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An $\mathcal{L}_2$-Consistent Data Transmission Sequence for Linear Systems

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***Abstract*— In this paper, we tackle the problem of selecting a sparse data transmission sequence for a networked control system while guaranteeing a certain $\mathcal{L}_2$-induced norm bound with respect to an exogenous disturbance input. As the main contribution of this work, for every periodic transmission sequence, we provide a transmission sequence counterpart, aperiodic in general, that results in fewer or at most the same number of transmissions while still guaranteeing an $\mathcal{L}_2$ control policy with the same $\mathcal{L}_2$-induced norm bound as that of the periodic transmission policy. In this sense, the proposed transmission sequence is called $\mathcal{L}_2$-consistent. Moreover, we show that as the horizon approaches infinity, the proposed $\mathcal{L}_2$-consistent transmission sequence counterpart of every periodic transmission sequence approaches a periodic transmission sequence with an equal or larger time period.**

I INTRODUCTION

Traditionally, sampled-data control is implemented by periodically transmitting the state or the output of the system to the controller and then updating the control input. However, periodic sampling and control may require excessive resources, which can be prohibitive in applications where the computation power of the controller is limited, or the bandwidth of the communication channels is small. Moreover, in any application, all these resources are constrained, especially when the control loops are closed over a shared communication network. Therefore, resource-aware control [1] has received a considerable amount of attention in recent years in pursuit of new sampling and control policies, which meet the constraints imposed in the networked control systems of the future.

There are many papers that propose data transmission policies not only to save transmissions but also to guarantee or optimize a certain performance criterion for the control system [2]–[4]. Moreover, it has been widely demonstrated in recent years that aperiodic control, such as event-triggered control (ETC) [1], [5]–[11], can significantly reduce the average communication frequency while keeping the closed-loop control performance within the desired values.

Within the performance criteria considered in the context of ETC, and for control systems in general, the $\mathcal{L}_2$-induced norm bound is an important performance measure [12]–[15]. It indicates the attenuation level of the disturbance input at the performance output of the system and highly depends on the sequence at which the state or the output vector is transmitted to the controller. Some of the previous studies investigated the $\mathcal{L}_2$ control design problem for the periodic data transmission to the controller [16], [17]. Moreover, there are some other studies establishing the $\mathcal{L}_2$ stability of the ETCs [12], [13] or providing guaranteed values for the $\mathcal{L}_2$-gain of the proposed ETCs [15]. In this paper, inspired by [18], we introduce a notion of $\mathcal{L}_2$-*consistency* for a data transmission sequence, where in comparison to its periodic transmission counterpart, it guarantees an $\mathcal{L}_2$ control policy with an equal or smaller $\mathcal{L}_2$-induced norm bound, while its average transmission frequency is lower or at most is equal to that of the periodic transmission counterpart. Our goal in this paper is to propose an $\mathcal{L}_2$-consistent data transmission sequence from the sensors to the controller for general discrete-time linear systems.

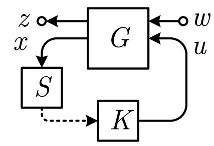

Fig. 1: Disturbance attenuation by an $\mathcal{L}_2$ state-feedback controller where G, K and S refer to the plant, the controller and the scheduler, respectively.

To achieve this goal, we approach the finite-time horizon $\mathcal{L}_2$ control design problem from the perspective of non-cooperative game theory [19]. As depicted in Fig. 1, we consider a data scheduler that sends the state information to the controller based on a time sequence, which should be determined to guarantee a certain $\mathcal{L}_2$-induced norm bound for the system. We start by determining an $\mathcal{L}_2$-induced norm bound for the periodic transmission time sequence with a single constant time period $\tau \in \mathbb{N}$. Then, for the τ-step periodic transmission sequence, we propose an $\mathcal{L}_2$-consistent aperiodic transmission sequence. We also investigate the problem when the time horizon approaches infinity. In this case, we show that the proposed $\mathcal{L}_2$-consistent transmission sequence counterpart of every periodic transmission sequence approaches a periodic transmission with an equal or larger time period.

The remainder of this paper is organized as follows: the problem of interest is introduced in Section II and the $\mathcal{L}_2$ control problem is solved in Section III by resorting to game theory. Then, an aperiodic $\mathcal{L}_2$-consistent transmission time sequence is introduced in Section IV. Finally, the effectiveness of the technique to provide efficient transmission sequences is demonstrated through numerical simulations

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in Section V. Section VI presents concluding remarks. The proof of the main theorem can be found in the appendix.

Notation: Let $\mathbb{N}_0$ indicate the set of non-negative integers, and $\mathbb{N}_r^s = \{t \in \mathbb{N}_0 | r \le t \le s\}$ for $r, s \in \mathbb{N}_0$.

II Problem setting

In this section, we illustrate the configuration of the networked control system in Section II-A and introduce the problem of interest in Section II-B.

II-A Networked control configuration

Consider a discrete-time linear time-invariant (LTI) system

$$x_{k+1} = Ax_k + Bu_k + Dw_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector at $k \in \mathbb{N}_0^K$, $u_k \in \mathbb{R}^m$ is the control input at $k \in \mathbb{N}_0^{K-1}$ and $w_{[0,K-1]} \in \ell^2([0, K-1], \mathbb{R}^l)$ is a sequence of the disturbance inputs. Let us also assume that (A, B) is stabilizable. For the sake of simplicity, we assume $x_0 = 0$, although, we can also define and solve the $\mathcal{L}_2$ control design problem for the system with an unknown initial condition[1] [20]. The disturbance generator is assumed to have access to the state vector at all times and therefore $w_k = \Phi_k(\mathcal{E}_k)$, where

$$\mathcal{E}_k := \{x_i | i \in \mathbb{N}_0^k\}. \tag{2}$$

However, we assume a state transmission time sequence $\eta \in \{0, 1\}^K$ from the sensors to the controller specified by $(\delta_0^\eta, \delta_1^\eta, \ldots, \delta_{K-1}^\eta) \in \{0, 1\}^K$, where for all $k \in \mathbb{N}_0^{K-1}$, x_k is transmitted to the controller if $\delta_k^\eta = 1$, otherwise it is not transmitted. Therefore, the control policy is denoted by $u_k := \Psi_k(\mathcal{F}_k^\eta)$, $k \in \mathbb{N}_0^{K-1}$, where

$$\mathcal{F}_k^\eta := \{x_i | i \in \mathbb{N}_0^k \wedge \delta_i^\eta = 1\}. \tag{3}$$

Furthermore, let us denote $T_\eta = \sum_{i=0}^{K-1} \delta_i^\eta$ as the total number of transmissions, and (t_i, τ_i) for all $i \in \mathbb{N}_0^{T_\eta - 1}$ as the pairs of the transmission times and the time interval up to the next transmission time, i.e. $t_i \in \mathbb{N}_0^{K-1}$ such that $t_0 < t_1 < \cdots < t_{T_\eta - 1}$ and $\{t_0, \ldots, t_{T_\eta - 1}\} = \{k \in \mathbb{N}_0^{K-1} | \delta_k^\eta = 1\}$ and $\tau_i = t_{i+1} - t_i$ for all $i \in \mathbb{N}_0^{T_\eta - 1}$ where $t_{T_\eta} = K$. The goal of the $\mathcal{L}_2$ control problem is to attenuate the effect of the disturbance input w_k on the performance output z_k of the system defined as

$$z_k = Ex_k + Fu_k, \tag{4}$$

where for the sake of simplicity, we select E and F such that $E^\intercal E = Q \ge 0$, $F^\intercal F = I$, $E^\intercal F = 0$ and $(A, Q^{\frac{1}{2}})$ to be observable. Therefore, $z_k^\intercal z_k = x_k^\intercal Q x_k + u_k^\intercal u_k$ at every $k \in \mathbb{N}_0^{K-1}$.

Definition 1: ($\mathcal{L}_2$*-induced norm bound*) Let $\eta \in \{0, 1\}^K$ be given as a data transmission sequence from the sensor to the controller of the discrete-time system (1) and suppose there exists a state-feedback control policy

$$u_k = \Psi_k(\mathcal{F}_k^\eta), \qquad k \in \mathbb{N}_0^{K-1}, \tag{5}$$

for which

$$\sum_{k=0}^{K-1} z_k^\intercal z_k \le \gamma^2 \sum_{k=0}^{K-1} w_k^\intercal w_k. \tag{6}$$

Moreover, let

$$\begin{aligned} \Gamma^\eta := \big\{ &\gamma \in \mathbb{R}_{>0} | \exists \text{ (5) for which (6) holds} \\ &\text{for every } w_{[0,K-1]} \in \ell^2([0, K-1], \mathbb{R}^l) \big\}. \end{aligned} \tag{7}$$

Then, any $\gamma \in \Gamma^\eta$ is an $\mathcal{L}_2$-induced norm bound of the system (1) for the data transmission sequence η to the controller and is denoted by γ^η.

□

The problem of finding a control policy (5) such that (6) is met for every sequence $w_{[0,K-1]} \in \ell^2([0, K-1], \mathbb{R}^l)$ will be denoted by the $\mathcal{L}_2$ control problem. Such policy always exists for sufficiently large $\gamma \in \mathbb{R}_{>0}$ and in particular the set Γ^η is non-empty.

II-B Problem statement

We denote the τ-step periodic transmission sequence for $\tau \in \mathbb{N}$ by p_τ where $\delta_k^{p_\tau} = 1$ if k is zero or a multiple of the sampling period τ and $\delta_k^{p_\tau} = 0$, otherwise. Next, we define the concept of an $\mathcal{L}_2$-consistent data transmission sequence.

Definition 2: ($\mathcal{L}_2$*-consistent data transmission sequence*) Let $K \in \mathbb{N}$ and $\tau \in \mathbb{N}_1^K$ be fixed. A data transmission sequence $\mu \in \{0, 1\}^K$ is called $\mathcal{L}_2$-consistent if in comparison to the τ-step periodic transmission sequence, it guarantees an $\mathcal{L}_2$ control policy with the same or smaller $\mathcal{L}_2$-induced norm bound, however, by using fewer or at most the same number of data transmissions. That is, for any $\gamma^{p_\tau} \in \Gamma^{p_\tau}$ as an $\mathcal{L}_2$-induced norm bound of the system for the τ-step periodic transmission sequence, there exists $\gamma^\mu \in \Gamma^\mu$ such that $\gamma^\mu \le \gamma^{p_\tau}$ while $T_\mu \le T_{p_\tau}$. □

The problem of interest in this work is to find an $\mathcal{L}_2$-consistent data transmission sequence from the sensor to the controller.

III $\mathcal{L}_2$ control design based on zero-sum game

This section starts by introducing a two player zero-sum quadratic dynamic game (ZQDG) and its saddle-point solution is given in Lemma 1. Then, Lemma 2 provides the condition for the existence of the ZQDG saddle point solution, which is beneficial for determining an $\mathcal{L}_2$-induced norm bound of the system. Finally, we illustrate the behaviour of the saddle-point existence condition at different transmission time for the periodic transmission sequence in Lemma 3. The proofs of Lemmas 1-5 are omitted due to space limitations.

Let us now denote $u = u_{[0,K-1]}$ and $w = w_{[0,K-1]}$ as the control and the disturbance inputs during the time window $\mathbb{N}_0^{K-1}$, respectively, and define the following performance index

$$J_0(u, w) := \sum_{k=0}^{K-1} x_k^\intercal Q x_k + u_k^\intercal u_k - \gamma^2 w_k^\intercal w_k. \tag{8}$$

It is well-known that the $\mathcal{L}_2$ control problem is solved by considering a two player zero-sum quadratic dynamic

[1] In this case, we add one extra time-step to the time-horizon and consider the initial condition as the disturbance of the previous time-step.

game (ZQDG) in which the controller acts as a minimizer and the disturbance generator acts as a maximizer of (8) (see, e.g., [16]). Therefore, we wish to find control and disturbance policies $u_k^* = \Psi_k(\mathcal{F}_k^\eta)$ and $w_k^* = \Phi_k(\mathcal{E}_k^\eta)$ for all $k \in \mathbb{N}_0^{K-1}$ where the pair (u^*, w^*) with $u^* = u^*_{[0,K-1]}$ and $w^* = w^*_{[0,K-1]}$ satisfies the following inequalities

$$J_0(u^*, w) \leq J_0(u^*, w^*) \leq J_0(u, w^*),$$

for every u and w, where $w = w_{[0,K-1]} \in \ell^2([0, K-1], \mathbb{R}^l)$. The pair of the control and the disturbance policies (Ψ_k, Φ_k) is called the saddle-point solution of the ZQDG.

In order to determine these policies, let us denote the augmented vectors of the control and the disturbance inputs in between every two successive transmission time-steps by $U_i = [u_{t_i}^\intercal, \dots, u_{t_{i+1}-1}^\intercal]^\intercal$ and $W_i = [w_{t_i}^\intercal, \dots, w_{t_{i+1}-1}^\intercal]^\intercal$, respectively, for all $i \in \mathbb{N}_0^{T_\eta - 1}$. Then, if the solution of the following optimization problem

$$J_i(U_i^*, W_i^*) = \min_{U_i(\mathcal{F}_{t_i}^\eta)} \max_{w_{t_i}(\mathcal{E}_{t_i})} \dots \max_{w_{t_{i+1}-1}(\mathcal{E}_{t_{i+1}-1})} \sum_{k=t_i}^{t_{i+1}-1} [x_k^\intercal Q x_k + u_k^\intercal u_k - \gamma^2 w_k^\intercal w_k] + J_{i+1}(U_{i+1}^*, W_{i+1}^*) \tag{9}$$

for all $i \in \mathbb{N}_0^{T_\eta - 1}$ exists (if it is bounded) where $J_{N_\eta} = 0$, it results in the saddle-point solution (U_i^*, W_i^*) of the dynamic game at time-steps $\{t_i, \dots, t_{i+1}-1\}$ where U_i^* is the state-feedback $\mathcal{L}_2$ control input for the given transmission sequence η. If this solutions exists, we say that he ZQDG (9) admits a saddle-point solution $(u^*, w^*) = \{(U_i^*, W_i^*) | i \in \mathbb{N}_0^{T_\eta - 1}\}$. Moreover, a bound for the $\mathcal{L}_2$-induced norm of the system can be determined based on the condition under which the ZQDG (9) attains a saddle-point solution. It is worthwhile to mention that $J_i(U_i^*, W_i^*)$ for all $i \in \mathbb{N}_0^{T_\eta - 1}$ is the saddle-point game value at every data transmission time.

Lemma 1: Assume that the ZQDG (9) admits a saddle-point solution for a given $\gamma \in \mathbb{R}_{>0}$ when the information available to the disturbance generator and the controller is described by (2) and (3), respectively, for an arbitrary given data transmission sequence η. Then, for every $k \in \mathbb{N}_0^{K-1}$,

$$\begin{aligned} u_k^* &= R_k \hat{x}_{k|k}, \\ w_k^* &= S_k x_k + (L_k - S_k)\hat{x}_{k|k}, \end{aligned} \tag{10}$$

are the saddle-point control and disturbance policies of the ZQDG (9), where

$$\begin{aligned} R_k &= -B^\intercal N_{k+1} \Lambda_k^{-1} A, \\ L_k &= \gamma^{-2} D^\intercal N_{k+1} \Lambda_k^{-1} A, \\ S_k &= \gamma^{-2} D^\intercal \Theta_{k+1} V_k^{-1} A, \end{aligned} \tag{11}$$

for $\Lambda_k = I + (BB^\intercal - \gamma^{-2} DD^\intercal) N_{k+1}$, $N_K = 0$ and

$$N_k = Q + A^\intercal N_{k+1} \Lambda_k^{-1} A, \tag{12}$$

for all $k \in \mathbb{N}_0^{K-1}$. Moreover, $V_k = I - \gamma^{-2} DD^\intercal \Theta_{k+1}$ where $\Theta_K = 0$ and for all $k \in \mathbb{N}_0^{K-1}$,

$$\Theta_k = \begin{cases} Q + A^\intercal \Theta_{k+1} V_k^{-1} A, & \text{if } \delta_k^\eta = 0 \\ N_k, & \text{otherwise.} \end{cases}$$

Furthermore, $\hat{x}_{k|k}$ is determined as follows

$$\begin{aligned} \hat{x}_{k+1|k} &= \Lambda_k^{-1} A \hat{x}_{k|k}, \\ \hat{x}_{k|k} &= \begin{cases} x_k, & \text{if } \delta_k^\eta = 1 \\ \hat{x}_{k|k-1}, & \text{otherwise.} \end{cases} \end{aligned} \tag{13}$$

□

The following lemma provides the saddle-point existence condition for the ZQDG (9).

Lemma 2: Consider that the disturbance generator and the controller have access to the information sets (2) and (3), respectively, for an arbitrary given data transmission sequence η. Then for a given $\gamma \in \mathbb{R}_{>0}$, the ZQDG admits a saddle-point solution if for every $i \in \mathbb{N}_0^{T_\eta - 1}$

$$\bar{M}_{t_i}^{\tau_i}(\gamma) > 0, \tag{14}$$

where

$$\bar{M}_{t_i}^{\tau_i}(\gamma) = \gamma^2 I_{\tau_i \times l} - \bar{D}_{\tau_i}^\intercal \bar{N}_{t_i}^{\tau_i} \bar{D}_{\tau_i}, \tag{15}$$

for which

$$\bar{D}_{\tau_i} = \begin{bmatrix} D & 0 & & 0 \\ AD & D & & 0 \\ & & \dots & \\ A^{\tau_i - 1} D & A^{\tau_i - 2} D & & D \end{bmatrix}_{\tau_i \times \tau_i}$$

and

$$\bar{N}_{t_i}^{\tau_i} = \begin{cases} N_{t_i+1}, & \text{if } \tau_i = 1 \\ \text{diag}(I_{\tau_i - 1} \otimes Q, N_{t_i + \tau_i}), & \text{otherwise,} \end{cases} \tag{16}$$

where N_k is determined based on (12). □

Based on Definition 1 and Lemma 2, any member of the following set

$$\Gamma^\eta = \{\gamma \in \mathbb{R}_{>0} | \bar{M}_{t_i}^{\tau_i}(\gamma) > 0, \ \forall i \in \mathbb{N}_0^{T_\eta - 1}\}. \tag{17}$$

is an $\mathcal{L}_2$-induced norm bound of the discrete-time system (1) for the state transmission time sequence η. In the following lemma, we illustrate the non-decreasing behaviour of the eigenvalues of $\bar{M}_{t_i}^\tau(\gamma)$ with respect to $i \in \mathbb{N}_0^{T_{p_\tau} - 1}$ for the τ-step periodic transmission sequence.

Lemma 3: For the τ-step periodic transmission sequence and any $\gamma \in \Gamma^{p_\tau}$, the eigenvalues sequence of the matrix sequence $\{\bar{M}_{t_i}^\tau(\gamma) | i \in \mathbb{N}^{T_{p_\tau} - 1}\}$ are non-decreasing with respect to $i \in \mathbb{N}_0^{T_{p_\tau} - 1}$ and therefore, $\bar{M}_{t_i}^\tau(\gamma) \leq \bar{M}_{t_{i+1}}^\tau(\gamma)$ for all $i \in \mathbb{N}_0^{T_{p_\tau} - 1}$ and $t_i = i\tau$. □

IV $\mathcal{L}_2$-CONSISTENT TRANSMISSION SEQUENCE

In this section, we propose an $\mathcal{L}_2$-consistent transmission sequence for the finite-time horizon problems in Section IV-A and then investigate its behaviour as time horizon approaches infinity in Section IV-B. Then, in Section IV-C, we extend the $\mathcal{L}_2$-consistent transmission sequence for the case when the parameters of the performance output are time-varying.

IV-A Finite-time horizon problems

In this section, we introduce an $\mathcal{L}_2$-consistent state transmission sequence to the controller based on Definition 2. The idea behind this sequence is inspired by the non-decreasing behaviour of the eigenvalues of $\bar{M}_{t_i}^{\tau}(\gamma)$ with respect to transmission time-step t_i for the periodic transmission sequence according to Lemma 3. This data transmission sequence is formed by selecting at every transmission time-step (t_i) the largest inter-transmission time interval (τ_i) so that $\bar{M}_{t_i}^{\tau_i}(\gamma)$ has its minimum eigenvalue in the closest proximity of zero.

Theorem 1: (*$\mathcal{L}_2$-consistent state transmission sequence*) Suppose that $\gamma^{p_\tau}\in\Gamma^{p_\tau}$ is an $\mathcal{L}_2$-induced norm bound of the system (1) for the τ-step periodic state transmission sequence. Then, for $\gamma=\gamma^{p_\tau}$, the transmission time sequence μ for which the transmission time-steps are determined as

$$t_{i+1}=t_i+(\nu_i-1) \tag{18}$$

for $t_0=0$ and

$$\nu_i=\min\{r\in\mathbb{N}|\bar{M}_{t_i}^{r}(\gamma)=\gamma^2 I_{r\times l}-\bar{D}_r^{\intercal}\bar{N}_{t_i}^{r}\bar{D}_r\leq 0\}$$

for all $i\in\mathbb{N}_0^{T_\mu-1}$ where,

$$\bar{N}_{t_i}^{r}=\begin{cases}N_{t_i+1}, & \text{if } r=1\\ \operatorname{diag}(I_{r-1}\otimes Q, N_{t_i+r}), & \text{otherwise}\end{cases} \tag{19}$$

is $\mathcal{L}_2$-consistent, i.e., it results in fewer or at most the same number of transmissions in comparison with the τ-step periodic state transmission sequence as γ^{p_τ} is the $\mathcal{L}_2$-induced norm bound of the system for both transmission sequences. □

The proof of Theorem 1 is provided in the Appendix.

Note that the value of γ^{p_τ} in Theorem 1 can be selected very close to the infimum value of the set Γ^{p_τ} which is actually defined as the $\mathcal{L}_2$-induced norm of the system [16].

IV-B Horizon approaches infinity

The $\mathcal{L}_2$-consistent state transmission sequence proposed in Theorem 1 has the ability to decrease the number of transmissions for *finite-time horizon* problems, while still guaranteeing the same value for the $\mathcal{L}_2$-induced norm bound of the system. It is also of interest to evaluate the effectiveness of the proposed $\mathcal{L}_2$-consistent state transmission sequence as the horizon approaches infinity. The following lemma shows that if the time horizon is long enough, then there is a time-step $\bar{L}\in\mathbb{N}_0^{K-1}$ where the $\mathcal{L}_2$-consistent state transmission sequence of Theorem 1 follows the periodic transmission sequence at all $k\in\mathbb{N}_0^{\bar{L}}$.

Lemma 4: Assume that the data transmission follows the $\mathcal{L}_2$-consistent state transmission sequence μ proposed in Theorem 1 for a given $\tau\in\mathbb{N}$ and $\gamma^{p_\tau}\in\mathbb{R}_{>0}$. Then for every time-step $\bar{L}\in\mathbb{N}_0$, there exists a time-step $\bar{K}>\bar{L}$ such that if $K>\bar{K}$ then for every $k\in\{0,\ldots,\bar{L}\}$, $\mu\in\{0,1\}^K$ follows the periodic transmission sequence with a fixed time period $\tau'\in\mathbb{N}$ which is equal or larger than τ, i.e. for all $k\leq\bar{L}$

$$\delta_k^{\mu}=\begin{cases}1, & \text{if } k=i\tau'\\ 0, & \text{otherwise,}\end{cases}$$

where $i\in\mathbb{N}_0$ and $\tau\leq\tau'$. □

Lemma 4 states essentially that for the long or infinite-time horizon problems, the $\mathcal{L}_2$-consistent state transmission sequence of Theorem 1 follows the periodic transmission sequence from the initial time-step with a period equal or larger than τ and as time approaches the final time-step, the transmission sequence may become aperiodic with inter-transmission times larger than τ.

IV-C Time-varying performance output

In this section, we consider a class of time-varying performance outputs for the system (1) as

$$z_k=E_k x_k+F_k u_k, \tag{20}$$

where $E_k^{\intercal}E_k=Q_k\geq 0$, $F_k^{\intercal}F_k=I$, $E_k^{\intercal}F_k=0$ and $(A, Q_k^{\frac{1}{2}})$ is observable for all $k\in\mathbb{N}_0^{K-1}$. In order to find the set of $\mathcal{L}_2$-induced norm bound (7) for the time-varying performance output (20) and any fixed transmission sequence η, we can follow (17) and Lemma 2 where (16) needs to be adapted as follows

$$\bar{N}_{t_i}^{\tau_i}=\begin{cases}N_{t_i+1}, & \text{if } \tau_i=1\\ \operatorname{diag}(Q_{t_i+1},\ldots,Q_{t_i+\tau_i-1},N_{t_i+\tau_i}), & \text{otherwise.}\end{cases}$$

Moreover, in the following lemma, we provide the condition under which the transmission sequence of Theorem 1 is $\mathcal{L}_2$-consistent for the time-varying performance output (20).

Lemma 5: The transmission sequence determined based on Theorem 1 where (19) is adapted to the time varying performance output (20) as

$$\bar{N}_{t_i}^{r}=\begin{cases}N_{t_i+1}, & \text{if } r=1\\ \operatorname{diag}(Q_{t_i+1},\ldots,Q_{t_i+r-1},N_{t_i+r}), & \text{otherwise,}\end{cases}$$

is $\mathcal{L}_2$-consistent if $Q_k\geq Q_{k+1}\geq 0$ for all $k\in\mathbb{N}_0^{K-2}$. □

Based on Lemma 5, we can design an $\mathcal{L}_2$-consistent transmission sequence similar to one proposed in Theorem 1 when the series of eigenvalues of $\{Q_k|k\in\mathbb{N}_0^{K-1}\}$ is a non-increasing function of time.

V Simulation results

V-A Scalar system

Consider $A=1, B=1$ and $D=5$ as the parameters of a scalar system, $Q=0.01$ and $K=37$. Then for every τ-step periodic transmission sequence where $\tau\in\{1,2,3,4\}$, we can select an $\mathcal{L}_2$-induced norm bound from the set Γ^{p_τ} given in (17). These selected values are $\gamma^{p_1}=4.7110, \gamma^{p_2}=4.8538, \gamma^{p_3}=5.0204, \gamma^{p_4}=5.2089$, which are all very close to the infimum value of the set Γ^{p_τ} for every $\tau\in\{1,2,3,4\}$. For these fixed $\mathcal{L}_2$-induced norm bounds, we determine their $\mathcal{L}_2$-consistent state transmission sequence counterparts based on Theorem 1, where the resulting total number of transmissions from the initial time-step is shown in Fig. 2(a). As it can be seen, for the given values of K, τ and γ^{p_τ}, transmissions follow a periodic pattern at time-steps close to the initial time-step. However, as time approaches the final time-step, the inter-transmission times become larger which results in

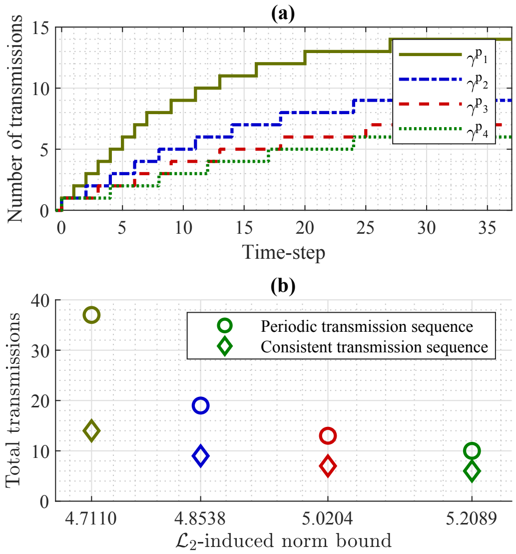

Fig. 2: a) Number of transmissions up to time-step k for the $\mathcal{L}_2$-consistent transmission sequence, b) total number of transmissions versus an $\mathcal{L}_2$-induced norm bound for the periodic and $\mathcal{L}_2$-consistent state transmission sequences.

fewer total transmissions. Fig. 2(b) shows the total number of transmissions with respect to the considered $\mathcal{L}_2$-induced norm bound of the system for both periodic transmission and its $\mathcal{L}_2$-consistent data transmission sequence counterpart. Based on that, the number of transmissions of the $\mathcal{L}_2$-consistent sequence is fewer than the one for the periodic sequence while they both guarantee the same $\mathcal{L}_2$-induced norm bound. Moreover, for this example, as the final time tends to infinity, the transmission reduction $(T_{p_\tau} - T_\mu)$ achieved by the $\mathcal{L}_2$-consistent sequence converges to a bounded value, which is shown in Fig. 3. It indicates that if for this system time horizon converges infinity, then the $\mathcal{L}_2$-consistent transmission sequence approaches periodic transmission where the time period is equal to τ, i.e. $\tau' = \tau$.

V-B System with delayed disturbance input

Consider the linear system (1) with $K > n$ and

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}_n, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

where $n \geq 2$ and $Q = \mathrm{diag}(1, 0, \ldots, 0)$. Based on these dynamics, it takes $n-1$ time-steps for every disturbance input to affect the first state (or, equivalently, the performance output) of the system while the control input affects the first

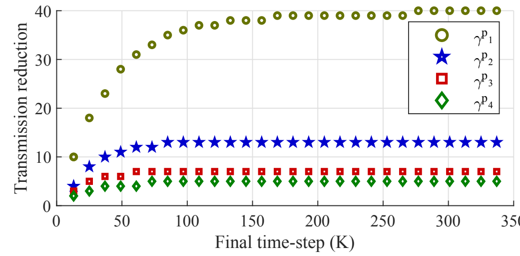

Fig. 3: Transmissions reduction achieved by the $\mathcal{L}_2$-consistent state transmission sequence as time horizon approaches infinity.

state directly. By using (12), we determine $N_K = 0$ and

$$N_k = \begin{cases} \mathrm{diag}(1, 0, \ldots, 0), & \text{if } k = K-1 \\ \mathrm{diag}(1, \frac{1}{2}, 0, \ldots, 0), & \text{if } k = K-2 \\ \ldots \\ \mathrm{diag}(1, \frac{1}{2}, \ldots, \frac{1}{2}, 0), & \text{if } k = K-(n-1) \\ \mathrm{diag}(1, \frac{1}{2}, \ldots, \frac{1}{2}), & \text{if } k \leq K-n \end{cases}$$

for any admissible value of $\gamma \in \mathbb{R}_{>0}$. Moreover, based on (17), for the τ-step periodic feedback transmission sequence, the set of $\mathcal{L}_2$-induced norm bound is given by

$$\Gamma^{p_\tau} = \begin{cases} (\frac{1}{\sqrt{2}}, \infty), & \text{if } \tau \leq n-1 \\ (1, \infty), & \text{otherwise}, \end{cases}$$

irrespective of the time horizon K. Determining the $\mathcal{L}_2$-consistent state transmission sequence based on Theorem 1 results in a periodic transmission sequence with $\tau = n-1$ for any $\gamma^{p_\tau} \in (\frac{1}{\sqrt{2}}, 1]$, while, it results in just one transmission at $k=0$ for $\gamma^{p_\tau} \in (1, \infty)$. Therefore,

$$T_\eta = \begin{cases} \frac{K-1}{n-1}, & \text{if } \gamma^{p_\tau} \in (\frac{1}{\sqrt{2}}, 1] \\ 1, & \text{if } \gamma^{p_\tau} \in (1, \infty), \end{cases}$$

and the fraction of the transmission reduction (TR) resulting by the $\mathcal{L}_2$-consistent transmission sequence is

$$TR = \frac{T_{p_\tau} - T_\mu}{\frac{K-1}{\tau}} = \begin{cases} 1 - \frac{\tau}{n-1}, & \text{if } \tau \leq n-1 \\ 1 - \frac{\tau}{K-1}, & \text{otherwise}. \end{cases}$$

This indicates that for some conditions, we can omit a large portion of the total transmissions irrespective of the time horizon. In these situations, the $\mathcal{L}_2$-consistent transmission sequence of Theorem 1 results in a periodic transmission with average inter-transmission time larger than τ.

VI Conclusions

In this work, we introduced a notion of $\mathcal{L}_2$-consistency for a data transmission sequence from the sensors to the controller, which guarantees an $\mathcal{L}_2$ control policy with the same or a smaller $\mathcal{L}_2$-induced norm bound as that of the τ-step periodic transmission policy, however with fewer or at most the same number of data transmissions. Then, based on the crucial observation that for the finite-time horizon problems, the eigenvalues of the weighting matrix of the saddle-point zero-sum quadratic dynamic game value have a constant or decreasing behaviour with respect to time, we propose an aperiodic $\mathcal{L}_2$-consistent data transmission sequence. In principle, the proposed transmission policy increases the inter-transmission time as time is approaching the final time-step and the eigenvalues of the saddle-point weighting matrix are becoming smaller. This results in the $\mathcal{L}_2$-consistency property. We also investigate the case where the horizon approaches infinity. For this case, the proposed $\mathcal{L}_2$-consistent transmission sequence counterpart of every periodic transmission sequence approaches a periodic transmission sequence with an equal or larger time period.

APPENDIX

A Theorem 1

Assume that $t_i \in \mathbb{N}_0^{K-1}$ is a transmission time-step generated by the proposed algorithm in this theorem. To prove the result, we just need to show that for this transmission time-step $\tau_i = \nu_i - 1 \geq \tau$. Equivalently, we can show it by establishing that at any $t_i \in \mathbb{N}_0^{K-1}$, $\bar{M}_{t_i}^r(\gamma) > 0$ for all $r \in \mathbb{N}_1^\tau$ and any $\gamma \in \Gamma^{p_\tau}$ where $\bar{M}_{t_i}^r(\gamma) = \gamma^2 \bar{I}_{r\times l} - \bar{D}_r^\intercal \bar{N}_{t_i}^r \bar{D}_r$. Since we know that $\bar{M}_{t_0}^\tau(\gamma) > 0$ and due to the non-increasing behaviour of N_k with respect to time, we can conclude that $\bar{M}_{t_i}^\tau(\gamma) > 0$ at any $t_i \in \mathbb{N}_0^{K-\tau}$. In the following, we show that $\bar{M}_{t_i}^r(\gamma) > 0$ for all $r \in \mathbb{N}_1^{\tau-1}$ given the condition that $\bar{M}_{t_i}^\tau(\gamma) > 0$. Let us partition $\bar{M}_{t_i}^\tau(\gamma)$ as follows

$$\bar{M}_{t_i}^\tau(\gamma) = \begin{bmatrix} \gamma^2 I_{(\tau-1)l} - \bar{R}_{\tau-1} & -\bar{S}_{\tau-1}^\intercal A^\intercal N_{t_i+\tau} D \\ -D^\intercal N_{t_i+\tau} A \bar{S}_{\tau-1} & \gamma^2 I - D^\intercal N_{t_i+\tau} D \end{bmatrix},$$

where $\bar{R}_{\tau-1} = \bar{D}_{\tau-1}^\intercal \bar{Q}_{t_i}^{\tau-1} \bar{D}_{\tau-1} + \bar{S}_{\tau-1}^\intercal A^\intercal N_{t_i+\tau} A \bar{S}_{\tau-1}$,

$$\bar{D}_\tau = \begin{bmatrix} \bar{D}_{\tau-1} & 0 \\ A\bar{S}_{\tau-1} & D \end{bmatrix}, \quad \bar{S}_{\tau-1} = \begin{bmatrix} A^{\tau-2} D & \dots & D \end{bmatrix}.$$

Then by using the Schur complement, guaranteing $\bar{M}_{t_i}^\tau(\gamma) > 0$ for any $\gamma \in \Gamma^{p_\tau}$ is equivalent to the inequalities $\Xi_1(\gamma) = \gamma^2 I - D^\intercal N_{t_i+\tau} D > 0$ and

$$\Xi_2(\gamma) = \gamma^2 I_{(\tau-1)l} - \bar{R}_{\tau-1} + (A\bar{S}_{\tau-1})^\intercal N_{t_i+\tau} D \left(\gamma^2 I - D^\intercal N_{t_i+\tau} D\right)^{-1} D^\intercal N_{t_i+\tau} A \bar{S}_{\tau-1} > 0.$$

By simplifying $\Xi_2(\gamma)$ and using the Lemma 6.2 in [21] we arrive at the following expression

$$\Xi_2(\gamma) = \gamma^2 I_{(\tau-1)l} - \bar{D}_{\tau-1}^\intercal \bar{Q}_{t_i}^{\tau-1} \bar{D}_{\tau-1} - \bar{S}_{\tau-1}^\intercal A^\intercal N_{t_i+\tau} (I - \gamma^{-2} D D^\intercal N_{t_i+\tau})^{-1} A \bar{S}_{\tau-1} > 0.$$

Moreover, since $\bar{D}_\tau^\intercal = [\bar{D}_{\tau-1}^\intercal \ \ \bar{S}_\tau^\intercal]$, we can represent the above equation as follows $\Xi_2(\gamma) = \gamma^2 I_{(\tau-1)l} - \bar{D}_{\tau-1}^\intercal \hat{N}_{t_i+\tau-1}^{\tau-1} \bar{D}_{\tau-1} > 0$, where

$$\hat{N}_{t_i+\tau-1}^{\tau-1} = \mathrm{diag}\big(Q_{t_i+1}, \dots, Q_{t_i+\tau-1} + A^\intercal N_{t_i+\tau} (I - \gamma^{-2} D D^\intercal N_{t_i+\tau})^{-1} A\big).$$

Furthermore, $\bar{M}_{t_i}^{\tau-1}(\gamma) = \gamma^2 I - \bar{D}_{\tau-1}^\intercal \bar{N}_{t_i+\tau-1}^{\tau-1} \bar{D}_{\tau-1}$ where

$$\bar{N}_{t_i+\tau-1}^{\tau-1} = \mathrm{diag}\Big(Q_{t_i}, \dots, Q_{t_i+\tau-1} + A^\intercal N_{t_i+\tau} \big(I + (BB^\intercal - \gamma^{-2} D D^\intercal) N_{t_i+\tau}\big)^{-1} A\Big).$$

Since $BB^\intercal N_{t_i+\tau} \geq 0$, then we can conclude that $\bar{N}_{t_i+\tau-1}^{\tau-1} \leq \hat{N}_{t_i+\tau-1}^{\tau-1}$ and therefore, for any $\gamma \in \Gamma^{p_\tau}$ we have $0 < \Xi_2(\gamma) \leq \bar{M}_{t_i}^{\tau-1}(\gamma)$ which indicates that if $\bar{M}_{t_i}^\tau(\gamma) > 0$ then $\bar{M}_{t_i}^{\tau-1}(\gamma) > 0$. Following an induction argument, we can also prove that $\bar{M}_{t_i}^r(\gamma) > 0$ holds for all $r \in \bar{N}_1^{\tau-2}$ which proves the theorem.

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