

# Output Feedback Control for a Class of Piecewise Linear Systems

A. Lj. Juloski, W. P. M. H. Heemels, S. Weiland

**Abstract**—In this paper we present an output feedback controller design method for a class of bi-modal piecewise linear systems. The proposed output feedback controller consists of a switching state observer and a static state feedback. The observer and the controller are designed separately using the concept of input-to-state stability (ISS). A sufficient small gain condition is derived that guarantees global asymptotic stability of the system in closed loop with the designed observer and state feedback. The derived theory is illustrated by an example.

## I. INTRODUCTION

In this paper we present a design method for output feedback controllers for a class of bi-modal piecewise linear systems. The systems of the considered class comprise two linear dynamics with the same input distribution matrices. We consider the case where the currently active mode of the system is not known, and can not be inferred directly from the measured outputs. We will design an output-feedback controller that consists of a switching state observer and a static state feedback, that uses the estimated state as input.

Observers of Luenberger-type for the considered class of bi-modal systems were first proposed in [1], [2]. As the currently active mode of the underlying system is not known, the dynamics of the observation error has four modes. Two modes of the error dynamics correspond to the situation when the system and the observer are in the same mode, while the other two modes correspond to the situation when the modes of the system and the observer are different. Moreover, the observation error dynamics is not autonomous, but depends explicitly on the state of the observed system. This means that the error dynamics can not be rendered globally asymptotically stable (GAS) in general. This is a fundamental difference with respect to the case when the currently active mode is known. In that case the error dynamics is autonomous and has the same number of modes as the original system. GAS observers can be designed as shown in [3]–[5].

The approach taken in [1], [2] was to design the observer so that the error asymptotically converges to a bounded set around the origin, where the bound of this set depends on the size of the state of the observed system. In the special case when the system dynamics is continuous over the switching plane, GAS observers may be obtained. In this particular case, the approach from [2] is equivalent to the

approach in [6]. The observer design proposed in [1], [2] was further extended in [7] by including observer state jumps, to improve the performance of the observer and make the design conditions less conservative.

In this paper we use the same observer structure as in [1], [2]. In contrast to [1], [2] in this paper we employ the formalism of input-to-state stability (ISS) [8]–[10] to study the observer error dynamics. The concept of ISS was originally developed for systems with Lipschitz continuous vector fields, and has recently been studied in the context of switched and hybrid systems [11]–[14]. In [11] one considers ISS for switching systems with an average dwell time property, while in [12] a generalized interconnection theory based on smooth Lyapunov functions is developed. In [13] ISS properties are considered for a system with smooth flow dynamics and state resets. In [14] the authors study ISS for discontinuous dynamical systems and adopt non-smooth Lyapunov functions. In this paper we employ the theoretical framework developed in [14], and focus on an output-based feedback design.

The observer is designed so that the error dynamics is input-to-state stable (actually, we only use the existence of an ISS-Lyapunov function for the observer), where the state of the observed system plays the role of the external (disturbance) input. The state estimate is fed back to the plant. The feedback gain is designed separately and independently from the observer. We design the gain such that the original system is input-to-state stable, where the observation error now plays the role of the external input. The system in closed loop with the observer and the feedback gain can now be seen as the interconnection of two ISS systems. As the closed loop system is autonomous we derive sufficient small gain conditions for the global asymptotic stability of the interconnected system.

The conditions for the stability of general interconnected ISS systems were first presented in [9], [10]. The approach of designing the observer and the controller separately (also referred to as “certainty equivalence design” or “separation principle”), and applying the ISS-interconnection approach to prove the stability of the closed loop system was used before, for example in [6], [15]. These results apply to continuous systems, and the designed observers are GAS (or even stronger the observation error converges to zero exponentially). As the class of systems that we consider is in general discontinuous we can not guarantee GAS observers.

As we adopt quadratic ISS-Lyapunov functions, the design of the observer and the feedback gain is considerably simplified. The observer can be designed by solving a set of linear matrix inequalities (LMI). The design of the controller amounts to solving a set of linear matrix inequalities, after

This work is partially supported by European project grants SICONOS (IST2001-37172) and HYCON Network of Excellence, contract number FP6-IST-511368

Aleksandar Juloski and Siep Weiland are with the Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, E-mail: a.juloski@tue.nl.

Maurice Heemels is with the Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

fixing a single scalar parameter. An approach where the observer and the feedback gains are designed simultaneously was presented in [16], [17]. However, in contrast to the method presented here, the approach of [16], [17] requires solving bi-linear matrix inequalities (BMI), via iterative and suboptimal numerical techniques.

The paper is structured as follows. We introduce the considered class of systems in section II. The design of the observer and the controller and the sufficient conditions for the global asymptotic stability of the interconnection are presented in section III. An illustrative example is given in section IV. Finally, we state the conclusions in section V.

## II. PROBLEM STATEMENT

We consider the following system:

$$\dot{x}(t) = \begin{cases} A_1x(t) + Bu(t), & \text{if } H^\top x(t) \leq 0 \\ A_2x(t) + Bu(t), & \text{if } H^\top x(t) \geq 0 \end{cases} \quad (1a)$$

$$y(t) = Cx(t), \quad (1b)$$

where  $x(t) \in \mathbb{R}^n$ ,  $y(t) \in \mathbb{R}^p$  and  $u(t) \in \mathbb{R}^m$  are the state, output and the input of the system, respectively at time  $t \in \mathbb{R}^+$ . The matrices are of the following dimensions:  $A_1, A_2 \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$  and  $H \neq 0 \in \mathbb{R}^n$ . The hyperplane defined by  $\ker H^\top$  separates the state space  $\mathbb{R}^n$  into the two half-spaces. The considered class of bimodal piecewise linear systems has identical input distribution matrix  $B$  and output distribution matrix  $C$  for both modes. The input  $u : \mathbb{R}^+ \rightarrow \mathbb{R}^m$  is assumed to be a Lebesgue integrable function. We will consider the solutions of the system (1) in the sense of Filippov [18].

Depending on the values of  $A_1$  and  $A_2$  we distinguish two situations:

- 1) the vector field of the system is continuous over the switching plane, i.e.  $A_1x = A_2x$ , when  $H^\top x = 0$ . It is straightforward to show that in this case:

$$A_2 = A_1 + GH^\top \quad (2)$$

for some vector  $G$  of appropriate dimensions. In this case equation (1a) can also be rewritten as:

$$\dot{x} = A_1x + G \max(0, H^\top x) + Bu.$$

Moreover, from (2) it follows that  $\text{rank}(\Delta A) = 1$ , where  $\Delta A$  is defined as

$$\Delta A := A_1 - A_2.$$

- 2) the vector field of the system is not continuous over the switching plane, i.e. a parametrization as in (2) does not exist.

The problem at hand is to design a stabilizing output feedback controller for the system (1).

## III. MAIN RESULTS

We will design a controller consisting of an observer and a state feedback, as depicted in figure 1. We aim to design an observer that is ISS from state to observer error and to render the system ISS from the observer error to state by proper choice of the state feedback gain  $K$ . The test for the stability of the closed loop system will be provided.

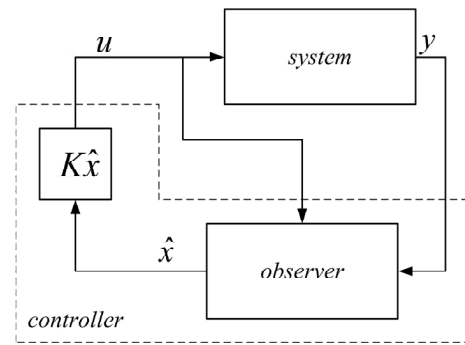


Fig. 1. Structure of the output feedback controller

### A. Preliminaries

With  $|\cdot|$  we will denote the usual Euclidean norm for vectors in  $\mathbb{R}^n$ , and with  $\|\cdot\|$  we denote the  $L_\infty$  norm for time functions. For a real valued, differentiable function  $V$ ,  $\nabla V$  denotes its gradient. For a positive semi-definite matrix  $A \in \mathbb{R}^n$   $\lambda_{\min}(A)$  and  $\lambda_{\max}(A)$  will denote its minimal and maximal eigenvalue. In matrices we denote by  $(*)$  at block position  $(i, j)$  the transposed matrix block at position  $(j, i)$ , e.g.

$$\begin{bmatrix} A & B \\ (*) & C \end{bmatrix} \text{ means } \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}.$$

The operator  $\text{col}(\cdot, \cdot)$  stacks its arguments into a column vector.

A function  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is of class  $\mathcal{K}$  if it is continuous, strictly increasing and  $\gamma(0) = 0$ . It is of class  $\mathcal{K}_\infty$  if, in addition, it is unbounded, i.e.  $\gamma(s) \rightarrow \infty$  as  $s \rightarrow \infty$ . A function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is of class  $\mathcal{KL}$  if, for each fixed  $t$ , the function  $\beta(\cdot, t)$  is of class  $\mathcal{K}$ , and for each fixed  $s$ , the function  $\beta(s, \cdot)$  is decreasing and tends to zero at infinity.

Consider the piecewise smooth system of the form:

$$\dot{x}(t) = f_i(x(t), u(t)) \text{ when } x(t) \in \Omega_i, \quad i = 1, \dots, N, \quad (3)$$

where  $f_i \in \mathbb{R}^n \times \mathbb{R}^m$  is a continuously differentiable function on an open domain containing  $\Omega_i$  for  $i = 1, \dots, N$ , and the sets  $\Omega_1, \dots, \Omega_N$  form a partitioning of  $\mathbb{R}^n \times \mathbb{R}^m$ , i.e.  $\Omega_i$  are closed sets with  $\text{cl}(\text{int } \Omega_i) = \Omega_i$ ,  $\text{int } \Omega_i \cap \text{int } \Omega_j = \emptyset$  for  $i \neq j$  and  $\bigcup_{i=1}^N \Omega_i = \mathbb{R}^n \times \mathbb{R}^m$ . We define the index set

$$I(x, u) = \{i \in \{1, \dots, N\} \mid \text{col}(x, u) \in \Omega_i\}.$$

**Definition III.1** A smooth ( $C^1$ ) function  $V$  is said to be an ISS-Lyapunov function for the system (3) if:

- there exist functions  $\psi_1, \psi_2$  of class  $\mathcal{K}_\infty$  such that:

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^n$$

- there exist functions  $\alpha$  and  $\chi$  of class  $\mathcal{K}$  such that for all  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$  the following implication holds:

$$\{|x| > \chi(|u|)\} \Rightarrow \nabla V_j(x) f_i(x, u) \leq -\alpha(|x|), \quad (4)$$

for all  $i \in I(x, u)$ .

Definition III.1 is similar to the definition from [8]–[10], the only difference being that we do not require  $f$  in (3) to

be locally Lipschitz. In this paper we will not use directly the ISS property of the system, but only the existence of an ISS Lyapunov function. In [14] it was proven that the existence of the ISS-Lyapunov function implies input-to-state stability of the system (3) using Filippov solutions. Moreover, [14] even allows for ISS-Lyapunov functions that are non-smooth and continuous.

### B. Observer design

As an observer for the system (1), we propose a continuous time bimodal system with the following structure:

$$\dot{\hat{x}} = \begin{cases} A_1\hat{x} + Bu + L_1(y - \hat{y}), & \text{if } H^\top \hat{x} + M^\top(y - \hat{y}) \leq 0 \\ A_2\hat{x} + Bu + L_2(y - \hat{y}), & \text{if } H^\top \hat{x} + M^\top(y - \hat{y}) \geq 0 \end{cases} \quad (5a)$$

$$\hat{y} = C\hat{x}, \quad (5b)$$

where  $\hat{x}(t) \in \mathbb{R}^n$  is the estimated state at time  $t$  and  $L_1, L_2 \in \mathbb{R}^{n \times p}$  and  $M \in \mathbb{R}^p$ .

The dynamics of the state estimation error,  $e := x - \hat{x}$ , is then described by

$$\dot{e} = \begin{cases} (A_1 - L_1C)e, & H^\top x \leq 0, \\ & H^\top \hat{x} + M^\top(y - \hat{y}) \leq 0 \\ (A_2 - L_2C)e + \Delta Ax, & H^\top x \leq 0, \\ & H^\top \hat{x} + M^\top(y - \hat{y}) \geq 0 \\ (A_1 - L_1C)e - \Delta Ax, & H^\top x \geq 0, \\ & H^\top \hat{x} + M^\top(y - \hat{y}) \leq 0 \\ (A_2 - L_2C)e, & H^\top x \geq 0, \\ & H^\top \hat{x} + M^\top(y - \hat{y}) \geq 0, \end{cases} \quad (6)$$

where  $x$  satisfies (1a) and  $\hat{x}$  satisfies (5a). By substituting  $\hat{x} = x - e$  in (6), we see that the right-hand side of the state estimation error dynamics is piecewise linear in the variable  $\text{col}(e, x)$ , and in the form (3). Note that the error dynamics in the first and the fourth mode of (6) is described by an  $n$ -dimensional autonomous state equation, while in the two other modes the external signal  $x$  is present.

### Theorem III.2 The function

$$V_e(e) = e^\top P_e e$$

where  $P_e = P_e^\top > 0$  is an ISS-Lyapunov function for the error dynamics (6) if there exist constants  $\lambda, \varepsilon_e, \mu_e > 0$  such that the following set of matrix inequalities is satisfied:

$$\begin{bmatrix} (A_1 - L_1C)^\top P_e & (*) \\ +P_e(A_1 - L_1C) + (\mu_e + 1)I & \\ -\Delta A^\top P_e + & \\ \frac{\lambda}{2}H(H^\top - M^\top C) & -\lambda HH^\top - \varepsilon_e I \end{bmatrix} < 0 \quad (7a)$$

$$\begin{bmatrix} (A_2 - L_2C)^\top P_e & (*) \\ +P_e(A_2 - L_2C) + (\mu_e + 1)I & \\ \Delta A^\top P_e + & \\ \frac{\lambda}{2}H(H^\top - M^\top C) & -\lambda HH^\top - \varepsilon_e I \end{bmatrix} < 0 \quad (7b)$$

Furthermore, Definition III.1 is satisfied for  $\psi_{e,1}(|e|) = \lambda_{\min}(P_e)|e|^2$ ,  $\psi_{e,2}(|e|) = \lambda_{\max}(P_e)|e|^2$ ,  $\chi_e(|x|) = \sqrt{\varepsilon_e}|x|$ , and  $\alpha_e(|e|) = \mu_e|e|^2$ .

*Proof:* Immediately, we have that:

$$\lambda_{\min}(P_e)|e|^2 \leq e^\top P_e e \leq \lambda_{\max}(P_e)|e|^2.$$

Hence,  $\psi_{e,1}(|e|) = \lambda_{\min}(P_e)|e|^2$ ,  $\psi_{e,2}(|e|) = \lambda_{\max}(P_e)|e|^2$ . For the function  $V_e(e) = e^\top P_e e$  to be an ISS-Lyapunov function for the dynamics (6), we have to guarantee that (4) holds for some functions  $\chi_e$  and  $\alpha_e$  of class  $\mathcal{K}$ .

If we select

$$\chi_e(|x|) = \sqrt{\varepsilon_e}|x|, \quad \alpha_e(|e|) = \mu_e|e|^2,$$

then (4) becomes equivalent to the following set of conditions:

$$e^\top \{(A_1 - L_1C)^\top P_e + P_e(A_1 - L_1C) + \mu_e I\} e \leq 0, \quad (8a)$$

if  $H^\top x \leq 0, H^\top x + (M^\top C - H^\top)e \leq 0, |e| \geq \chi_e(|x|)$

$$\begin{bmatrix} e \\ x \end{bmatrix}^\top \begin{bmatrix} (A_2 - L_2C)^\top P_e + & P_e \Delta A \\ P_e(A_2 - L_2C) + \mu_e I & 0 \\ \Delta A^\top P_e & \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0, \quad (8b)$$

if  $H^\top x \leq 0, H^\top x + (M^\top C - H^\top)e > 0, |e| \geq \chi_e(|x|)$

$$\begin{bmatrix} e \\ x \end{bmatrix}^\top \begin{bmatrix} (A_1 - L_1C)^\top P_e + & -P_e \Delta A \\ P_e(A_1 - L_1C) + \mu_e I & 0 \\ -\Delta A^\top P_e & \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0, \quad (8c)$$

if  $H^\top x > 0, H^\top x + (M^\top C - H^\top)e \leq 0, |e| \geq \chi_e(|x|)$  and

$$e^\top \{(A_2 - L_2C)^\top P_e + P_e(A_2 - L_2C) + \mu_e I\} e \leq 0, \quad (8d)$$

if  $H^\top x > 0, H^\top x + (M^\top C - H^\top)e > 0, |e| \geq \chi_e(|x|)$ .

Condition  $|e| \geq \chi_e(|x|)$  yields the following quadratic constraint:

$$|e|^2 \geq \varepsilon_e |x|^2 \quad (9)$$

Constraint (9) can be combined with (8a)-(8d), using  $S$ -procedure [19], [20], to ensure (4).

The constraints can be relaxed further (see [1], [2]), by using the fact that (8a)-(8d) combined with (9) need to hold only in the regions of  $(e, x)$  space where the corresponding dynamics is active. Regions of the  $\text{col}(e, x)$  space where the second and the third linear dynamics of the error (6) is active can be covered with the following quadratic constraint:

$$\begin{bmatrix} e \\ x \end{bmatrix}^\top \begin{bmatrix} 0 & -\frac{1}{2}(H - C^\top M)H^\top \\ (*) & HH^\top \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} \leq 0 \quad (10)$$

Inequality (10) is derived by multiplying the mode constraints:

$$x^\top H(H^\top(x - e) + M^\top C e) \leq 0.$$

Constraints (9) and (10), combined with (8b)-(8c), using  $S$ -procedure, yield inequalities (7a) and (7b). Observe that (8a) and (8d) are implied by (7a) and (7b), which completes the proof. ■

**Remark III.3** Theorem III.2 is an analysis result on error dynamics (6), but it can also be used for the observer synthesis. The inequalities (7b)-(7a) are nonlinear matrix inequalities in  $\{P_e, L_1, L_2, M, \lambda, \mu_e, \varepsilon_e\}$ , but are linear in  $\{P_e, L_1^\top P_e, L_2^\top P_e, \lambda M, \lambda, \mu_e, \varepsilon_e\}$ , and thus can be efficiently solved using available software packages (such as the free software LMItool) [21].

Note that (8a)-(8d) can not be satisfied in the entire  $(e, x)$  space unless  $\Delta A = 0$ . Hence, the relaxations (9) and (10) are essential for the feasibility of the design.

**Remark III.4** Ideally, one would like to design an observer with  $\varepsilon_e = 0$ , as this would mean that the observer error dynamics is globally asymptotically stable, regardless of the influence of external input  $x$ . However, the constraints (7b) and (7a) can be satisfied with  $\varepsilon_e = 0$  only if the dynamics of the system (1) is continuous over the switching plane  $H^\top x = 0$  (see [1], [2]). In the case when the dynamics is discontinuous  $\varepsilon_e > 0$  is needed.

**Remark III.5** Also note that if there exists a feasible solution for the system of inequalities

$$P_e > 0, \quad \mu_e > 0 \quad (11a)$$

$$(A_1 - L_1 C)^\top P_e + P_e (A_1 - L_1 C) + (\mu_e + 1)I < 0 \quad (11b)$$

$$(A_2 - L_2 C)^\top P_e + P_e (A_2 - L_2 C) + (\mu_e + 1)I < 0 \quad (11c)$$

(which implies that the pairs  $(A_1, C)$  and  $(A_2, C)$  are detectable), a (possibly large)  $\varepsilon_e$  can always be found such that (7b) and (7a) are feasible. Conditions (11) are exactly the conditions required for the observer design in the case when the current mode is known [3]–[5]. Moreover, in the case of the continuous vector field, if we choose  $L_1 = L_2$  and  $M = 0$  conditions (11) are also sufficient for the feasibility of (7) [22].

### C. Controller design

As a controller for the system (1) we propose the following controller:

$$u = K\hat{x}, \quad (12)$$

where  $K \in \mathbb{R}^n$ .

By using

$$\hat{x} = x - e$$

the system dynamics with the controller (12) takes the following form:

$$\dot{x} = \begin{cases} (A_1 + BK)x - BKe, & H^\top x \leq 0 \\ (A_2 + BK)x - BKe, & H^\top x \geq 0 \end{cases} \quad (13)$$

We have the following result:

**Theorem III.6** *The function*

$$V_x(x) = x^\top P_x x$$

where  $P_x = P_x^\top > 0$  is an ISS-Lyapunov function for the system (13) if there exist a vector  $K$  and constants  $\hat{\varepsilon}_x, \hat{\mu}_x > 0$  such that the following set of matrix inequalities is satisfied:

$$\begin{bmatrix} (A_1 + BK)^\top P_x + & -P_x BK \\ P_x (A_1 + BK) + (\hat{\mu}_x + 1)P_x & -\hat{\varepsilon}_x P_x \end{bmatrix} < 0 \quad (14a)$$

$$\begin{bmatrix} (A_2 + BK)^\top P_x + & -P_x BK \\ P_x (A_2 + BK) + (\hat{\mu}_x + 1)P_x & -\hat{\varepsilon}_x P_x \end{bmatrix} < 0 \quad (14b)$$

Furthermore,  $\psi_{x,1}(|x|) = \lambda_{\min}(P_x)|x|^2$ ,  $\psi_{x,2}(|x|) = \lambda_{\max}(P_x)|x|^2$ ,  $\chi_x(|e|) = \sqrt{\varepsilon_x}|e|$ , and  $\alpha_x(|x|) = \mu_x|x|^2$ , where

$$\varepsilon_x = \frac{\lambda_{\max}(P_x)}{\lambda_{\min}(P_x)} \hat{\varepsilon}_x, \quad \mu_x = \lambda_{\min}(P_x) \hat{\mu}_x. \quad (15)$$

*Proof:* To prove that  $V_x(x) = x^\top P_x x$  is indeed an ISS-Lyapunov function for (13), we first note that

$$\begin{aligned} \psi_{x,1}(|x|) &= \lambda_{\min}(P)|x|^2 \leq V_x(x) \leq \\ &\leq \lambda_{\max}(P)|x|^2 = \psi_{x,2}(|x|). \end{aligned}$$

To show (4), we observe that

$$|x| > \chi_x(|e|) \Rightarrow x^\top P_x x \geq \hat{\varepsilon}_x e^\top P_x e.$$

Hence, to satisfy (4) it suffices to show that

$$x^\top P_x x > \hat{\varepsilon}_x e^\top P_x e \Rightarrow \dot{V}(x, e) \leq -\hat{\mu}_x x^\top P_x x. \quad (16)$$

As (16) is implied by (14a),(14b), and

$$-\hat{\mu}_x x^\top P_x x \leq -\hat{\mu}_x \lambda_{\min}(P)|x|^2 = \alpha_x(|x|)$$

the proof is complete. ■

**Remark III.7** Matrix inequalities (14a),(14b) are bilinear in the variables. By pre- and post-multiplying the whole matrix inequalities (14a) and (14b) with the matrix  $\text{diag}\{P_x^{-1}, P_x^{-1}\}$ , the following matrix inequalities are obtained:

$$\begin{bmatrix} P_x^{-1} A_i^\top + P_x^{-1} K^\top B^\top + & -BK P_x^{-1} \\ A_i P_x^{-1} + BK^\top P_x^{-1} + \mu_x P_x^{-1} & \varepsilon_x P_x^{-1} \end{bmatrix} < 0 \quad (17)$$

for  $i = 1, 2$ .

By fixing values of  $\varepsilon_x$  and  $\mu_x$  (17) become linear matrix inequalities in  $\{P_x^{-1}, P_x^{-1} K^\top\}$ . As we will see in the following section, the value of  $\mu_x$  does not influence the stability conditions (cf. stability condition (18) below), as long as  $\mu_x > 0$ . One recipe for solving (17) is then: fix  $\mu_x$  at some small positive value and minimize  $\varepsilon_x$ .

D. Interconnection

Theorems III.2 and III.6 give the means to design the observer gains  $L_1, L_2, M$  and the feedback gain  $K$ , so that the system and the observer separately have quadratic ISS-Lyapunov functions. The observer (5) is designed such that the observation error dynamics  $e$  is input-to-state stable with respect to the system state  $x$ , as an external input. The system is designed such that the state  $x$  is input-to-state stable with respect to the observation error  $e$ , as an external input. This situation is depicted in figure 2.

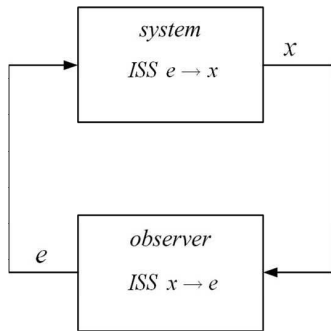


Fig. 2. Interconnection of the system and the observer

The question arises under which conditions the autonomous closed loop system is globally asymptotically stable. The following theorem gives sufficient conditions for global asymptotic stability of the interconnection of the system and the output-feedback controller, as depicted in figures 1 and 2.

**Theorem III.8** Consider the system (1), the observer (5) and the controller (12).

Suppose that the observer is designed according to theorem III.2, i.e. there exists an ISS-Lyapunov function for the observer error dynamics (6) of the form

$$V_e(e) = e^T P_e e$$

such that (7b),(7a) are satisfied.

Suppose that the controller is designed according to theorem III.6, i.e. there exists an ISS Lyapunov function for the closed loop system dynamics (13), of the form

$$V_x(x) = x^T P_x x$$

such that (14a), (14b) are satisfied.

Then the closed loop system is globally asymptotically stable adopting Filippov's solutions if the following condition is satisfied:

$$\frac{\lambda_{max}(P_e)}{\lambda_{min}(P_e)} \frac{\lambda_{max}(P_x)}{\lambda_{min}(P_x)} \varepsilon_e \varepsilon_x < 1 \tag{18}$$

The proof follows from the general interconnection theorem in [14].

IV. EXAMPLE

Consider the bi-modal system (1) with the following values of the parameters:

$$A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.3 & -1 \\ 1 & 0.3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = [1 \ 0], \quad H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The system dynamics is discontinuous over the switching plane. Note that the switching is driven by the value of the second component of the state  $x_2$ , while  $x_1$  is measured. Hence, the mode can not be directly inferred from the measurements. The system is unstable in the open loop. The simulation of the system trajectory, with initial condition  $x(0) = [1 \ -1]^T$  and zero input is depicted in figure 3.

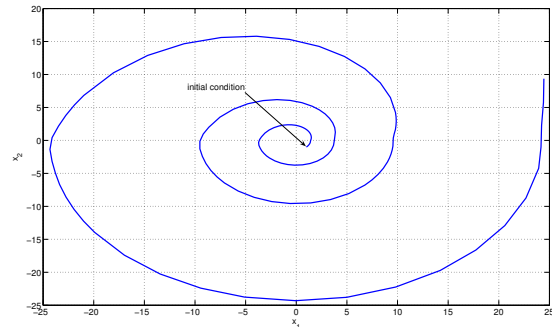


Fig. 3. Trajectory of the system in the open loop with the initial condition  $x(0) = [1 \ -1]^T$ .

The controller was designed with

$$K = [-0.6405 \ -0.7546]$$

$$P_x = 10^{11} \cdot \begin{bmatrix} 4.4156 & -0.6454 \\ -0.6454 & 4.1532 \end{bmatrix}, \quad \frac{\lambda_{max}(P_x)}{\lambda_{min}(P_x)} = 1.3633,$$

and  $\varepsilon_x = 0.2$ . The observer was designed with

$$L_1 = \begin{bmatrix} 1.4880 \\ -1.3837 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 2.2453 \\ -1.7692 \end{bmatrix}, \quad M = 0.17.$$

$$P_e = \begin{bmatrix} 3.4028 & 1.4192 \\ 1.4192 & 1.8382 \end{bmatrix}, \quad \frac{\lambda_{max}(P_e)}{\lambda_{min}(P_e)} = 4.2410,$$

and  $\varepsilon_e = 0.7$

The value of the interconnection criterion is

$$\frac{\lambda_{max}(P_e)}{\lambda_{min}(P_e)} \frac{\lambda_{max}(P_x)}{\lambda_{min}(P_x)} \varepsilon_e \varepsilon_x = 0.8094 < 1.$$

Hence, the closed loop system is globally asymptotically stable.

The trajectory of the system in the closed loop is depicted in figure 4. The initial condition for the system was  $x(0) = [1 \ -1]^T$  and the observer initial condition  $\hat{x}(0) = [-1 \ 1]^T$ . The system and the observer start in different modes. The trajectory of the observer, with the same initial conditions is depicted in figure 5.

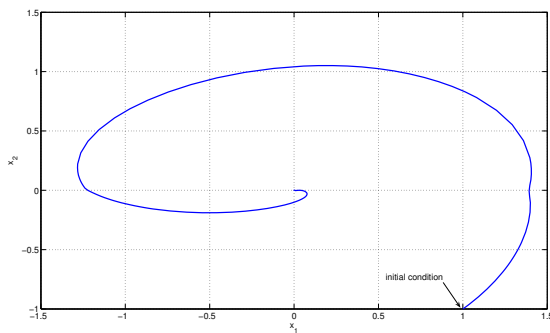


Fig. 4. Trajectory of the system in the closed loop with initial condition  $x(0) = [1 \ -1]^T$  and the observer initial condition  $\hat{x}(0) = [-1 \ 1]^T$ .

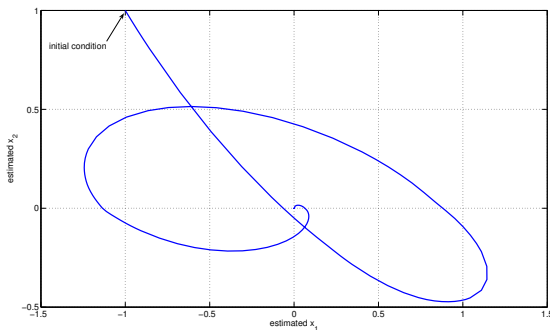


Fig. 5. Trajectory of the estimated states in the closed loop with the system initial condition  $x(0) = [1 \ -1]^T$  and the observer initial condition  $\hat{x}(0) = [-1 \ 1]^T$ .

## V. CONCLUSIONS

In this paper we have presented an output feedback controller design for a class of bi-modal piecewise linear systems. The designed controller consists of a state observer and a static state feedback law. We have used the machinery of input-to-state stability to design both the controller and the observer. A sufficient small gain condition for the global asymptotic stability of the system in closed loop with the observer based controller has been derived.

Robustness of the presented design with respect to the disturbances such as noise and model mismatch will be investigated in the future work. The guiding idea is to exploit the ISS properties of the component subsystems to establish the ISS of the closed loop system with respect to disturbances. Future work will also focus on the broadening of the class of hybrid systems to which the presented techniques are applicable.

## REFERENCES

- [1] A. Juloski, W. Heemels, and S. Weiland, "Observer design for a class of piecewise affine systems," in *Proc. of Conference on Decision and Control 2002*, Las Vegas, USA, 2002, pp. 2606–2611.
- [2] —, "Observer design for a class of piecewise linear systems," *International Journal of Robust and Nonlinear Control*, 2007, in press.

- [3] A. Alessandri and P. Coletta, "Design of Luenberger observers for a class of hybrid linear systems," in *Proc. of Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, M. Di Benedetto and A. Sangiovanni-Vincentelli, Eds., vol. 2034. Rome: Springer, 2001, pp. 7–18.
- [4] —, "Switching observers for continuous-time and discrete-time linear systems," in *Proc. of the American Control Conference*, 2001, pp. 2516–2521.
- [5] G. Julia Bara, J. Daafouz, F. Kratz, and C. Iung, "State estimation for a class of hybrid systems," in *Proc. of 4th Int. Conf. on Automation of Mixed Processes: Hybrid Dynamic Systems*, Dortmund, Germany, 2000, pp. 313–316.
- [6] M. Arcak and P. Kokotović, "Observer based control of systems with slope-restricted nonlinearities," *IEEE Transactions on Automatic Control*, vol. 46, no. 7, pp. 1146–1150, 2001.
- [7] S. Pettersson, "Switched state jump observers for switched systems," in *Proceedings of the 16th IFAC World Congress, Prague, Czech Republic*, 2005.
- [8] E. D. Sontag, "The ISS philosophy as a unifying framework for stability-like behavior," in *Nonlinear Control in the Year 2000*, ser. Lecture Notes in Control and Information Sciences, A. Isidori, F. Lamnabhi-Lagarrigue, and W. Respondek, Eds. Springer-Verlag, 2000, pp. 443–468.
- [9] Z. Jiang, I. Mareels, and Y. Wang, "A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems," *Automatica*, vol. 32, pp. 1211–1215, 1996.
- [10] Z. Jiang, A. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Mathematics of Control, Signals & Systems*, vol. 7, pp. 95–120, 1994.
- [11] L. Vu, D. Chatterjee, and D. Liberzon, "ISS of switched systems and applications to switching adaptive control," in *44th IEEE Conference on Decision and Control*, Seville, Spain, 2005, pp. 120–125.
- [12] D. Liberzon and D. Nesić, "Stability analysis of hybrid systems via small-gain theorems," 2006, pp. 421–435.
- [13] C. Cai and A. Teel, "Results on input-to-state stability for hybrid systems," in *44th IEEE Conference on Decision and Control*, Seville, Spain, 2005, pp. 5403–5408.
- [14] W. Heemels, S. Weiland, and A. Juloski, "Input-to-state stability of discontinuous dynamical systems with an observer-based control application," in *Lecture Notes in Computer Science. Proceedings of the Hybrid Systems: Computation and Control 2007*, PISA, Italy, 2007.
- [15] M. Arcak, "Certainty equivalence output feedback design with circle criterion observers."
- [16] L. Rodrigues, A. Hassibi, and J. How, "Output feedback controller synthesis for piecewise-affine systems with multiple equilibria," in *Proc. of American Control Conference*, 2000, pp. 1784–1789.
- [17] L. Rodrigues and J. How, "Observer-based control of piecewise affine systems," in *Proc. of Conference on Decision and Control*, 2001.
- [18] A. Filippov, *Differential Equations with Discontinuous Righthand Sides*, ser. Mathematics and its Applications. Dordrecht, The Netherlands: Kluwer, 1988.
- [19] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Control Theory*, ser. Studies in Applied Mathematics. SIAM, 1994, vol. 15.
- [20] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 555–559, 1998.
- [21] L. El Ghaoui and J. Commeau, "LMITOOL: a package for LMI optimization in Matlab user's guide." 1999, web site: <http://robotics.eecs.berkeley.edu/elghaoui/lmitool/lmitool.html>.
- [22] A. Pavlov, N. van de Wouw, and H. Nijmeijer, "Convergent piecewise affine systems: analysis and design Part 1: continuous case," in *Proceedings of the Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05*, 2005.