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Comparison of stability characterisations for networked control systems

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***Abstract*— This paper presents linear matrix inequalities for stability analysis for networked control systems (NCSs) that incorporates various network phenomena: time-varying sampling intervals, packet dropouts and time-varying delays that may be both smaller and larger than the sampling interval. The problem is approached from a discrete-time modelling perspective. A comparison is made between the use of Parameter Dependent Lyapunov functions and Lyapunov-Krasovskii functions for stability analysis. Examples illustrate the developed theory.**

***Index Terms*— network control systems, time-varying delay, sampled-data control, linear matrix inequalities (LMIs), stability analysis.**

I. INTRODUCTION

The literature on modeling, analysis and controller design of networked control systems (NCSs) expanded rapidly over the last decade [1], [26]. The use of networks offers many advantages (low installation and maintenance costs, flexibility, etc.). However, it also induces side effects such as time-varying delays, aperiodic sampling or packet dropouts which have undesirable effects on the system performance and stability. Both continuous-time and discrete-time models have been developed for the purpose of the stability analysis of NCSs. For continuous-time NCS models, several approaches can be distinguished: the work in [7], [18], [24], [25] on the Lyapunov-Krasovskii functional method, the robust parametric modeling of the delay operator [14] and the impulsive delay differential equations NCS approach [17]. The majority of NCS models are based on the exact discretization of the continuous-time linear plant over a sample interval (see [2], [5], [6], [9], [15], [20], [26] and the references therein). In the literature, the authors treat the case when the variation of the delay is smaller than the sampling period. In this case, the analysis/control design problems can be addressed by using robust control methods for parametric uncertainties [2], [11] or by applying the Lyapunov-Krasovskii function (LKF) approach [19], [21]–[23] (for the LKF approach in discrete-time, see [8]). In this context, the main problems are the conservatism inherent to the use of upper boundings in the increment of the LKF and the reduced applicability of the results since they are able to deal only with delay variations smaller than the sampling interval. Generalizing such models to the case of large delay variations, packet dropouts and time-varying sampling intervals is not a trivial task.

In the current paper, we propose a discrete-time NCS model that can deal simultaneously with packet dropouts and time-varying delays smaller and larger than a possibly time-varying sampling interval. This model is obtained using exact discretization over a sampling interval and also takes into account the complicated case in which the delay variations may be larger than the sampling interval. The possibility of packet dropouts is modeled explicitly. Based on this model, stability analysis conditions in terms of linear matrix inequalities (LMIs) will be derived, using both a common quadratic and a parameter dependent Lyapunov approach. Note that recently, in [12], a simplified event-based discrete-time model has been proposed using the system's representation at both sampling and actuation times. The advantage of the model presented in this paper in comparison to this event-based model is that it generally leads to a discrete-time representation of a smaller dimension. Moreover, it generalizes several of the models that exist in the literature to the case in which all the mentioned network effects appear simultaneously. This enables the theoretical comparison with existing approaches. A discussion on the stability characterization based on LKFs and on parameter dependent Lyapunov functions (PDLF) will be given. This discussion is inspired by the results in [13] in which a comparison between LKFs and Lyapunov functions for switched systems is presented in the case of difference equations with time-varying delays in the state. The approach in [13] can deal only with delays that are a multiple of the sampling time, and therefore it does not apply to continuous-time systems as the NCSs studied here. We show that the stability analysis based on the most general LKF of a quadratic type is always more conservative than the novel stability characterization presented here. This result applies to the context of NCSs in which we are faced with an interaction between continuous-time systems and discrete-time controllers under different perturbing networked effects. In particular, we will show that the existence of general LKFs as used in the literature, implies also the existence of a Lyapunov function in our framework.

This paper is structured as follows: In Section II we present our NCS model. Section III is dedicated to the theoretical comparison of stability characterizations. It also presents LMI-based stability analysis methods that are illustrated by numerical examples in Section IV. Section V closes the paper with concluding remarks.

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II. NCS MODELING

In this section, the discrete-time model of a NCS including delays larger than the uncertain, and time-varying sampling interval and packet dropouts is presented. The NCS consists of a linear continuous-time plant

$$\dot{x}(t) = Ax(t) + Bu^*(t), \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, and a discrete-time static time-invariant controller, which are connected over a communication network that induces network delays (e.g. sensor-to-controller delay, controller-to-actuator delay). The state measurements are sampled resulting in the sampling time instants s_k given by: $s_k = \sum_{i=0}^{k-1} h_i$, $\forall k \geq 1$, and $s_0 = 0$, which are non-equidistantly spaced in time due to the time-varying sampling intervals h_k, $h_{\min} \leq h_k \leq h_{\max}$. The sequence of sampling instants is strictly increasing, i.e. $s_0 < s_1 < s_2 < \ldots$. We denote by $x_k := x(s_k)$ the k^{th} sampled value of x and by u_k the control value corresponding to x_k. Packet drops may occur. They are modeled by the parameter m_k ($m_k = 1$ if x_k and/or u_k is lost, $m_k = 0$ otherwise). We assume that the number of subsequent dropouts is upper bounded by $\overline{\delta}$. The zero-order-hold (ZOH) function transforms the discrete-time input u_k to a continuous-time control $u^*(t)$ being the actual actuation signal of the plant. We assume that the sensor acts in a time-driven fashion and that both the controller and the actuator act in an event-driven fashion (i.e. responding instantaneously to newly arrived data). Furthermore, we consider that not all the data are used due to packet dropouts and message rejection and that the most recent control input remains active in the plant if a packet is dropped. Under these assumptions, all delays (sensor-to-controller, controller-to-actuator and control computation delays) can be captured by a single delay τ_k, $\tau_{\min} < \tau_k < \tau_{\max}$ (see [26]). Define $\underline{d} := \lfloor \frac{\tau_{\min}}{h_{\max}} \rfloor$, the largest integer smaller than or equal to $\frac{\tau_{\min}}{h_{\max}}$ and $\overline{d} := \lceil \frac{\tau_{\max}}{h_{\min}} \rceil$, the smallest integer larger than or equal to $\frac{\tau_{\max}}{h_{\min}}$. Then, the control action $u^*(t)$ in the sampling interval $[s_k, s_{k+1})$ is described by

$$u^*(t) = u_{k+j-\overline{d}-\overline{\delta}} \text{ for } t \in [s_k + t_j^k, s_k + t_{j+1}^k), \tag{2}$$

where $t_j^k \in [0, h_k]$, $j = 0, \ldots, \overline{d} + \overline{\delta} - \underline{d}$ represent the actuation update instants in the considered sampling interval. The discrete-time NCS model can be given by:

$$x_{k+1} = e^{Ah_k} x_k + \sum_{j=0}^{\overline{d}+\overline{\delta}-\underline{d}} \int_{h_k - t_{j+1}^k}^{h_k - t_j^k} e^{As} ds B u_{k+j-\overline{d}-\overline{\delta}}. \tag{3}$$

The parameters t_j^k can be bounded in the interval $[0, h_{max}]$. We denote by $t_{j,\min}$ and $t_{j,\max}$, the minimum and maximum values of the t_j^k parameters, respectively. Explicit methods for the computation of these bounds as functions of $\delta, h_{min}, h_{max}, \tau_{min}$ and τ_{max} are given in [3]. Let θ_k denote the vector of uncertain parameters consisting of the sampling interval and the actuation instants at discrete time k: $\theta_k := (h_k, t_1^k, \ldots, t_{\overline{d}+\overline{\delta}-\underline{d}}^k)$ The set in which these parameters evolve is denoted by:

$$\begin{aligned} \Theta \;=\; & \{\theta_k \in \mathbb{R}^{\overline{d}+\overline{\delta}-\underline{d}+1} \,| h_k \in [h_{\min}, h_{\max}], \\ & t_j^k \in [t_{j,\min}, t_{j,\max}],\ 1 \leq j \leq \overline{d} + \overline{\delta} - \underline{d}, \\ & 0 \leq t_1^k \leq \ldots \leq t_{\overline{d}+\overline{\delta}-\underline{d}}^k \leq h_k\}. \end{aligned} \tag{4}$$

System (3) represents a discrete-time system with multiple delays in the input. Moreover, the system matrices are time-varying according to the uncertain parameters $\theta_k \in \Theta$. In the following section, we will show how to characterize the stability of this system based on LMIs and compare this to the Lyapunov-Krasovskii Function (LKF) approach.

III. STABILITY CHARACTERIZATIONS AND RELATIONS WITH LKF-BASED THEORY

In this section we discuss the stability characterization for the NCS (3) with a state feedback of the form

$$u_k = -\overline{K} x_k. \tag{5}$$

We can without loss of generality assume that $\overline{K}$ has a full row rank. When it is not the case, it is always possible to write the controller in the form

$$u_k = \begin{pmatrix} u_k^{a\,T} & u_k^{b\,T} \end{pmatrix}^T = \begin{pmatrix} I & G^T \end{pmatrix}^T K_a x_k = \begin{pmatrix} I & G^T \end{pmatrix}^T u_k^a,$$

where K_a has full row rank (using a permutation of the inputs) and we obtain a model similar to (3) that does satisfy the full row rank condition on the feedback gain, with K_a instead of $\overline{K}$, $B \begin{pmatrix} I & G^T \end{pmatrix}^T$ instead of B and u_k^a instead of u_k.

To render the model (3) with the feedback (5) suitable for analysis, we consider an equivalent delay-free model, based on a lifted state vector $\xi_k = \begin{pmatrix} x_k^T & u_{k-1}^T & \ldots & u_{k-\overline{d}-\overline{\delta}}^T \end{pmatrix}^T$ that includes past system inputs. This leads to the lifted model

$$\xi_{k+1} = \tilde{A}_1(\theta_k) \xi_k, \tag{6}$$

where $\tilde{A}_1(\theta_k) =$

$$\begin{pmatrix} \Lambda(\theta_k) & \widetilde{M}_{\overline{d}+\overline{\delta}-1}(\theta_k) & \widetilde{M}_{\overline{d}+\overline{\delta}-2}(\theta_k) & \ldots & \widetilde{M}_1(\theta_k) & \widetilde{M}_0(\theta_k) \\ -\overline{K} & 0 & 0 & \ldots & 0 & 0 \\ 0 & I & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 \\ 0 & \ldots & \ldots & 0 & I & 0 \end{pmatrix}$$

with $\Lambda(\theta_k) = e^{Ah_k} - \tilde{M}_{\overline{d}+\overline{\delta}}(\theta_k)\overline{K}$ and

$$\widetilde{M}_j(\theta_k) = \begin{cases} \displaystyle\int_{h_k - t_{j+1}^k}^{h_k - t_j^k} e^{As} ds B & \text{if } 0 \leq j \leq \overline{d} + \overline{\delta} - \underline{d}, \\ 0 & \text{if } \overline{d} + \overline{\delta} - \underline{d} < j \leq \overline{d} + \overline{\delta}. \end{cases} \tag{7}$$

The goal of this section is twofold: to derive LMI-based conditions for stability of NCS and to prove that characterizing the stability of the closed-loop NCS (3) using the lifted model (6) and (parameter dependent) quadratic Lyapunov functions is less conservative than the methods available in the literature based on discrete-time LKF. In order to show the latter, we will use an alternative lifted state space model as an intermediate step in the proof. This model uses the state vector $\chi_k = \left(x_k^T \ x_{k-1}^T \ \dots \ x_{k-\overline{d}-\overline{\delta}}^T\right)^T$, i.e.

$$\chi_{k+1} = \tilde{A}_2(\theta_k)\chi_k, \tag{8}$$

where $\tilde{A}_2(\theta_k) =$

$$\begin{pmatrix} \Lambda(\theta_k) & -\widetilde{M}_{\overline{d}+\overline{\delta}-1}(\theta_k)\overline{K} & -\widetilde{M}_{\overline{d}+\overline{\delta}-2}(\theta_k)\overline{K} & \dots & -\widetilde{M}_0(\theta_k)\overline{K} \\ I & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & I & 0 \end{pmatrix}.$$

This second lifted model is important since it is easy to show that if there exists a LKF (even the most general LKF that can be obtained using quadratic terms), then there exists a parameter dependent quadratic Lyapunov function for (8), as well. This relation will be described in detail at the end of the section. First we will show that the existence of a parameter dependent Lyapunov function for (8) is equivalent to the existence of a parameter dependent Lyapunov function for (6). This issue is relevant since it would formally prove that we can base the stability analysis for the NCS (3) with state feedback controller (5) on (6) without loosing stability properties that could be obtained via (8). Note that the state dimension of ξ_k in (6) is smaller than the dimension of χ_k in (8), which clearly has both modeling and numerical advantages.

A. *Equivalence of stability characterizations for the two lifted models*

Let us discuss the equivalence of the lifted models (8) and (6) with respect to stability and Lyapunov functions in more detail. Clearly, for a given *constant* parameter θ, the stability of (8) is equivalent to the stability of (6) and vice versa. Moreover, since for linear time-invariant systems the existence of a quadratic Lyapunov is a necessary and sufficient stability condition, there exists a quadratic Lyapunov function for (8) if and only if there exists one for (6) when θ is constant. However, assuming that there exists a quadratic Lyapunov function for one of the systems, (8) or (6), there is no constructive method available in the literature for deducing a Lyapunov function for the other one. As follows we will provide such a constructive method. Moreover, we will even consider a more general case of this problem as (8) and (6) are uncertain systems that vary over time as θ_k is changing. In this case, quadratic Lyapunov functions are known to be sufficient only for characterizing stability, but not necessary. The question now is whether, in the time-varying uncertain case, the existence of a quadratic Lyapunov function for system (8) is equivalent to the existence of a quadratic Lyapunov function for (6). In Theorem 2, we will answer this question and we will show that there exists a quadratic-like Lyapunov function for system (6) if and only if there exists one for the alternative representation (8). To prove this result for any parameter dependent quadratic Lyapunov function, the following lemma will be needed.

Lemma 1: Consider the matrix $R \in \mathbb{R}^{q\times p}$ and the matrices $A(\theta) \in \mathbb{R}^{p\times p}$ that depend continuously on $\theta \in \Theta$, where $\Theta \subset \mathbb{R}^l$ is a compact set. Define the matrices

$$\bar{A}(\theta) = \begin{pmatrix} A(\theta) & 0 \\ R & 0 \end{pmatrix} \in \mathbb{R}^{(p+q)\times(p+q)}, \tag{9}$$

for $\theta \in \Theta$. The following statements are equivalent:

- There exist symmetric positive definite matrices $P(\theta) \in \mathbb{R}^{p+q\times p+q}$, $\theta \in \Theta$ such that
$$\bar{A}(\theta_1)^T P(\theta_2)\bar{A}(\theta_1) - P(\theta_1) < 0, \ \forall\, \theta_1, \theta_2 \in \Theta. \tag{10}$$
- There exist symmetric positive definite matrices $Q(\theta) \in \mathbb{R}^{p\times p}$, $\theta \in \Theta$ such that
$$A(\theta_1)^T Q(\theta_2)A(\theta_1) - Q(\theta_1) < 0, \ \forall\, \theta_1, \theta_2 \in \Theta. \tag{11}$$

Moreover, there exists a common solution $P(\theta) = P > 0$, for all $\theta \in \Theta$ to (10) if and only if there exists a common solution $Q(\theta) = Q > 0$ to (11).

Proof: Suppose that (10) holds for some matrices $P^T(\theta) = P(\theta) > 0, \forall \theta \in \Theta$. Decompose the matrices as follows:

$$P(\theta) = \begin{pmatrix} P_1(\theta) & P_2(\theta) \\ P_2^T(\theta) & P_3(\theta) \end{pmatrix}$$

in accordance with the matrix $\bar{A}(\theta)$. By expanding (10) we obtain for all $\theta_1, \theta_2 \in \Theta$ that the matrix

$$\begin{pmatrix} \begin{bmatrix} A^T(\theta_1)P_1(\theta_2)A(\theta_1) - P_1(\theta_1) + R^T P_2(\theta_2)A(\theta_1) \\ + A^T(\theta_1)P_2(\theta_2)R + R^T P_3(\theta_2)R \end{bmatrix} & -P_2(\theta_1) \\ -P_2^T(\theta_1) & -P_3(\theta_1) \end{pmatrix}$$

is negative definite. This is equivalent (using the Schur lemma) to

$$\begin{aligned} & A^T(\theta_1)P_1(\theta_2)A(\theta_1) - P_1(\theta_1) + R^T P_2(\theta_2)A(\theta_1) + \\ & A^T(\theta_1)P_2(\theta_2)R + R^T P_3(\theta_2)R + \\ & P_2(\theta_1)P_3^{-1}(\theta_1)P_2^T(\theta_1) < 0. \end{aligned}$$

Adding and subtracting $A^T(\theta_1)P_2(\theta_2)P_3^{-1}(\theta_2)P_2^T(\theta_2)A(\theta_1)$ to the previous inequality implies for all $\theta_1, \theta_2 \in \Theta$ that

$$A^T(\theta_1)Q(\theta_2)A(\theta_1) - Q(\theta_1) + W(\theta_1, \theta_2) < 0,$$

where $Q(\theta) = P_1(\theta) - P_2(\theta)P_3^{-1}(\theta)P_2^T(\theta)$, $\forall\, \theta \in \Theta$, and

$$\begin{aligned} W(\theta_1, \theta_2) \ &= \ (P_2(\theta_2)A(\theta_1) + P_3(\theta_2)R)^T \times P_3^{-1}(\theta_2) \\ &\quad \times (P_2(\theta_2)A(\theta_1) + P_3(\theta_2)R). \end{aligned}$$

As $W(\theta_1, \theta_2) \geq 0$ and $Q(\theta) > 0$ (since $P_3(\theta) > 0$ and $Q(\theta)$ is the Schur complement of $P(\theta)$), clearly $Q(\theta)$, $\theta \in \Theta$, satisfy condition (11). Notice that when the matrices $P(\theta)$ are constant, i.e.

$$P(\theta) = P = \begin{pmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{pmatrix}, \ \forall \theta \in \Theta,$$

the corresponding matrices $Q(\theta)$ that satisfy (11) are constant as well, $Q(\theta) = Q = P_1 - P_2P_3^{-1}P_2^T$, $\theta \in \Theta$.

To prove the converse, assume that (11) holds. Then, due to the continuity of A with respect to θ and to the compactness of Θ, there exists $\epsilon > 0$ such that for all $\theta_1, \theta_2 \in \Theta$

$$\begin{pmatrix} A^T(\theta_1)Q(\theta_2)A(\theta_1) - Q(\theta_1) + \epsilon R^T R & 0 \\ 0 & -\epsilon I \end{pmatrix} < 0.$$

This inequality shows that the matrices $P(\theta)$ defined as

$$P(\theta) = \begin{pmatrix} Q(\theta) & 0 \\ 0 & \epsilon I \end{pmatrix} > 0,\ \theta \in \Theta$$

satisfy (10). Clearly, when $Q(\theta) = Q,\ \theta \in \Theta$, the common matrix

$$P(\theta) = P = \begin{pmatrix} Q & 0 \\ 0 & \epsilon I \end{pmatrix} > 0,\ \theta \in \Theta$$

satisfies the inequality (10), which completes the proof. ■

Theorem 2: Consider the NCS (3) with state feedback controller (5) and the two representations (6) and (8). The following statements are equivalent :

- There exist symmetric positive definite matrices $P(\theta)$, $\theta \in \Theta$ such that

  $$\tilde{A}_1^T(\theta_k) P(\theta_{k+1}) \tilde{A}_1(\theta_k) - P(\theta_k) < 0, \tag{12}$$

  for all $\theta_k, \theta_{k+1} \in \Theta$, thus

  $$V(\xi_k) = \xi_k^T P(\theta_k) \xi_k \tag{13}$$

  is a parameter dependent Lyapunov function for system (6).
- There exist symmetric positive definite matrices $Q(\theta)$, $\theta \in \Theta$ such that

  $$\tilde{A}_2^T(\theta_k) Q(\theta_{k+1}) \tilde{A}_2(\theta_k) - Q(\theta_k) < 0, \tag{14}$$

  for all $\theta_k, \theta_{k+1} \in \Theta$, thus

  $$V(\chi_k) = \chi_k^T Q(\theta_k) \chi_k \tag{15}$$

  is a parameter dependent Lyapunov function for system (8). Moreover, system (6) has a common quadratic Lyapunov function $V(\xi_k) = \xi_k^T P \xi_k$ if and only if system (8) has a common quadratic Lyapunov function $V(\chi_k) = \chi_k^T Q \chi_k$.

Proof: Since the state feedback matrix $\overline{K}$ has full row rank there exists a matrix $S \in \mathbb{R}^{(n-m)\times n}$ such that the matrix $\begin{pmatrix} \overline{K}^T & S^T \end{pmatrix}^T$ is invertible. Define the matrices $\tilde{A}_3(\theta_k) =$

$$\left(\begin{array}{c|cccc|c|c|c}
\Lambda(\theta_k) & \widetilde{M}_{\overline{d}+\overline{\delta}-1}(\theta_k) & \dots & \dots & \widetilde{M}_1(\theta_k) & \widetilde{M}_0(\theta_k) & 0 & 0 \\ \hline
-\overline{K} & 0 & \dots & \dots & 0 & 0 & 0 & 0 \\ \hline
0 & I & 0 & \dots & 0 & 0 & 0 & 0 \\
\vdots & 0 & \ddots & & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & & \ddots & 0 & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & I & 0 & 0 & 0 \\ \hline
S & 0 & \dots & \dots & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & \dots & \dots & 0 & 0 & I & 0
\end{array}\right)$$

$$W = \left(\begin{array}{c|cccc|c}
I & 0 & \dots & \dots & 0 & 0 \\ \hline
0 & -\overline{K} & 0 & \dots & 0 & 0 \\
\vdots & 0 & \ddots & & \vdots & \vdots \\
\vdots & \vdots & & \ddots & 0 & \vdots \\
0 & 0 & \dots & 0 & -\overline{K} & 0 \\ \hline
0 & 0 & \dots & \dots & 0 & -\overline{K} \\ \hline
0 & S & 0 & \dots & 0 & 0 \\
\vdots & 0 & \ddots & & \vdots & \vdots \\
\vdots & \vdots & & \ddots & 0 & \vdots \\
0 & 0 & \dots & 0 & S & 0 \\ \hline
0 & 0 & \dots & \dots & 0 & S
\end{array}\right).$$

Notice that $\tilde{A}_3(\theta_k) W = W \tilde{A}_2(\theta_k) =$

$$\left(\begin{array}{c|cccc|c}
\Lambda(\theta_k) & -\widetilde{M}_{\overline{d}+\overline{\delta}-1}(\theta_k)\overline{K} & \dots & \dots & -\widetilde{M}_1(\theta_k)\overline{K} & -\widetilde{M}_0(\theta_k)\overline{K} \\ \hline
-\overline{K} & 0 & \dots & \dots & 0 & 0 \\ \hline
0 & -\overline{K} & 0 & \dots & 0 & 0 \\
\vdots & 0 & \ddots & & \vdots & \vdots \\
\vdots & \vdots & & \ddots & 0 & \vdots \\
0 & 0 & \dots & 0 & -\overline{K} & 0 \\ \hline
S & 0 & \dots & \dots & 0 & 0 \\ \hline
0 & S & 0 & \dots & 0 & 0 \\
\vdots & 0 & \ddots & & \vdots & \vdots \\
\vdots & \vdots & & \ddots & 0 & \vdots \\
0 & 0 & \dots & 0 & S & 0
\end{array}\right).$$

This implies that $\tilde{A}_3(\theta_k)$ is similar to $\tilde{A}_2(\theta_k)$. It is easy to show that (14) holds if and only if there exists symmetric positive definite matrices $\tilde{P}(\theta_k) = W^{-T} Q(\theta_k) W^{-1}$ such that $\tilde{A}_3^T(\theta_k) \tilde{P}(\theta_{k+1}) \tilde{A}_3(\theta_k) - \tilde{P}(\theta_k) < 0$. Furthermore, notice that $\tilde{A}_3(\theta_k)$ can be expressed as

$$\tilde{A}_3(\theta_k) = \left(\begin{array}{c|c|c}
\tilde{A}_1(\theta_k) & 0 & 0 \\ \hline
S \mid 0 & 0 & 0 \\ \hline
0 & I & 0
\end{array}\right).$$

Then, apply Lemma 1 with

$$A(\theta) := \left(\begin{array}{c|c}
\tilde{A}_1(\theta_k) & 0 \\ \hline
S \mid 0 & 0
\end{array}\right) \quad \text{and } R := \left(\begin{array}{c|c} 0 & I \end{array}\right).$$

Next apply Lemma 1 again for $A(\theta) := \tilde{A}_1(\theta_k)$ and $R := (S\ 0)$ in order to complete the proof. ■

B. Relations with the Lyapunov-Krasovskii stability characterization

For discrete-time uncertain systems with delay in the input such as (3), several stability results exist based on Lyapunov-Krasovskii functions (LKFs). Using an adequate partition of the Lyapunov matrix

$$Q(\theta_k) = \begin{pmatrix}
Q^{0,0}(\theta_k) & Q^{0,1}(\theta_k) & \dots & Q^{0,\overline{d}+\overline{\delta}}(\theta_k) \\
Q^{0,1}(\theta_k) & Q^{1,1}(\theta_k) & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
Q^{0,\overline{d}+\overline{\delta}}(\theta_k) & \dots & \dots & Q^{\overline{d}+\overline{\delta},\overline{d}+\overline{\delta}}(\theta_k)
\end{pmatrix}, \tag{16}$$

it can be shown that the Lyapunov function (15) is equivalent to the LKF

$$V(x_k, \dots, x_{k-\overline{d}-\overline{\delta}}) = \sum_{i=0}^{\overline{d}+\overline{\delta}} \sum_{j=0}^{\overline{d}+\overline{\delta}} x_{k-i}^T Q^{i,j}(\theta_k) x_{k-j}, \tag{17}$$

which is the most general LKF that can be obtained using quadratic forms. Any of the quadratic LKFs found in the literature (see [19], [21]–[23]) are a particular case of (17). As a consequence of Theorem 2, we know that there exists a Lyapunov function (15) for (8) iff there exists one of the form (13) for (6), i.e. iff the equations (12) are satisfied. Consequently, condition (12) represents a necessary and sufficient condition for the existence of the most general form of LKFs that can be obtained using quadratic terms as in (17). Hence, using a stability characterization based on the model (6) is less conservative than the stability analysis results based on quadratic LKF that are available in the literature [19], [21]–[23].

In the next subsection, we will present a constructive LMI method for stability analysis using characterizations based on parameter dependent Lyapunov functions such as in (13).

C. LMI stability conditions

To derive the LMI stability conditions, one has to deal with the non-linear representation of the *exponential uncertainties* (7) in the model (6). Several methods exist in the literature for dealing with such uncertainties. The basic idea is to embed this uncertainty in a more classical parametric uncertainty, by considering a polytopic approximation of the convexhull of $\tilde{A}_1(\theta_k)$. Such an embedding aims at finding a set of ζ matrices $H_i, i = 1, \ldots, \zeta$, such that $\tilde{A}_1(\theta_k) \in \mathcal{H} = co\{H_1, \ldots, H_\zeta\}$, for all $\theta_k \in \Theta$. Analytical methods for deriving such a representation can be found in literature based on the Taylor series expansion [11], Jordan decomposition [2] or the application of the Caley-Hamilton lemma [10]. Using such an over-approximation with a finite number of ζ vertices, H_j, a finite number of LMI stability conditions can be obtained based on the reformulation of the stability analysis method developed in [4] for polytopic systems.

Theorem 3: Consider the NCS model (6). If there exist matrices $P_j = P_j^T > 0$, $j = 1, 2, \ldots, \zeta$, that satisfy

$$H_j^T P_l H_j - P_j < 0,$$

for all $j, l \in \{1, 2, \ldots, \zeta\}$, then the closed-loop NCS (6) is globally asymptotically stable (GAS).

This theorem is based on the existence of a Lyapunov function that changes according to the unknown parameters θ_k, i.e. $V(\xi_k) = \xi_k^T P(\theta_k)\xi_k$, with $P(\theta_k) = \sum_{j=1}^{\zeta} \mu_j(\theta_k) P_j$, where $\mu_j(\theta_k)$ represent the barycentric coordinates of $\tilde{A}_1(\theta_k)$ in the polytope $\mathcal{H}$, i.e. $0 \leq \mu_j(\theta_k) \leq 1, \sum_{j=1}^{\zeta} \mu_j(\theta_k) = 1$. Using the results from the previous section, this shows that, if the LMIs in Theorem 3 are satisfied, they imply the existence of a LKF of the form (17). Notice that using this approach we avoid the conservative upper boundings in the difference of the LKF, which are usually encountered in the literature to arrive at LKF-based stability conditions in LMI form. The case of a common quadratic Lyapunov function (CQLF) $V(\xi) = \xi_k^T P \xi_k$ is a particular case of this theorem by taking $P_j = P, \ \forall \ j = 1, \ldots, \zeta$.

Similarly to the results in [3], [11], it can be shown that the previous theorem also guarantees the stability of continuous-time NCS.

IV. Illustrative examples

In this section we will present several examples that illustrate the approach presented in this paper and compare it with other approaches in the literature. The convex embedding has been constructed based on the Taylor expansion [11].

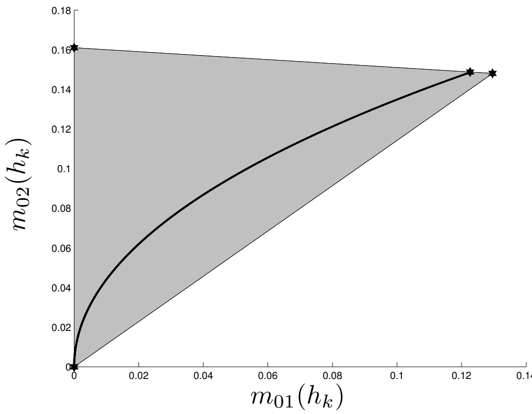

Fig. 1. Illustration of the construction of the convex embedding using the Taylor expansion for the system in Example 1. In this case we have only one exponential uncertainty $M_0(\theta_k) = \int_0^{h_k} e^{As} ds B$ in equation (7) that represents a 2D vector $M_0(h_k) = (m_{01}(h_k), m_{02}(h_k))^T$ that depends only on the sampling period h_k. This uncertainty is represented by the dark curve in the figure. The polytope represents the convex polytope and the vertex are marked by stars.

A. Variation on the sampling period ($\tau_{min} = \tau_{max} = \overline{\delta} = 0$)

We consider now the case where the system (1) is described by the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -0.1 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}.$$

and the controller in (5) is given with $\overline{K} = -(3.75 \ 11.5)$. For this example, the results in [26], [7] and [16] indicated the that the system is stable for any time-varying sampling period in the interval $(0, 0.0593]$, $(0, .86]$ and $(0, 1.36]$, respectively. We construct a convex embedding with 4 vertices based on a 3th-order Taylor expansion (a graphical illustration of this embedding is given in Figure 1). Using the approach provided here, we can show that the system is GAS for $h \in (0.01, 1.72]$. In fact, for a constant sampling interval $h = 1.73$ the equivalent discrete-time system is unstable[1], which illustrates that the stability characterisation proposed in this paper is hardly conservative in this example.

B. Comparison with the LKF approach

Consider a NCS (1) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(i.e. a double integrator), $\overline{K} = \begin{pmatrix} 0.0032 & 0.1085 \end{pmatrix}$, a constant sampling interval $h_{min} = h_{max} = h = 10$ and $\tau_k < h$. We consider the case of $\overline{\delta} = 0$, i.e. packet dropouts do not occur. The system is GAS for a constant time delay up to $\tau_{max} = 7.05$. In [22] it has been shown that the system is stable for a time varying-delay in the interval $(0.2, 1.4]$. Using the approach proposed here we can show that the system is stable for any time-varying delay in the interval $(0, 6.49]$. Alternatively, we can show that the systems is stable for any delay in $(6.89, 7.05]$.

[1] This interval has also been recently confirmed by [9].

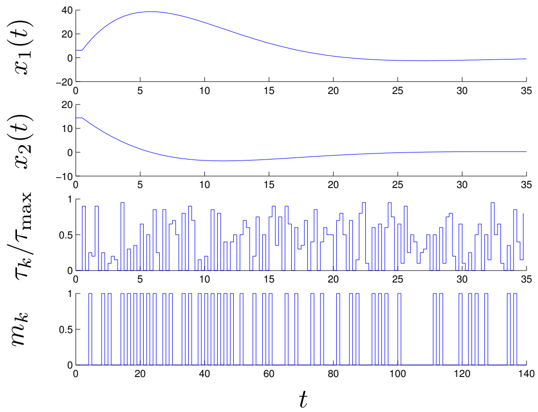

Fig. 2. Time response with delay and packet-dropouts for $h = 0.25$, $\tau_{\max} = h$ and $\overline{\delta} = 1$.

C. Large delays and packet dropouts

Consider the system from the previous example with $K = \begin{pmatrix} 0.0363 & 0.2525 \end{pmatrix}$. We consider time-delays larger than the constant sampling interval $h_{min} = h_{max} = h$. In this case $\tau_{\max} = 2.8h$ and $\tau_{\min} = 0$. Note that the same results hold also for the situation with packet dropouts $\overline{\delta} = 1$ and $\tau_{\max} = 1.8h$ or $\overline{\delta} = 2$ and $\tau_{\max} = 0.8h$. In this case the GAS can be shown using the Theorem 3 for sampling intervals up to $h = 1.1$s. A simulation with both delay and packet dropouts is presented in Figure 2 for $h = 0.25$, $\tau_{\max} = h$ and $\overline{\delta} = 1$ which shows GAS of the NCS as confirmed by our theory.

V. Conclusions

A discrete-time NCS model, based on an exact discretization of the continuous-time linear plant at the sampling instants, is presented. This model includes various relevant network phenomena: the presence of time-varying delays that may be larger than the sampling interval, message rejection, packet dropouts, and variations in the sampling interval. Based on this model, stability characterizations using parameter dependent Lyapunov functions are proposed. It is proven that the stability characterizations presented here are less conservative than the methods available in the literature based on Lyapunov-Krasovskii functions. Exploiting the developed model and the proposed stability characterizations, stability conditions are derived in terms of LMIs. Examples illustrate that these stability results are less conservative than those found in the literature.

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