the equilibrium of (3), i.e. if the state is estimated with $e(T_0) = 0$ for some $T_0 > 0$, then $e(t) = 0$ for all $t > T_0$.

Remark 3.5 If there exists a feasible solution for the system of equations formed by requiring that the members at the upper left corners or (11a),(11b) are negative definite an ε can always be found so that (11a),(11b) are feasible. In other words, sufficient condition for feasibility of (11a),(11b) is that the Luenberger observers for each of the modes are stable, with a common P.

Remark 3.6 The equation

$$P \geq I \tag{15}$$

can be added to the (11a),(11b) without changing feasibility. Namely, if $\{P, L_1, L_2, \lambda_1, \lambda_2, \mu_1, \mu_2, \varepsilon\}$ is the feasible solution of (11a),(11b), so is the scaled set $\{\frac{1}{\gamma_1} P, L_1, L_2, \frac{1}{\gamma_1}\lambda_1, \frac{1}{\gamma_1}\lambda_2, \frac{1}{\gamma_1}\mu_1, \frac{1}{\gamma_1}\mu_2, \varepsilon\}$, and $P^* = \frac{1}{\gamma_1} P \geq I$. The second part of the double inequality (12) follows from:

$$P - \gamma_2 I < 0. \tag{16}$$

Remark 3.7 Condition (14) can be stated in a more general form, when $\|\cdot\|$ is replaced by $\|\cdot\|_Q$. An interesting case is when $\|e\|$ is replaced by $\|e\|_P$. Then, given certain $\eta_{spec} > 0$, existence of an observer that achieves bound $e_{max} \leq \eta_{spec} x_{max}$ follows from the feasibility of bi-linear equations similar in form to (11a), (11b), (15) with $\varepsilon = \eta_{spec}$. The drawback is that bilinear equations can not be solved in an efficient way.

Remark 3.8 Equations (11a),(11b) are bi-linear in variables. When ε is fixed, by using the same change of variables as in remark 3.2 we get the set of linear matrix inequalities.

Any feasible solution of the equations (11a),(11b), (15) is solution for the problem 2a, with $\eta = \sqrt{\gamma_2}\varepsilon$. Algorithm that addresses problem 2b follows from theorem 3.3, and remarks 3.5,3.6,3.8. Under the conditions of remark 3.5 an ε can be found when (11a),(11b) cease to be feasible. For a somewhat bigger value of ε an optimization problem:

$$\min \gamma_2$$

under (11a),(11b),(15),(16) is solved. Then $e_{max} < \sqrt{\gamma_2}\varepsilon x_{max}$. Another problem that occurs is that the above minimization problem frequently gives observer gains L_1, L_2 of an unacceptably high magnitude. The "size" of the gains can be indirectly included in the optimization problem, by including quantity:

$$L_1^\top P L_1 + L_2^\top P L_2,$$

in the optimization criterion, with appropriate weighting factors. Previous equation can be transformed into an LMI using Schur complements.

4 Analysis of sliding modes

All derivations so far were done with the implicit assumption that sliding modes do not occur neither in the original system, nor in the designed observer. In the discussion that follow we consider sliding modes along the switching plane ($H^\top x = 0$ for the system, $H^\top \hat{x} = 0$ for the observer) under the assumption that we have constructed an observer that satisfies equations (11), and we are going to show that the estimation error remains eventually bounded. The mode of the system (resp. observer) where $H^\top x < 0$ (resp. $H^\top x > 0$) is referred to as the first mode (resp. second mode).

First, consider the case where sliding mode occurs in the designed observer,along the plane $H^\top \hat{x} = 0$. Then, the dynamics of the observer is given by the convex combination of the constituting linear dynamics (see, for instance [11]):

$$\begin{aligned} \dot{\hat{x}} &= \lambda\{A_1\hat{x} + Bu + L_1(y - \hat{y})\} + \\ &\quad (1-\lambda)\{A_2\hat{x} + Bu + L_2(y - \hat{y})\}, \\ \hat{y} &= C\hat{x} \end{aligned} \tag{17}$$

where $\lambda \in [0, 1]$. Consider next the situation where the system is in the first mode. Then the error dynamics is given by:

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} = \lambda\{(A_1 - L_1C)e\} + \\ &\quad (1-\lambda)\{(A_2 - L_2C)e + \Delta A x\} \end{aligned} \tag{18}$$

which is the convex combination of first and second mode of the error dynamics (3). Since $\dot{V}(e)$ is negative when (14) holds, for both modes, it is also negative for their convex combinations under (14). Hence, the error is eventually bounded, as proven in theorem 3.3. A similar argument holds in the second case (i.e. when the system is in the second mode, and observer is in the sliding motion).

Consider now the case when a sliding mode exists on the switching plane of the system. Then, the system dynamics is given by the convex combination of the constituting linear dynamics:

$$\begin{aligned} \dot{x} &= \mu\{A_1x + Bu\} + (1-\mu)\{A_2x + Bu\} \\ y &= Cx \end{aligned} \tag{19}$$

where $\mu \in [0, 1]$. If the observer is in the first mode, the error dynamics is given by:

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} = \mu\{(A_1 - L_1C)e\} + \\ &\quad (1-\mu)\{(A_1 - L_1C)e - \Delta A x\} \end{aligned} \tag{20}$$

which is a convex combination of the first and the third mode of error dynamics. Hence, $\dot{V}(e)$ is negative when (14) holds. A similar argument holds for the case when the system is in sliding mode, and the observer is in the second mode.

Consider now the situation where there are sliding modes in both system and the observer. Then the dynamics of the system is given by (19), and the dynamics of the observer is given by (17). The error dynamics is given by:

$$\begin{aligned} \dot{e} &= (\mu - \lambda)\{(A_2 - L_2C)e + \Delta A x\} + \\ &\quad (1-\mu)\{(A_2 - L_2C)e\} + \lambda\{(A_1 - L_1C)e\} \end{aligned}$$

if $\mu - \lambda \geq 0$, or

$$\begin{aligned} \dot{e} &= (\lambda - \mu)\{(A_1 - L_1C)e - \Delta A x\} + \\ &\quad (1-\lambda)\{(A_2 - L_2C)e\} + \\ &\quad \mu\{(A_1 - L_1C)e\} \end{aligned} \tag{21}$$

if $\lambda - \mu \geq 0$. We see that the error dynamics is again given as a convex combination of the modes of the error dynamics (3), and is, by a similar argument as in the previous cases, eventually bounded.

To summarize, from the previous analysis we conclude that the estimation error under sliding modes is eventually bounded.

5 Example

In this section, the presented theory will be illustrated by means of the following bimodal system with:

$$A_1 = \begin{bmatrix} -1 & -0.2 \\ 0.2 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0.2 \\ -0.2 & 0.3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

We see that the switching is driven by the first state variable x_1, while x_2 is measured. Hence, the discrete mode can not be reconstructed directly from the measurements.

Linear matrix inequalities were solved using the LMI-tool. For the value of $\varepsilon = 0.1$, the following feasible solution was obtained:

$$L_1 = \begin{bmatrix} 2.09 \\ 4.38 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 2.41 \\ 5.78 \end{bmatrix}.$$

with $\gamma_2 = 1.3998$, and $\eta x_{max} \approx 0.12$

For the purpose of simulation an input that takes values in $\{-1, 0, +1\}$, with a period of 1s was applied to the system. The initial conditions for the system were chosen as $x(0) = \begin{bmatrix} -1 & -1 \end{bmatrix}^T$, and for the observer $\hat{x}(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. The simulation results are shown in figure 1,and figure 2. In the figure 1 we see that the sliding patch exists approximately in the interval $[4.5, 5.5]$. From figure 3 we see that the observer error remains within the determined bounds, as predicted by the analysis.

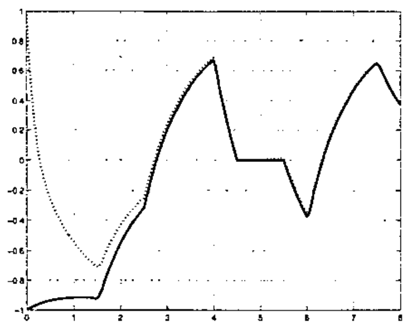

Figure 1: System (solid) and observer (dotted) response for the state x_1

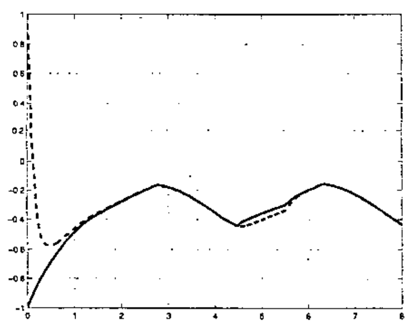

Figure 2: System (solid) and observer (dotted) response for the state x_2

6 Conclusions

We have presented an observer design procedure for a class of bi-modal piece-wise affine systems. The proposed observer is of Luenberger type, but, unlike the classical Luenberger observer, the estimation error dynamics is not autonomous. Sufficient conditions for global asymptotic stability of the estimation error were derived, but are shown to be feasible only for continuous bi-modal piecewise affine systems. Relaxed conditions, applicable for the larger class of systems are derived, and the obtained observers guarantee certain estimation error, relative to the state, asymptotically. It is proven that the desired properties are retained under presence of sliding modes.

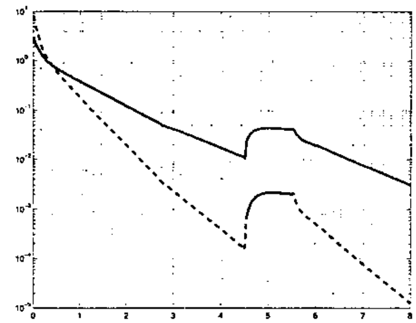

Figure 3: Norm of the error $\|e\|$ (solid); Lyapunov function of the error $e^T Pe$ (dotted) on log scale

It remains an open problem whether it is possible to get global asymptotic stability of the estimation error with the proposed observer structure, in the general case of discontinuous bi-modal piece-wise affine system.

The focus of future work will be on utilizing the obtained observers for feedback stabilization of the class of bi-modal piece-wise affine systems, as well as on broadening the class of piece-wise affine (and general hybrid) systems to which the presented techniques are applicable.

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