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Stability Analysis of Networked Control Systems with Direct-Feedthrough Terms: Part II – The Linear Case

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Abstract— **We consider networked control systems (NCSs) composed of a linear plant and a linear controller with direct-feedthrough terms, i.e., terms that directly connect the plant's input and output from/to the controller with each other and the controller's input and output from/to the plant with each other. The presence of such direct-feedthrough terms generates non-trivial difficulties in terms of the modeling and the analysis of NCSs. In particular, a novel stability analysis is required to address standard scheduling protocols such as the sampled-data (SD), try-once-discard (TOD), and round-robin (RR) protocols. Hereto, we will take a renewed look at the concept of uniformly globally exponentially stable (UGES) scheduling protocols for these standard scheduling protocols as used in literature, such that the direct-feedthrough terms can be incorporated in the system configuration. The application of our results are illustrated using the benchmark example of a batch reactor.**

I. Introduction

In many applications, including manufacturing plants, vehicles, and aircraft, communication is needed for the exchange of information and control signals between spatially distributed system components, such as supervisory computers, controllers, and intelligent input-output (I/O) devices. When sensor and actuator data is communicated over a shared (wired or wireless) packet-based communication network, the system is called a *networked control system* (NCS). Such NCSs have received considerable attention in recent years [1]–[3]. This interest is motivated by the many advantages their flexible architectures offers, such as reduced installation and maintenance costs, when compared to conventional control systems in which sensor and actuation data is transmitted over dedicated point-to-point (wired) links, see, e.g., [4]. Additionally, wireless communication is able to overcome the physical limitations of employing wired links, which is very appealing in, for instance, intelligent transportation, see, e.g., [5].

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On the other side, the usage of packet-based networked communication comes also with the inevitable network-induced imperfections, such as varying delays, dropouts, varying transmission intervals, and so on. Sensor and actuator data need to be quantized and cannot be transmitted continuously, but only at discrete time instants. Moreover, as the communication network is often shared by multiple sensors and actuators, there is a need for so-called scheduling protocols, which govern the access of the nodes to the network. To deal with all these networked-induced phenomena, several modeling frameworks have been developed. In this paper we are particularly interested in the framework where the NCS is modeled as a hybrid inclusion. Based on this hybrid framework, conditions to guarantee overall stability or $\mathcal{L}_p$-gain performance of the NCS have been derived, see [6]–[9]. In addition to this general setup, many extensions can be found in [10]–[13], and the references therein.

Unfortunately however, it appears that the aforementioned results of [6]–[9] always avoided the inclusion of so-called direct-feedthrough terms, i.e., terms that directly connect the plant's input and output from/to the controller with each other and the controller's input and output from/to the plant with each other, when *both* actuator and sensor signals are transmitted over the communication network. This is because these feedthrough terms lead to non-trivial difficulties in terms of modeling and analysis, a phenomenon that we will refer to as the *direct-feedthrough problem* in this paper. In particular, the presence of the feedthrough terms significantly modifies the model of the networked-induced error at transmissions, and, as such, requires a novel (stability) analysis to address standard scheduling protocols such as the sampled data (SD), try-once-discard (TOD), and round-robin (RR) protocols. As a result, the classes of NCSs that can be analyzed using the results of [6]–[9] cannot handle at present classical controllers such as, for instance, Proportional-Integral(-Derivative) (PI(D)) regulators.

Given the importance of PI(D) control and other control/plant structures with feedthrough terms, in [14] this direct-feedthrough problem was first addressed by showing that, for the case where *only* the controller contained feedthrough terms, stability of nonlinear NCSs could still be guaranteed when using standard scheduling protocols. In this paper, we will study NCSs with direct-feedthrough terms in *both* the plant as well as the controller, which introduces additional difficulties as we will see below. Therefore, as a starting point, we focus in this work on linear NCSs for which, based on [6]–[8], we will briefly provide a recap on the stability analysis results concerning uniform global exponential stability (UGES), however now slightly altered to take

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into account the presence of the direct-feedthrough terms. In addition, and more importantly, we will also revisit the concept of UGES scheduling protocols as introduced in [7] for the standard SD, TOD, and RR scheduling protocols and show that, under certain conditions, the direct-feedthrough terms can be incorporated in the NCS configuration and the stability analysis for UGES of [6]–[8] can still be applied. To illustrate the usefulness of our results we apply them to the benchmark numerical example of a batch reactor.

II. Notation

The sets of non-negative integers is denoted by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, and the set of non-negative real numbers by $\mathbb{R}_{\geq 0}$. For vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$, we denote by $(v_1, v_2, \ldots, v_n)$ the vector $\begin{bmatrix} v_1^\top & v_2^\top & \ldots & v_n^\top \end{bmatrix}^\top$, and by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and the usual inner product, respectively. Moreover, we use the notation $r^+ = r(t^+) = \lim_{\tau \downarrow t} r(\tau)$ where r is any left-continuous mapping from $\mathbb{R}$ to $\mathbb{R}^n$. The n by n identity and zero matrices are denoted by I_n and 0_n, respectively. When the dimensions are clear from the context, these notations are simplified to I and 0. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and, in addition, it is unbounded.

III. System description: The NCS model

In this section, the considered class of systems is introduced, where we in particular focus on the influence and impact of the direct-feedthrough terms on the NCS configuration as introduced in literature.

A. Networked control configuration

Consider the NCS as shown in Fig. 1, where the continuous-time plant $\mathcal{P}$ is given by

$$\mathcal{P} : \begin{bmatrix} \dot{x}_p \\ y \end{bmatrix} = \begin{bmatrix} A_\mathbf{P} & B_\mathbf{P} \\ C_\mathbf{P} & C_\mathbf{Y} \end{bmatrix} \begin{bmatrix} x_p \\ \hat{u} \end{bmatrix} \tag{1}$$

with the initial condition $x_p(0) = x_{0,p} \in \mathbb{R}^{m_{x_p}}$ and where $x_p \in \mathbb{R}^{m_{x_p}}$ denotes the state, $\hat{u} \in \mathbb{R}^{m_u}$ the most recently received control input, and $y \in \mathbb{R}^{m_y}$ the output to the controller.

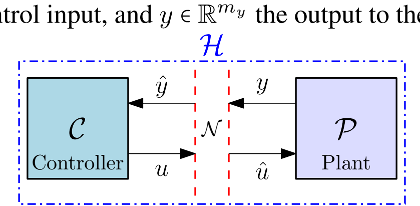

Fig. 1. The NCS setup as described in [6]–[8]. The plant $\mathcal{P}$ has its own network $\mathcal{N}$ to communicate with its controller $\mathcal{C}$. The overall "networked" (hybrid) system $\mathcal{H}$ is the combination of the plant $\mathcal{P}$, its controller $\mathcal{C}$, and its network $\mathcal{N}$.

As shown in Fig. 1, the plant $\mathcal{P}$ is controlled by its own controller $\mathcal{C}$, which communicate with each other via the communication network $\mathcal{N}$. The controller $\mathcal{C}$ is described by

$$\mathcal{C} : \begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} A_\mathbf{C} & B_\mathbf{C} \\ C_\mathbf{C} & C_\mathbf{U} \end{bmatrix} \begin{bmatrix} x_c \\ \hat{y} \end{bmatrix}, \tag{2}$$

where $x_c \in \mathbb{R}^{m_c}$ denotes the controller state, $\hat{y} \in \mathbb{R}^{m_y}$ the most recently received output measurement of the plant, and $u \in \mathbb{R}^{m_u}$ the controller output.

Note now that the difference between the setups in [6]–[12] and (1)-(2) is given by the matrices $C_\mathbf{U}$ and $C_\mathbf{Y}$ from which at least one of them is nonzero while *both* the control input u as well as the output to the controller y are transmitted over the communication network $\mathcal{N}$.

To complete the description of the NCS setup, it has to be explained how the communication network $\mathcal{N}$ operates. This network $\mathcal{N}$ has a collection of sampling/transmission times t_j, $j \in \mathbb{N}$, which satisfy $0 \leq t_1 < t_2 < \ldots$. In the considered setup, similar to [6]–[11], it is assumed that the transmission times satisfy $\delta \leq t_{j+1} - t_j \leq \tau_{mati}$ for all $j \in \mathbb{N}$, where $\delta > 0$ is a certain (arbitrarily small) constant to prevent Zeno behavior and τ_{mati} denotes the *maximally allowable transmission interval* (MATI). For the NCS, (parts of) the output y and input u are sampled and transmitted over the network $\mathcal{N}$ to the controller $\mathcal{C}$ and/or plant $\mathcal{P}$, respectively, at such a transmission time t_j. The network $\mathcal{N}$ might be subdivided in several (sensor and/or actuator) nodes, where each node corresponds to a subset of the entries $y/\hat{y}$ and/or $u/\hat{u}$. A scheduling protocol determines which of the nodes in the network is granted access to the network at a transmission time. After a node is granted access to the network, it collects and transmits the values of the corresponding entries in $y(t_j)$ and $u(t_j)$, which results in an update according to

$$\begin{aligned} \hat{y}\left(t_j^+\right) &= y(t_j) + h_y(j, e(t_j)) \\ \hat{u}\left(t_j^+\right) &= u(t_j) + h_u(j, e(t_j)), \end{aligned} \tag{3}$$

where the functions $h := (h_y, h_u)$ models the (local) network protocol and where e denotes the network-induced error defined by

$$e := \begin{bmatrix} e_y \\ e_u \end{bmatrix} = \begin{bmatrix} \hat{y} - y \\ \hat{u} - u \end{bmatrix}. \tag{4}$$

Finally, it is assumed that $\hat{y}$ and $\hat{u}$ are constant in between two successive transmissions, i.e., we assume a zero-order-hold (ZOH) fashion.

Note that, from this point forward, we will use the shorthand notations $\hat{y}^+ = \hat{y}\left(t_j^+\right)$, $\hat{u}^+ = \hat{u}\left(t_j^+\right)$, $y = y(t_j)$, $\hat{y} = \hat{y}(t_j)$, $u = u(t_j)$, $\hat{u} = \hat{u}(t_j)$, $x_p = x_p(t_j)$, $x_c = x_c(t_j)$, $e^+ = e\left(t_j^+\right)$, and $e = e(t_j)$ in all of the equations.

B. Updating the network-induced error e

Because of the presence of the direct-feedthrough terms $C_\mathbf{U}$ and $C_\mathbf{Y}$ in the networked interconnection, we have that u and y depend on the networked values $\hat{y}$ and $\hat{u}$, respectively. As a result, an update of $\hat{y}$ and $\hat{u}$ also results in a change of the values of y and u, i.e., we have that

$$y^+ = C_\mathbf{P} x_p + C_\mathbf{Y} \hat{u}^+ \quad \text{and} \quad u^+ = C_\mathbf{C} x_c + C_\mathbf{U} \hat{y}^+. \tag{5}$$

As a consequence, we encounter some non-trivial difficulties regarding the modeling of the update equation for the networked-induced error e. To put this into more context, consider the following analysis. Using the expressions of (1)-(2) and (4) it can be obtained that the errors e_y and e_u themselves are given by

$$\begin{aligned} e_y &= \hat{y} - y = \hat{y} - C_\mathbf{P} x_p - C_\mathbf{Y} \hat{u} \\ e_u &= \hat{u} - u = \hat{u} - C_\mathbf{C} x_c - C_\mathbf{U} \hat{y}. \end{aligned} \tag{6}$$

$$\mathbf{A} = \begin{bmatrix} A_{\mathbf{P}} + B_{\mathbf{P}} \left(I_{m_u} - C_{\mathbf{U}} C_{\mathbf{Y}}\right)^{-1} C_{\mathbf{U}} C_{\mathbf{P}} & B_{\mathbf{P}} \left(I_{m_u} - C_{\mathbf{U}} C_{\mathbf{Y}}\right)^{-1} C_{\mathbf{C}} \\ B_{\mathbf{C}} \left(I_{m_y} - C_{\mathbf{Y}} C_{\mathbf{U}}\right)^{-1} C_{\mathbf{P}} & A_{\mathbf{C}} + B_{\mathbf{C}} \left(I_{m_y} - C_{\mathbf{Y}} C_{\mathbf{U}}\right)^{-1} C_{\mathbf{Y}} C_{\mathbf{C}} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -C_{\mathbf{P}} & 0 \\ 0 & -C_{\mathbf{C}} \end{bmatrix} \mathbf{A},$$
$$\mathbf{E} = \begin{bmatrix} B_{\mathbf{P}} \left(I_{m_u} - C_{\mathbf{U}} C_{\mathbf{Y}}\right)^{-1} C_{\mathbf{U}} & B_{\mathbf{P}} \left(I_{m_u} - C_{\mathbf{U}} C_{\mathbf{Y}}\right)^{-1} \\ B_{\mathbf{C}} \left(I_{m_y} - C_{\mathbf{Y}} C_{\mathbf{U}}\right)^{-1} & B_{\mathbf{C}} \left(I_{m_y} - C_{\mathbf{Y}} C_{\mathbf{U}}\right)^{-1} C_{\mathbf{Y}} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} -C_{\mathbf{P}} & 0 \\ 0 & -C_{\mathbf{C}} \end{bmatrix} \mathbf{E}. \tag{13}$$

Consider now the situation that we have an update of our networked values at transmission time t_j, $j \in \mathbb{N}$, according to (3), i.e.,

$$\begin{aligned} \hat{y}^+ &= y + h_y(j,e) = C_{\mathbf{P}} x_p + C_{\mathbf{Y}} \hat{u} + h_y(j,e) \\ \hat{u}^+ &= u + h_u(j,e) = C_{\mathbf{C}} x_c + C_{\mathbf{U}} \hat{y} + h_u(j,e). \end{aligned} \tag{7}$$

By using (5)-(7), we derive that this update of the networked values leads to the error being updated according to

$$\begin{aligned} e_y^+ &= \hat{y}^+ - C_{\mathbf{P}} x_p - C_{\mathbf{Y}} \hat{u}^+ \\ &= C_{\mathbf{Y}} \hat{u} + h_y(j,e) - C_{\mathbf{Y}} C_{\mathbf{C}} x_c - C_{\mathbf{Y}} C_{\mathbf{U}} \hat{y} - C_{\mathbf{Y}} h_u(j,e) \\ &= h_y(j,e) - C_{\mathbf{Y}} h_u(j,e) + C_{\mathbf{Y}} e_u \\ e_u^+ &= \hat{u}^+ - C_{\mathbf{C}} x_c - C_{\mathbf{U}} \hat{y}^+ \\ &= C_{\mathbf{U}} \hat{y} + h_u(j,e) - C_{\mathbf{U}} C_{\mathbf{P}} x_p - C_{\mathbf{U}} C_{\mathbf{Y}} \hat{u} + C_{\mathbf{U}} h_y(j,e) \\ &= h_u(j,e) - C_{\mathbf{U}} h_y(j,e) + C_{\mathbf{U}} e_y. \end{aligned}$$

Hence, we have that, in general, the update equation of the error e can be described by using an update function $h_{df} : \mathbb{R}^{m_e} \to \mathbb{R}^{m_e}$, i.e,

$$e^+ = h(j,e) + \underbrace{\begin{bmatrix} 0_{m_y} & C_{\mathbf{Y}} \\ C_{\mathbf{U}} & 0_{m_u} \end{bmatrix} (e - h(j,e))}_{:= h_{df}(h(j,e),e)}. \tag{8}$$

From this result it thus follows that it is not straightforward to describe the update of the error e similar to the situation *without* direct-feedthrough terms, i.e., $C_{\mathbf{Y}} = 0$ and $C_{\mathbf{U}} = 0$, as described in [6]–[12], which resulted in

$$e^+ = h(j,e). \tag{9}$$

More precisely, we can conclude that the update property of (9) as studied in many previous papers, see [6]–[12], can a priori be lost because of the perturbative term described by the update function h_{df} induced by the feedthrough terms $C_{\mathbf{Y}}$ and $C_{\mathbf{U}}$. In particular, one must now keep in mind that a jump of a component of e may not represent an update of this component due to its transmission nature modeled by (8), which is a fundamental difference with the update property (9) considered in previous works where feedthrough-terms were ignored. As such, a careful analysis is needed for the considered NCSs described by (1)-(3). However, we would still like to base our analysis on previous results in literature. Therefore, similar to [6]–[12], we will first rewrite the modeling setup in the form of a hybrid system [15].

C. A hybrid modeling framework

Based on the above setup, the triple $(\mathcal{P}, \mathcal{C}, \mathcal{N})$ can be rewritten into the format of a hybrid system $\mathcal{H}$, as described in [6]–[8], where each jump of the hybrid system corresponds to an update of the networked values according to (3). To do so, we need to eliminate the control variables $u/\hat{u}$ and $y/\hat{y}$ from the state dynamics, or in other words, we need to express the networked values in terms of the state $x := (x_p, x_c)$ and the error e. Based on (1), (2), and (4), we obtain by using Neumann series, see, e.g., [16], that

$$\begin{aligned} \hat{u} &= \left(I_{m_u} - C_{\mathbf{U}} C_{\mathbf{Y}}\right)^{-1} \left(C_{\mathbf{C}} x_c + C_{\mathbf{U}} C_{\mathbf{P}} x_p + C_{\mathbf{U}} e_y + e_u\right) \\ \hat{y} &= \left(I_{m_y} - C_{\mathbf{Y}} C_{\mathbf{U}}\right)^{-1} \left(C_{\mathbf{P}} x_p + C_{\mathbf{Y}} C_{\mathbf{C}} x_c + C_{\mathbf{Y}} e_u + e_y\right), \end{aligned} \tag{10}$$

provided that the inverses as in (10) exist and

$$\lim_{n \to \infty} \left(C_{\mathbf{U}} C_{\mathbf{Y}}\right)^n = 0 \quad \text{and} \quad \lim_{n \to \infty} \left(C_{\mathbf{Y}} C_{\mathbf{U}}\right)^n = 0.$$

Hence, to construct our hybrid model, we need the following assumption on the interconnection itself.

Assumption 1: For the NCS described by (1)-(3) the interconnection is well-posed in the sense that it holds that

$$\max_i \left|\lambda_i \left(C_{\mathbf{U}} C_{\mathbf{Y}}\right)\right| < 1, \tag{11}$$

or, equivalent, $\min\left(\left|C_{\mathbf{U}} C_{\mathbf{Y}}\right|, \left|C_{\mathbf{Y}} C_{\mathbf{U}}\right|\right) < 1$, where λ_i denotes the i-th eigenvalue. ■

Assumption 1 can be related to a small-gain type of condition for feedback systems between the $\hat{u}$- and $\hat{y}$-systems in (10). Note that Assumption 1 is always satisfied when only the plant or controller has feedthrough terms (in which case $C_{\mathbf{Y}} = 0$ or $C_{\mathbf{U}} = 0$), see also Remark 2.

Now, by using (10) we can eliminate the control variables in the state dynamics and, by using the ZOH assumption, we can also derive an expression for the error dynamics by directly using (1), (2), and (6), which yields

$$\begin{bmatrix} \dot{e}_y \\ \dot{e}_u \end{bmatrix} = \begin{bmatrix} -C_{\mathbf{P}} \dot{x}_p \\ -C_{\mathbf{C}} \dot{x}_c \end{bmatrix}.$$

As a result, similar to [6]–[12], by introducing the timer $\tau \in \mathbb{R}_{\geq 0}$, which measures the amount of time since the last transmission, and the counter $\kappa \in \mathbb{N}$, which counts the number of transmissions and is needed to implement certain scheduling protocols, we obtain for the triple $(\mathcal{P}, \mathcal{C}, \mathcal{N})$ the following hybrid model

$$\mathcal{H} : \begin{cases} \left.\begin{aligned} \dot{x} &= \mathbf{A}x + \mathbf{E}e \\ \dot{e} &= \mathbf{C}x + \mathbf{F}e \\ \dot{\tau} &= 1 \\ \dot{\kappa} &= 0 \end{aligned}\right\} \text{when } \tau \in [0, \tau_{mati}] \\ \left.\begin{aligned} x^+ &= x \\ e^+ &= h(\kappa,e) + h_{df}(h(\kappa,e),e) \\ \tau^+ &= 0 \\ \kappa^+ &= \kappa + 1 \end{aligned}\right\} \text{when } \tau \in [\delta, \infty) \end{cases} \tag{12}$$

where the various matrices are given by (13) and with the new state of the hybrid system $\xi := (x, e, \tau, \kappa) \in \mathbb{X} := \mathbb{R}^{m_x} \times \mathbb{R}^{m_e} \times \mathbb{R}_{\geq 0} \times \mathbb{N}$. Using this hybrid modeling framework, stability in the sense of UGES for the NCS can now be analyzed.

IV. Stability analysis

In this section, we analyze the stability of the system described by (12) as defined next.

Definition 1: For the overall system $\mathcal{H}$ that satisfies Assumption 1 given by (12), the set

$$\mathcal{E} = \{\xi \in \mathbb{X} \,|\, x = 0 \wedge e = 0\} \tag{14}$$

is said to be *uniformly globally exponentially stable* (UGES) if there exists a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \to \mathbb{R}_{\geq 0}$, which can be taken of the form $\beta(r,t,j) = Mr\exp(-\rho(t+j))$ for some $M \geq 0$ and $\rho > 0$, such that for any initial condition $\xi(0,0) \in \mathbb{X}$, all corresponding maximal solutions ξ are complete and satisfy for all $(t,j) \in \operatorname{dom}\xi$

$$\left|(x(t,j), e(t,j))\right| \leq \beta\left(\left|(x(0,0), e(0,0))\right|, t, j\right). \quad \blacksquare$$

Remark 1: Note that in Definition 1 we used the solution concept as introduced in [15] for describing the NCS in terms of the hybrid system (12). For more information and a detailed analysis the interested reader is referred to [15].

Based now on the results in [7], [8], we will provide LMI-based conditions that guarantee UGES of the set (14). However, to do so, the way of viewing the scheduling protocols first needs to be re-examined.

A. UGES scheduling protocols

One of the most important aspects in the analysis approach of [7] is the introduction of the concept of UGES scheduling protocols. Hereto, the update equation of the error is modeled as a discrete-time system of the form

$$e(i+1) = p_f\left(i, e(i)\right) \tag{15}$$

induced by a certain function $p_f : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}^{m_e}$. Hence, when there are no direct-feedthrough terms, i.e., $C_{\mathbf{Y}} = 0$ and $C_{\mathbf{U}} = 0$, and we thus have that the update of the error is described by (9), then the function p_f is given by the scheduling protocol function h itself, i.e., for all $i \in \mathbb{N}$ and $e \in \mathbb{R}^{m_e}$

$$p_f\left(i, e\right) = h\left(i, e\right). \tag{16}$$

Similarly, when the direct-feedthrough terms are present in the interconnection, we have that the update of the error is given by (8) and, hence, for all $i \in \mathbb{N}$ and $e \in \mathbb{R}^{m_e}$

$$p_f\left(i, e\right) = h\left(i, e\right) + h_{df}(h(i,e), e). \tag{17}$$

Consider now the following definition.

Definition 2: Let $W : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}_{\geq 0}$ be given and suppose that there exist constants $\lambda \in (0,1)$ and $\underline{\alpha}_W^c, \overline{\alpha}_W^c > 0$ such that the following conditions hold for all $i \in \mathbb{N}$ and all $e \in \mathbb{R}^{m_e}$:

$$\underline{\alpha}_W^c |e| \leq W(i,e) \leq \overline{\alpha}_W^c |e| \tag{18a}$$

$$W\left(i+1, p_f\left(i,e\right)\right) \leq \lambda W(i,e). \tag{18b}$$

Then the discrete-time system of (15) is said to be UGES with Lyapunov function W. ■

When Definition 2 is satisfied by the discrete-time system (15) with (16), as was the case in [7], then it is even said that the scheduling protocol function h is UGES with Lyapunov function W. As shown in [7], various scheduling protocols exist in this case that satisfy this definition of UGES scheduling protocols, including the try-once-discard (TOD), sampled-data (SD), and round-robin (RR) protocols.

However, as we are in this work dealing with an update of the error according to (8) rather than (9), this concept of UGES scheduling protocols needs to be slightly altered. In particular, we need to take into account that in this paper, as a result of the update function h_{df}, the update of the error e depends on the system matrices $C_{\mathbf{Y}}$ and $C_{\mathbf{U}}$, while in [7] (15) did not depend on the controller/plant parameters at all. Hence, for a given scheduling protocol, the corresponding system (15) with (17) might be UGES in some cases, while in others it is not. As such, consider the following definition.

Definition 3: The scheduling protocol modeled by the scheduling protocol function $h : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}^{m_e}$ is said to be UGES with Lyapunov function W for the given system of (1)-(3) when the discrete-time system of (15) with (17) is UGES according to Definition 2 with Lyapunov function W. ■

In Section V, we will show that various well-known scheduling protocols from [6], [7] are also UGES according to Definition 3 under appropriate conditions on the direct-feedthrough matrices $C_{\mathbf{U}}$ and $C_{\mathbf{Y}}$.

B. LMI-based condition for UGES

As shown before in, for instance, [10]–[13], it is possible to formulate LMI-based conditions such that UGES of the set (14) is guaranteed for a NCS without direct-feedthrough terms. In this subsection, we provide in a similar fashion LMI-based conditions for the NCS *with* direct-feedthrough terms given by (1)-(3) and modeled by (12).

Theorem 1: Consider the system $\mathcal{H}$ of (12) that satisfies Assumption 1. Assume there exist a function $W : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}_{\geq 0}$, a symmetric positive definite matrix $X_{\mathbf{T}}$, and strictly positive real numbers M, $\underline{\alpha}_W^c$, $\overline{\alpha}_W^c$, $\lambda \in (0,1)$, and $0 < \varepsilon < \gamma$ such that

- the scheduling protocol function $h : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}^{m_e}$ is UGES according to Definition 3 with Lyapunov function W and the constants $\underline{\alpha}_W^c, \overline{\alpha}_W^c$, and λ
- for all $\kappa \in \mathbb{N}$, and for almost all $e \in \mathbb{R}^{m_e}$ it holds that

$$\left|\frac{\partial W(\kappa, e)}{\partial e}\right| \leq M \tag{19}$$

- $\begin{bmatrix} \mathbf{A}^\top X_{\mathbf{T}} + X_{\mathbf{T}}\mathbf{A} + \varepsilon^2 I_{m_x} + M^2 \mathbf{C}^\top \mathbf{C} & X_{\mathbf{T}}\mathbf{E} \\ \mathbf{E}^\top X_{\mathbf{T}} & -{\underline{\alpha}_W^c}^2[\gamma^2 - \varepsilon^2] I_{m_e} \end{bmatrix} \leq 0.$

If now τ_{mati} satisfies for $L = M{\underline{\alpha}_W^c}^{-1}\left|\mathbf{F}\right|$ the bound

$$\tau_{mati} \leq \begin{cases} \frac{1}{Lr}\arctan\left(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}-1\right)+1+\lambda}\right) & \gamma > L \\ \frac{1}{L}\frac{1-\lambda}{1+\lambda} & \gamma = L \\ \frac{1}{Lr}\operatorname{arctanh}\left(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}-1\right)+1+\lambda}\right) & \gamma < L \end{cases}$$

with $r = \sqrt{\left|\left(\frac{\gamma}{L}\right)^2 - 1\right|}$, then the set $\mathcal{E}$ of (14) is UGES. ■

Note that Theorem 1 is merely an application of [8, Theorem 1]. Now, in order to use Theorem 1 to verify stability of the linear hybrid system given by (12), it is necessary to have a scheduling protocol function h that is UGES according to Definition 3 such that (18) and (19) are satisfied. In the next section, this problem is analyzed.

V. UGES SCHEDULING PROTOCOLS FOR THE NCS WITH DIRECT-FEEDTHROUGH TERMS

In this section, we focus on some of the most well-known scheduling protocols and show under which conditions those protocols are UGES according to Definition 3, although we envision that many other protocols are UGES in the sense of Definition 3 as well.

A. Sampled-data protocol

First we consider the sampled-data (SD) protocol, see, e.g., [6] or [7], which is modeled for $i \in \mathbb{N}$ and $e \in \mathbb{R}^{m_e}$ by

$$h(i,e) = 0. \tag{20}$$

From the results obtained in Section III-B, we know that the function p_f in (15) corresponding to the NCS with direct-feedthrough terms of (1)-(3) is given by (17), which results for the SD protocol in

$$p_f(i,e) = C_{\mathbf{YU}}e, \text{ with } C_{\mathbf{YU}} := \begin{bmatrix} 0_{m_y} & C_{\mathbf{Y}} \\ C_{\mathbf{U}} & 0_{m_u} \end{bmatrix}.$$

Hence, UGES of the discrete-time system (15) in the sense of Definition 3 follows now directly when we have that for all eigenvalues λ_i of $C_{\mathbf{YU}}$ it holds that

$$|\lambda_i(C_{\mathbf{YU}})| < 1.$$

As such, for the SD protocol, we present the following result.

Proposition 1: Under Assumption 1, the SD protocol, modeled by the scheduling protocol function (20), is UGES according to Definition 3 with Lyapunov function $W(e) = \sqrt{e^\top Pe}$ for the NCS given by (1)-(3), where the real symmetric matrix $P \succ 0$ is computed by solving

$$C_{\mathbf{YU}}^\top PC_{\mathbf{YU}} - \rho P \preceq 0 \tag{21}$$

for a certain constant $\rho \in (0,1)$. Moreover, (18a) is then satisfied for $\underline{\alpha}_W^c = \sqrt{\lambda_{\min}(P)}$ and $\overline{\alpha}_W^c = \sqrt{\lambda_{\max}(P)}$, (18b) for $\lambda = \sqrt{\rho}$, and (19) for $M = \sqrt{\lambda_{\max}(P)}$, where $\lambda_{\min}(P)$/ $\lambda_{\max}(P)$ denote the smallest/largest eigenvalue of P. ■

Proposition 1 thus implies that, for any NCS given by (1)-(3) satisfying Assumption 1, a solution to the LMI (21) exists, and, hence, Theorem 1 can always be used to verify stability of the linear NCS given by (1)-(3) with the SD protocol as the scheduling protocol.

Remark 2: From the analysis for the SD protocol it can thus be concluded that the condition (11) implies that, when either $C_{\mathbf{Y}}$ or $C_{\mathbf{U}}$ is absent in the dynamics, i.e., $C_{\mathbf{Y}} = 0$ or $C_{\mathbf{U}} = 0$ and, hence, $C_{\mathbf{Y}}C_{\mathbf{U}} = 0$, the other-ones matrix norm can grow arbitrarily large and still the SD protocol will ensure UGES of (15).

B. Try-once-discard protocol

Secondly, we consider the TOD protocol as introduced in [6]. As such, we have the situation that there are l nodes competing for access to the network and, hence, the error vector can be partitioned as $e = (e_1, e_2, \ldots, e_l)$. The scheduling protocol function h for the TOD protocol is for $i \in \mathbb{N}$ and $e \in \mathbb{R}^{m_e}$ given by [7]

$$h(i,e) = (I - \Psi(e))e \tag{22}$$

with $\Psi(e) = \operatorname{diag}\{\psi_1(e)I_{m_1}, \psi_2(e)I_{m_2}, \ldots, \psi_l(e)I_{m_l}\}$, where I_{m_j} are identity matrices of dimension m_j with $\sum_{j=1}^{l} m_j = m_y + m_u = m_e$ and

$$\psi_j(e) = \begin{cases} 1, & \text{if } j = \min\left(\arg\max_j |e_j|\right) \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

As a result, we can state the following result for the TOD protocol.

Proposition 2: Under Assumption 1, the TOD protocol, modeled by the scheduling protocol function of (22) with (23), is UGES according to Definition 3 with Lyapunov function $W(e) = \sqrt{e^\top Pe}$ for the NCS given by (1)-(3) if there exist a matrix $P \succ 0$, a constant $\rho \in (0,1)$, and nonnegative constants β_j^k, $l \in \bar{l}$ and $j \in \bar{l}\backslash\{k\}$, such that

$$A_k^\top PA_k - \rho P + \sum_{j=1, j\neq k}^{l} \beta_j^k Q_{kj} \preceq 0 \tag{24}$$

holds for all for all $k \in \bar{l}$ with the matrices $Q_{kj} := \operatorname{diag}\{0_{m_1}, \ldots, 0_{m_{k-1}}, I_{m_k}, 0_{m_{k+1}}, \ldots, 0_{m_{j-1}}, -I_{m_j}, 0_{m_{j+1}}, \ldots, 0_{m_l}\}$ and where the matrix A_k is given by

$$\begin{aligned} A_k := &\operatorname{diag}\{I_{m_1}, \ldots, I_{m_{k-1}}, 0_{m_k}, I_{m_{k+1}}, \ldots, I_{m_l}\} \\ &+ C_{\mathbf{YU}} \operatorname{diag}\{0_{m_1}, \ldots, 0_{m_{k-1}}, I_{m_k}, 0_{m_{k+1}}, \ldots, 0_{m_l}\}. \end{aligned} \tag{25}$$

Moreover, (18a) is then satisfied for $\underline{\alpha}_W^c = \sqrt{\lambda_{\min}(P)}$ and $\overline{\alpha}_W^c = \sqrt{\lambda_{\max}(P)}$, (18b) for $\lambda = \sqrt{\rho}$, and (19) for $M = \sqrt{\lambda_{\max}(P)}$. ■

Since (24) is a set of LMIs, they are amenable for computational verification [17], implying that UGES of the TOD protocol can easily be verified. Moreover, it can be shown that stability of the RR protocol can be analyzed in a similar fashion using the periodic Lyapunov lemma, see, e.g., [18], [19]. However, note that Proposition 2 only provides sufficient conditions as, in contrast to the SD protocol, we can build counter-examples (which we did not provide here because of space-limitations) that show that satisfying the well-posedness condition from Assumption 1 is not sufficient to guarantee UGES of the TOD protocol according to Definition 3.

Remark 3: Note that the SD protocol is a special case of the TOD protocol. Indeed, with $l = 1$ we have that in (25) $A_k = C_{\mathbf{YU}}$ for $k = 1$, implying that for this special case (24) indeed simplifies to (21).

VI. NUMERICAL EXAMPLE

Consider the benchmark example of the unstable batch reactor, see [6]–[9]. The linearized model of an unstable batch reactor and its controller is a two-input-two-output NCS system that can be captured by the model of (1)-(2) where

$$A_{\mathbf{P}} = \begin{pmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{pmatrix},$$

$$B_{\mathbf{P}} = \begin{pmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{pmatrix}, \quad C_{\mathbf{P}} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B_{\mathbf{C}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_{\mathbf{C}} = \begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}, \quad C_{\mathbf{U}} = \begin{pmatrix} 0 & -2 \\ 5 & 0 \end{pmatrix},$$

$A_{\mathbf{C}} = 0$, and $C_{\mathbf{Y}} = 0$, see, e.g., [6], [7]. Since the matrix $C_{\mathbf{U}} \neq 0$ we indeed have direct-feedthrough terms. If we now assume that *both* the output y to the controller as well as the controller output u itself are transmitted over a communication network, the network-induced error is updated according to (8), implying that the stability analysis of [7], [8] can no longer be applied and, hence, the results as presented in this work should be used to guarantee UGES of the set $\mathcal{E}$ in (14). Note hereby that the setup of the NCS indeed satisfies Assumption 1 since $C_{\mathbf{Y}} = 0$ and indeed Theorem 1 can be applied.

For the scheduling protocol we use the SD protocol, which is a valid choice as Proposition 1 is satisfied. Using the given matrices $C_{\mathbf{Y}}$ and $C_{\mathbf{U}}$, it is possible to compute the constants $\underline{\alpha}_W^c$, $\overline{\alpha}_W^c$, λ, and M such that the conditions of (18) and (19) are satisfied by using (21), which gives $\rho = 0.5$ and

$$P = \begin{pmatrix} 3.8678 & -3.6105{\cdot}10^{-2} & 1.0011{\cdot}10^{-3} & 1.1922{\cdot}10^{-3} \\ -3.6105{\cdot}10^{-2} & 3.4502 & 1.3669{\cdot}10^{-3} & 1.2047{\cdot}10^{-3} \\ 1.0011{\cdot}10^{-3} & 1.3669{\cdot}10^{-3} & 3.1348{\cdot}10^{-1} & 1.8706{\cdot}10^{-3} \\ 1.1922{\cdot}10^{-3} & 1.2047{\cdot}10^{-3} & 1.8706{\cdot}10^{-3} & 5.7145{\cdot}10^{-2} \end{pmatrix}$$

with $\lambda_{\min}(P) = 5.7130{\cdot}10^{-2}$ and $\lambda_{\max}(P) = 3.8709$. Hence, $\underline{\alpha}_W^c = 0.23902$, $\overline{\alpha}_W^c = M = 1.9675$, and $\lambda = \frac{1}{2}\sqrt{2}$. Now by using Theorem 1 the τ_{mati} bound can be computed by minimizing γ, see also [8], such that UGES of the set $\mathcal{E}$ is guaranteed. We can thus find that $L = 147.10$, $\gamma = 159.14$, and $\tau_{mati} \leq 1.1201 \cdot 10^{-3}$.

In Fig. 2 the results of a simulation for the numerical example of the batch reactor are shown for $\tau_{mati} = 1.1201 \cdot 10^{-3}$, $\delta = 1 \cdot 10^{-4}$, and initial condition $x = \begin{bmatrix} 1 & -3 & 0 & 0 & 2 & 1 \end{bmatrix}^\top$. As can be seen from the figure, the overall system is indeed (exponentially) stable, even with the error increasing at some transmission/inter-event times.

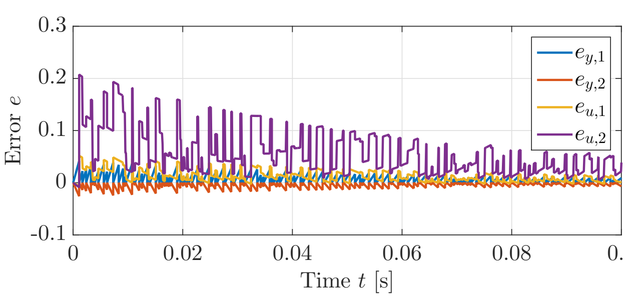

Fig. 2. Simulation results for the error e of the numerical example of the batch reactor with $\tau_{mati} = 1.1201 \cdot 10^{-3}$.

VII. Concluding Remarks

In this paper we considered the so-called direct-feedthrough problem for linear networked control systems. We have shown that the conditions proposed in [7] and [8], which ensure a uniform global exponential stability property, can still be used under suitable assumptions on the direct-feedthrough matrices. Hereto, the concept of UGES scheduling protocols as introduced in [7] had to be adapted for the influence of the direct-feedthrough terms on the networked-induced error. Based on this new concept, we have shown that the well-known sampled-data (SD), try-once-discard (TOD), and round-robin (RR) protocols still can be used in the NCS setup with direct-feedthrough terms under suitable assumptions. Finally, we have illustrated the application of our new results by means of the benchmark example of the batch reactor.

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