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A Generalized Hybrid Lyapunov Proof for Networked Control Systems: Improving the Maximum Allowable Transmission Interval

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***Abstract*— A popular design framework for networked control systems (NCSs) is the emulation-based approach combined with hybrid dynamical systems analysis techniques. In the rich literature regarding this framework, various bounds on the maximal allowable transmission interval (MATI) are provided to guarantee stability properties of the NCS using Lyapunov-based arguments for hybrid systems. In this work, we provide a generalization of these Lyapunov-based proofs, showing how the existing results for the MATI can be improved by only considering a different, more general hybrid Lyapunov function, while not altering the conditions in the analysis itself.**

I. INTRODUCTION

Networked control systems (NCSs) are systems in which the sensors, controllers, and actuators of the plant are physically distributed and communicate via (packet-based) digital channels. This architecture is motivated by the many benefits it offers with respect to conventional (wired) control systems, including greater flexibility and low cost, see, e.g., [1]. On the other hand, when, for instance, the frequency of packet transmissions is insufficient, the NCS's architecture can lead to instability. Moreover, as the communication network is often shared by multiple sensor, controller, and actuator nodes, there is a need for so-called scheduling protocols that govern the access of these nodes to the network.

To deal with these kind of network-induced phenomena, a popular two-step design framework for NCSs is the so-called emulation-based method as advocated in [2] combined with hybrid systems analysis tools, reflected in the works [3]–[9]. In this approach, first a (stabilizing) controller is designed for the plant while ignoring the communication constraints, i.e., ideal communication is assumed. In the second step, conditions on the network, e.g., bounds on transmission intervals and delays, are provided to guarantee closed-loop stability and performance of the NCS. To be more precise, as it was shown in [2], [3], the stability of NCSs is largely determined by the scheduling protocol used and the so-called maximal allowable transmission interval (MATI). Hence, the problem of characterizing the length of the MATI for a given protocol is an important concept in the analysis of NCSs.

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Ideally, one would want to limit the amount of communication as much as possible, i.e., obtain a as high as possible bound on the MATI. In [3], the authors were able to improve the initial MATI bounds given in [2] by, among others, introducing the concept of UGES scheduling protocols, which effectively summarizes the properties of scheduling protocols. Subsequently, in [4], the bound on the MATI was even further improved by using a Lyapunov-based approach for hybrid systems to analyze the stability of the NCS. In fact, it turned out that the obtained results in [4] proved to be very powerful in the analysis of NCSs, see, e.g., [5]–[8]. However, recently we showed in [9] that guaranteed larger 'stabilizing' values for the MATI can be obtained when knowledge of positive lower bounds on the transmission interval, i.e., minimal allowable transmission intervals (MIATIs), is exploited. Instrumental in this work was the concept of (minimal) dwell-time, see, e.g., [10]–[12]. In particular, the analysis showed that, by adapting the (hybrid) Lyapunov function as used in the proof of [4, Th. 1], one can obtain improved bounds on the MATI to ensure a uniform global exponential stability property, under the same conditions as presented in [4, Th. 1]. This observation leads to the insight that exploiting different hybrid Lyapunov functions than what was used in [4] can directly lead to improved MATI bounds for NCSs.

Therefore, we investigate in this work if the MATI bounds can be even further improved by studying a generalization of the Lyapunov-based proofs for NCSs as in [4] and [9]. To this end, we provide a new, more general construction for the hybrid Lyapunov function and investigate how the Lyapunov function itself can be appropriately designed within this general construction such that stability of the NCS is guaranteed. This includes the possible designs for the Lyapunov function given in [4] and [9] as special cases. Along the way, we show how existing results for the MATI from [4] can be guaranteed improved by only considering this different, more general hybrid Lyapunov function while *not* altering the conditions in the analysis itself. Finally, we compare the results for all the different analyzes by means of a numerical example.

Notation: The set of real numbers is denoted by $\mathbb{R}$ and the sets of non-negative real numbers and integers by $\mathbb{R}_{\geq 0}$ and $\mathbb{N}$, respectively. For vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$, we denote by $(v_1, v_2, \ldots, v_n)$ the vector $\begin{bmatrix} v_1^\top & v_2^\top & \cdots & v_n^\top \end{bmatrix}^\top$, and by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and the usual inner product, respectively. We use the notation $r^+(t) = r(t^+) = \lim_{\tau \downarrow t} r(\tau)$ for $r : \mathbb{R} \to \mathbb{R}^n$, provided the limit exists. The n by n identity and zero matrices are denoted by I_n and 0_n, respectively. For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, $\lambda_{\max}(A)$ denotes the largest eigenvalue of A.

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II. System description

In this section, we introduce the NCS setup and a hybrid model describing the overall dynamics.

A. Networked control configuration

We consider the 'standard' NCS as shown in Fig. 1, where the continuous-time plant $\mathcal{P}$ communicates with the controller $\mathcal{C}$ via the network $\mathcal{N}$. The plant and controller dynamics are given by

$$\mathcal{P}:\begin{cases}\dot{x}_p = f_p(x_p,\hat{u})\\ y = g_p(x_p)\end{cases} \text{ and } \mathcal{C}:\begin{cases}\dot{x}_c = f_c(x_c,\hat{y})\\ u = g_c(x_c)\end{cases}, \quad (1)$$

where $x_p \in \mathbb{R}^{m_{x_p}}$ and $x_c \in \mathbb{R}^{m_c}$ denote the plant and controller state, respectively, $u \in \mathbb{R}^{m_u}$ the control input and $\hat{u} \in \mathbb{R}^{m_u}$ the most recently received control input by the plant, and $y \in \mathbb{R}^{m_y}$ the output and $\hat{y} \in \mathbb{R}^{m_y}$ the most recently received output of the plant. We assume that f_p and f_c are continuous, and g_p and g_c continuously differentiable.

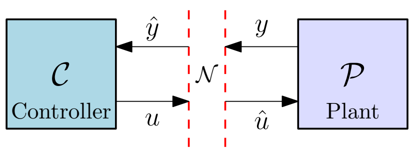

Fig. 1. The NCS setup as described in [2]–[4] and [9].

For the network $\mathcal{N}$, we assume that it has a collection of transmission times t_j, $j \in \mathbb{N}$, which satisfy

$$\tau_{miati} \le t_{j+1} - t_j \le \tau_{mati}, \quad j \in \mathbb{N}, \quad (2)$$

where τ_{mati} denotes the *maximal allowable transmission interval* (MATI) and τ_{miati} the *minimal allowable transmission interval* (MIATI), such that $0 < \tau_{miati} \le \tau_{mati}$, see also [2]–[4] and [9]. The upper bound τ_{mati} is used to guarantee stability properties of the NCS, see [2]–[4], while the lower bound on the transmission intervals $\tau_{miati} > 0$ represents physical hardware limitations, and prevents Zeno behavior. However, such a MIATI can actually be exploited in the stability analysis too, leading to higher MATIs, see Section IV-B below and [9].

In addition, the network $\mathcal{N}$ might also be subdivided in several (sensor and/or actuator) nodes, where each node corresponds to a subset of the entries $y/\hat{y}$ and/or $u/\hat{u}$. At a transmission time t_j, (parts of) the output y and the input u are sampled and transmitted to the controller $\mathcal{C}$ and the plant $\mathcal{P}$, respectively, which results in an update of the networked values according to

$$\begin{aligned}\hat{y}\left(t_j^+\right) &= y\left(t_j\right) + h_y\left(j, e\left(t_j\right)\right)\\ \hat{u}\left(t_j^+\right) &= u\left(t_j\right) + h_u\left(j, e\left(t_j\right)\right),\end{aligned} \quad (3)$$

where the function $h := (h_y, h_u)$ with $h : \mathbb{N}_0 \times \mathbb{R}^{m_e} \to \mathbb{R}^{m_e}$ models the scheduling protocol that determines which (sensor and/or actuator) node is granted access to the network at time t_j, see [2]–[4], and where $e := (e_y, e_u) = (\hat{y}-y, \hat{u}-u)$ denotes the network-induced error. We assume that $\hat{y}$ and $\hat{u}$ are constant in between two successive transmissions (i.e., the network nodes operate in a similar manner to a zero-order-hold (ZOH)). However, this can easily be modified, if desired, see [3].

B. Hybrid model

The above NCS setup can be rewritten in the hybrid system formalism[1] advocated in [3]. To do so, similar to [3] and [4], we introduce the timer $\tau \in \mathbb{R}_{\ge 0}$, which keeps track of the time elapsed since the last transmission and resets to zero after a transmission has occurred, and the counter $\kappa \in \mathbb{N}$, which keeps track of the number of transmissions. Using these auxiliary variables, the NCS can be expressed as

$$\mathcal{H}:\begin{cases}\dot{\xi} = F(\xi) & \text{when } \tau \in [0, \tau_{mati}]\\ \xi^+ = G(\xi) & \text{when } \tau \in [\tau_{miati}, \tau_{mati}]\end{cases} \quad (4)$$

with the full state of the hybrid system $\xi := ((x_p, x_c), e, \tau, \kappa) \in \mathbb{X} := \mathbb{R}^{m_x} \times \mathbb{R}^{m_e} \times \mathbb{R}_{\ge 0} \times \mathbb{N}$ and where $F(\xi) := (f(x,e), g(x,e), 1, 0)$ and $G(\xi) := (x, h(\kappa, e), 0, \kappa+1)$ with $x := (x_p, x_c) \in \mathbb{R}^{m_x}$, and

$$f(x,e) = \begin{bmatrix} f_p\left(x_p, g_c(x_c) + e_u\right)\\ f_c\left(x_c, g_p(x_p) + e_y\right)\end{bmatrix}$$

$$g(x,e) = \begin{bmatrix} -\frac{\partial g_p}{\partial x_p} f_p\left(x_p, g_c(x_c) + e_u\right)\\ -\frac{\partial g_c}{\partial x_c} f_c\left(x_c, g_p(x_p) + e_y\right)\end{bmatrix}.$$

We are interested in the stability of this hybrid model (4).

Definition 1: *For the system $\mathcal{H}$ given by* (4)*, the set*

$$\mathcal{E} := \{\xi = (x, e, \tau, \kappa) \in \mathbb{X} \mid x = 0 \text{ and } e = 0\} \quad (5)$$

is said to be uniformly globally exponentially stable *(UGES) if there exist constants $K, c > 0$ such that all maximal solutions ξ to $\mathcal{H}$ are complete and satisfy $\left|(x(t,j), e(t,j))\right| \le K\left|(x(0,0), e(0,0))\right| e^{-c(t+j)}$ for all $(t,j) \in \operatorname{dom}\xi$.*

III. Stability Analysis

In [4], an instrumental stability analysis has been presented for NCSs based on Lyapunov arguments for hybrid systems. To be more precise, the proof was based on the construction of a hybrid Lyapunov function $U : \mathbb{X} \to \mathbb{R}_{\ge 0}$ that is locally Lipschitz in its first two arguments and satisfies for some constants $\underline{\alpha}_U, \overline{\alpha}_U, \varepsilon_U > 0$ for (almost) all $\xi \in \mathbb{X}$

$$\underline{\alpha}_U \left|\xi_c\right|^2 \le U(\xi) \le \overline{\alpha}_U \left|\xi_c\right|^2 \quad (6a)$$

$$\langle \nabla U(\xi), F(\xi)\rangle \le -\varepsilon_U \left|\xi_c\right|^2, \text{ when } \tau \in [0, \tau_{mati}] \quad (6b)$$

$$U(G(\xi)) - U(\xi) \le 0, \quad \text{when } \tau \in [\tau_{miati}, \tau_{mati}] \quad (6c)$$

with $F(\xi)$ and $G(\xi)$ as in (4) and $\xi_c := (x, e)$. The conditions of (6) translate to the Lyapunov function U being radially unbounded, strictly decreasing during flows of the hybrid system (4) (i.e., in between transmission times) and to be not increasing at jumps of the hybrid system (4) (i.e., when an update of the networked values occurs).

As a result of this Lyapunov-based analysis, conditions were obtained on the MATI such that UGES of the set $\mathcal{E}$ for the NCS is guaranteed. Subsequently, it was shown in [9] that these results for the MATI can be improved when knowledge about the MIATI is explicitly exploited. Instrumental in this analysis was the construction of a more general Lyapunov function based on the concept of dwell-time [10]–[12].

[1] For details and terminology on hybrid systems of the form (4), see [13].

In this work, we investigate if the obtained results in [4] and [9] can be even further improved by considering an even more general Lyapunov function, which satisfies the conditions (6). To this end, in line with the works [3]–[7] and [9], we first make the following general assumption.

Assumption 1: *There exist a function $W : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, a locally Lipschitz function $V : \mathbb{R}^{m_x} \to \mathbb{R}_{\geq 0}$, a continuous function $H : \mathbb{R}^{m_x} \to \mathbb{R}$, and constants $\lambda \in (0,1)$, $L \geq 0$, $\underline{\alpha}_W, \overline{\alpha}_W, \underline{\alpha}_V, \overline{\alpha}_V, \beta_V, \gamma > 0$, and $0 < \varepsilon < 1$ such that the following hold:*

1) For all $\kappa \in \mathbb{N}$ and $e \in \mathbb{R}^{m_e}$

$$\underline{\alpha}_W |e| \leq W(\kappa, e) \leq \overline{\alpha}_W |e| \tag{7a}$$

$$W(\kappa + 1, h(\kappa, e)) \leq \lambda W(\kappa, e). \tag{7b}$$

2) For all $\kappa \in \mathbb{N}$, $x \in \mathbb{R}^{m_x}$, and almost all $e \in \mathbb{R}^{m_e}$

$$\left\langle \frac{\partial W(\kappa, e)}{\partial e}, g(x, e) \right\rangle \leq L W(\kappa, e) + H(x). \tag{8}$$

3) For all $x \in \mathbb{R}^{m_x}$

$$\underline{\alpha}_V |x|^2 \leq V(x) \leq \overline{\alpha}_V |x|^2. \tag{9}$$

4) For all $\kappa \in \mathbb{N}$, $e \in \mathbb{R}^{m_e}$, and almost all $x \in \mathbb{R}^{m_x}$

$$\begin{aligned} &\langle \nabla V(x), f(x, e) \rangle \\ &\quad \leq -\varepsilon^2 |x|^2 + \underline{\alpha}_W^2 \left(\gamma^2 - \varepsilon^2\right) |e|^2 - H^2(x). \end{aligned} \tag{10}$$

This assumption is essentially the same as the main assumption [4, Assumption 1]. Moreover, these conditions (7)-(10) have been proven to be systematically checkable for various classes of NCSs, see also, e.g., [6]–[9].

In view of Assumption 1 and inspired by [4] and [9], we propose for (4) the hybrid Lyapunov function

$$U(\xi) = \phi_V(\tau) V(x) + \gamma \phi_W(\tau) W^2(\kappa, e) \tag{11}$$

where $\phi_V, \phi_W : [0, \tau_{mati}] \to \mathbb{R}_{\geq 0}$ are some almost everywhere differentiable functions which are yet to be designed. For this Lyapunov function, we can state the following result.

Theorem 1: *Consider system* (4) *and suppose that Assumption 1 holds. For a given value of $\tau_{miati} > 0$, if the MATI $\tau_{mati} \geq \tau_{miati}$ is such that there exist constants $\underline{\phi}_V, \overline{\phi}_V, \underline{\phi}_W, \overline{\phi}_W \in \mathbb{R}_{>0}$ and functions $\phi_V, \phi_W : [0, \tau_{mati}] \to \mathbb{R}_{\geq 0}$ satisfying for all $\tau \in [0, \tau_{mati}]$*

$$\underline{\phi}_V \leq \phi_V(\tau) \leq \overline{\phi}_V \quad \text{and} \quad \underline{\phi}_W \leq \phi_W(\tau) \leq \overline{\phi}_W, \tag{12a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \phi_V(\tau) \leq \varepsilon^2 \eta \underline{\alpha}_V^{-1} \phi_V(\tau), \tag{12b}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \phi_W(\tau) \leq -2L\phi_W(\tau) - \gamma \left(\varepsilon_\phi \phi_V(\tau) + \frac{\phi_W^2(\tau)}{\phi_V(\tau)} \right) \tag{12c}$$

and for all $\tau \in [\tau_{miati}, \tau_{mati}]$

$$\phi_V(0) \leq \phi_V(\tau) \tag{12d}$$

$$\phi_W(0) \leq \frac{1}{\lambda^2} \phi_W(\tau) \tag{12e}$$

with $\varepsilon_\phi := 1 - \gamma^{-2} \varepsilon^2 \eta > 0$ for some constant $\eta \in [0,1)$, then the set $\mathcal{E}$ given by (5) *is UGES.*

The proof is given in Appendix I. Observe that, when ϕ_V is designed such that additionally

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \phi_V(\tau) \geq 0 \quad \text{for all } \tau > \tau_{miati} \tag{13}$$

holds and, as ϕ_W is always decreasing by virtue of (12c), it follows from (12e) that in this case we can determine the MATI as the point in time when $\phi_W(\tau_{mati}) = \lambda^2 \phi_W(0)$. In particular, when the conditions of Theorem 1 are verified with (13), the MATI can be computed as the amount of time it takes for ϕ_W to decrease from $\phi_W(0)$ to $\lambda^2 \phi_W(0)$.

We analyze in the next section for which appropriate designs of the functions ϕ_V and ϕ_W (and, implicitly, under which condition on τ_{mati}) the conditions (12) are satisfied.

Remark 1: *The construction for the Lyapunov function as given by* (11) *indeed generalizes prior constructions from [4] and [9] by means of the functions ϕ_V and ϕ_W. In fact, by choosing $\phi_V = 1$ and ϕ_W as in [4, Claim 1], we recover the Lyapunov function as used in [4], while choosing for some constant $\eta \in [0,1)$,*

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \phi_V(\tau) = \frac{\varepsilon^2 \min\{1, \underline{\alpha}_W^2\}}{\max\{\overline{\alpha}_V, \gamma \overline{\phi}_W \overline{\alpha}_W^2\}} \eta \phi_V(\tau), \quad \tau \in [0, \tau_{mati}]$$

and $\phi_W(\tau) = \phi_V(\tau) \phi_W^*(\tau)$ where ϕ_W^* is as in [4, Claim 1], we recover the result from [9].

IV. Designing the functions ϕ_W and ϕ_V

In general, we want to have the slowest decrease of ϕ_W as possible (as this results in larger MATI bounds), and, hence, $\frac{\mathrm{d}}{\mathrm{d}\tau} \phi_W(\tau)$ is often taken as its upper bound, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \phi_W(\tau) = -2L\phi_W(\tau) - \gamma \left(\varepsilon_\phi \phi_V(\tau) + \frac{\phi_W^2(\tau)}{\phi_V(\tau)} \right). \tag{14}$$

In addition, we choose $\phi_W(0) = \lambda^{-1}$ (as was also done in [4]), implying that $\phi_W(\tau_{mati}) = \lambda$, although this can be modified based on the results in [9], see also Remark 2 below. This leaves us with only the freedom of designing ϕ_V. In the following, we discuss various designs for the function ϕ_V based on (12b), (12d), and (13) as above.

A. Taking ϕ_V as a constant

The most simple choice for ϕ_V is to take it as a positive constant, i.e., $\phi_V(\tau) = \tilde{\phi}_V > 0$ for all $\tau \in [0, \tau_{mati}]$ (and, hence, $\frac{\mathrm{d}}{\mathrm{d}\tau} \phi_V(\tau) = 0$), which satisfies (12a), (12b) and (12d) and therefore guarantees UGES of the set $\mathcal{E}$. In this case, we almost recover the Lyapunov function as used in [4] (N.B.: when we take $\tilde{\phi}_V = 1$ and $\eta = 0$ we exactly recover the case of [4], see also Remark 1). To obtain now the highest possible value for τ_{mati} we have to choose $\tilde{\phi}_V$ appropriately. In particular, there is a trade-off, i.e., taking a high value for $\tilde{\phi}_V$ results in the term $\phi_W^2(\tau) \tilde{\phi}_V^{-1}$ being smaller, but $\varepsilon_\phi \tilde{\phi}_V$ being larger in (14) and vice versa when $\tilde{\phi}_V$ is taken small. Fortunately, as $\tilde{\phi}_V$ is a constant (cf. (13) holds), in a similar fashion as in [4] an explicit solution for ϕ_W and an expression for τ_{mati} can be computed, see Appendix II. As a result, we can also find the value of $\tilde{\phi}_V$ that leads to the maximum MATI. As such, we can obtain the following result.

Proposition 1: *Under Assumption 1 and for a given value of $\tau_{miati} > 0$, if $\tau_{mati} > \tau_{miati}$ satisfies the bound*

$$\tau_{mati} \leq \begin{cases} \frac{1}{L\tilde{r}} \arctan\left(\frac{\tilde{r}(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}\sqrt{\varepsilon_\phi}-1\right)+1+\lambda}\right), & \gamma^2\varepsilon_\phi > L^2 \\ \frac{1}{L\sqrt{\varepsilon_\phi}}\frac{1-\lambda}{1+\lambda}, & \gamma^2\varepsilon_\phi = L^2 \\ \frac{1}{L\tilde{r}} \operatorname{arctanh}\left(\frac{\tilde{r}(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}\sqrt{\varepsilon_\phi}-1\right)+1+\lambda}\right), & \gamma^2\varepsilon_\phi < L^2 \\ \frac{1}{\gamma\sqrt{\varepsilon_\phi}} \arctan\left(\frac{(1+\lambda)(1-\lambda)}{2\lambda}\right), & L = 0 \end{cases} \tag{15}$$

with $\tilde{r} := \sqrt{\left|\left(\frac{\gamma}{L}\right)^2 \varepsilon_\phi - 1\right|}$ *and* $\varepsilon_\phi = 1 - \gamma^{-2}\varepsilon^2\eta > 0$ *for some constant* $\eta \in [0,1)$*, then the set* $\mathcal{E}$ *in* (5) *is UGES for* (4).

From the computation of (15) in Appendix II, it follows that choosing $\tilde{\phi}_V = 1/\sqrt{\varepsilon_\phi}$ results in the largest bound for the MATI. Observe now that we exactly recover the result from [4, Th. 1] when we choose $\eta = 0$ (as this results in $\varepsilon_\phi = 1$ and $\tilde{\phi}_V = 1$), see also the discussion above. This observation also implies that, similar to the result of [9, Th. 1], we *always* obtain a higher bound for the MATI than can be obtained using the result from [4] when we choose $\eta \in (0,1)$, as in this case $\varepsilon_\phi < 1$. This observation is a direct result of the function ϕ_W decreasing slower with respect to the function as in [4, Claim 1]. In practice, we take η as close as possible to 1 since this results in the highest value for τ_{mati}.

Remark 2: *Similar to [9], we can exploit information concerning the existence of the MIATI to even further improve the bound* (15)*. In particular, we can combine the results of Proposition 1 with the results from [9, Th. 1] to obtain even higher bounds for the MATI. That is, instead of assuming that* $\phi_W(0) = \lambda^{-1}$ *and* $\phi_W(\tau_{mati}) = \lambda$*, we can choose* $\phi_W(0) = \sigma^{-1}$ *and* $\phi_W(\tau_{mati}) = \sigma$ *where the value for* $\sigma < \lambda$ *is computed according to the same lines as [9, Th. 1]. However, as in this case* ε_U *is very small due to* η *being chosen as close as possible to* 1*, see* (21)*, the obtained extra improvement is in general negligible.*

B. An increasing function ϕ_V and exploiting a MIATI

When we simulate the function ϕ_W with ϕ_V constant for various fixed values of $\tilde{\phi}_V$ for the example of Section V, we observe that the decrease rate of ϕ_W changes over time. In particular, choosing a higher value for $\tilde{\phi}_V$ results in a relative slower decrease rate for small values of τ but increases it when τ becomes larger. A smaller value for $\tilde{\phi}_V$ has the opposite effect. Based on this observation, one would like to have the dynamics of ϕ_V changing with time as this might result in an even further improvement of the MATI bound with respect to the results of Proposition 1. To this end, we investigate if we can obtain higher bounds on the MATI by considering various designs for ϕ_V. Several choices can be envisioned as listed next.

1) Another simple choice is to take the dynamics of ϕ_V to be described by its upper bound, i.e., $\frac{\mathrm{d}}{\mathrm{d}\tau}\phi_V(\tau) = \varepsilon^2\eta\overline{\alpha}_V^{-1}\phi_V(\tau)$. Note that in this case ϕ_V is an increasing function, and, hence, it also satisfies (12d) automatically.

2) Since we know that the hybrid system (4) will not jump until τ_{miati} time units have passed, we can design ϕ_V such that it decreases first, as long as it increases in time again to satisfy the design constraints. In particular, we consider the function given by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\phi_V(\tau) = \begin{cases} -\varepsilon^2\eta\overline{\alpha}_V^{-1}\phi_V(\tau), \text{ when } e^{\beta_V(\tau)}\phi_V(\tau) \geq \phi_V(0) \\ \varepsilon^2\eta\overline{\alpha}_V^{-1}\phi_V(\tau), \text{ otherwise} \end{cases} \tag{16}$$

with $\beta_V(\tau) := \varepsilon^2\eta\overline{\alpha}_V^{-1}(\tau_{miati} - \tau)$. Hence, the larger τ_{miati} is, the more the function ϕ_V can decrease.

3) Based on the observations as discussed above, we might want the function ϕ_V to be increasing in the beginning, but decreasing when τ becomes larger. Therefore, we also analyze the function ϕ_V designed as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\phi_V(\tau) = \begin{cases} \varepsilon^2\eta\overline{\alpha}_V^{-1}\phi_V(\tau), \text{ when } e^{-\beta_V(\tau)}\phi_V(\tau) \leq \phi_V(0) \\ -\varepsilon^2\eta\overline{\alpha}_V^{-1}\phi_V(\tau), \text{ when } e^{-\beta_V(\tau)}\phi_V(\tau) > \phi_V(0) \\ \qquad\qquad\qquad \text{and } \phi_V(\tau) \geq \phi_V(0) \\ 0, \text{ otherwise.} \end{cases} \tag{17}$$

Note that, to satisfy (12d), the function ϕ_V cannot attain a lower value than $\phi_V(0)$ for all $\tau \geq \tau_{miati}$, explaining the constant value for ϕ_V for all $\tau \geq \tau_{miati}$.

4) For any $\tau \in \mathbb{R}_{\geq 0}$, we know that in (14) the term $\varepsilon_\phi\phi_V(\tau) + \phi_W^2(\tau)\phi_V^{-1}(\tau)$ attains its minimal value when we design the function ϕ_V to be given by $\phi_V(\tau) = \frac{1}{\sqrt{\varepsilon_\phi}}\phi_W(\tau)$. Hence, to obtain the highest bound on the MATI, ϕ_V should also be a decreasing function for all τ with a decrease rate in the order of the decrease rate of ϕ_W. However, such a function would not satisfy (12d), implying that any decrease should be compensated by a increase of the function ϕ_V. As such, we consider the function description for ϕ_V given by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\phi_V(\tau) = \begin{cases} 0, \text{ when } \tau \geq \tau_{miati} \\ \frac{1}{\sqrt{\varepsilon_\phi}}\frac{\mathrm{d}}{\mathrm{d}\tau}\phi_W(\tau), \text{ when } \phi_V(\tau) \geq \phi_W(\tau) \\ \qquad\qquad \text{and } e^{\beta_V(\tau)}\phi_V(\tau) \geq \phi_V(0) \\ \varepsilon^2\eta\overline{\alpha}_V^{-1}\phi_V(\tau), \text{ otherwise.} \end{cases} \tag{18}$$

Observe that by means of the designs 2) – 4), similar to [9], we explicitly investigate if the presence of the MIATI can be exploited to improve the MATI.

Although (13) is satisfied for all the choices for ϕ_V, as a result of the time-varying character of the function ϕ_V in all of the above situations, computing an analytical expression for the MATI is not an easy task. Therefore, we analyze whether or not choosing such a complex design for the function ϕ_V is justified by a significant improvement of the MATI by means of the numerical example in the next section.

V. Comparison of Results: Numerical Example

To make a comparison of our results with the ones from [4] and [9], we consider the same numerical example as in [9] of stabilizing an open-loop unstable plant $\mathcal{P}$ with an output-feedback controller $\mathcal{C}$ given by

$$\mathcal{P}: \dot{x}_p = A_{\mathbf{P}}x_p + B_{\mathbf{P}}u \quad \text{and} \quad \mathcal{C}: u = -K\hat{x}_p \tag{19}$$

with $A_{\mathbf{P}} = \frac{1}{5}\begin{pmatrix}-4 & 1 \\ -2 & 3\end{pmatrix}$, $B_{\mathbf{P}} = \begin{pmatrix}-1 \\ 2\end{pmatrix}$, and $K = \begin{pmatrix}-0.2 & 0.5\end{pmatrix}$.

TABLE I

THE BOUND ON τ_{mati} FOR VARIOUS VALUES OF ε WITH $\eta = 0.9999$ AND $\phi_V(0) = 1/\sqrt{\varepsilon_\phi}$. THE HIGHEST VALUES OBTAINED ARE MARKED GREEN.

ε	$\tau_{mati}^{[4]}$	$\tau_{mati}^{[9]}$	τ_{mati}^{const}	τ_{mati}^{inc}	$\tau_{mati}^{miati,1}$	$\tau_{mati}^{miati,2}$	$\tau_{mati}^{miati,3}$	Improvement
2	0.044489	0.045051	0.046419	0.046398	0.046418	0.046420	0.046428	4.358 %
1	0.074297	0.074803	0.076829	0.076813	0.076828	0.076830	0.076836	3.417 %
0.5	0.10712	0.10811	0.10932	0.10930	0.10932	0.10932	0.10934	2.064 %
0.2	0.13369	0.13410	0.13447	0.13446	0.13447	0.13447	0.13447	0.588 %
0.1	0.13993	0.14009	0.14017	0.14016	0.14017	0.14017	0.14017	0.168 %

It follows that we assume that only the plant state x_p is transmitted over the network, i.e., we have for the error that $e = \hat{x}_p - x_p$ with the number of nodes in the network $\ell := 2$. As a result, we have that the closed-loop dynamics are given by $f(x,e) = \mathbf{A}x + \mathbf{E}e$ and $g(x,e) = \mathbf{C}x + \mathbf{F}e$ with $\mathbf{A} := A_\mathbf{P} - B_\mathbf{P}K$, $\mathbf{E} := -B_\mathbf{P}K$, $\mathbf{C} := -\mathbf{A}$, and $\mathbf{F} := -\mathbf{E}$.

Observe now that all the various MATI bounds as discussed in this work are computed based on the same conditions, i.e., the values for $\lambda \in (0,1)$, $L \geq 0$, $\underline{\alpha}_W, \overline{\alpha}_W, \underline{\alpha}_V, \overline{\alpha}_V > 0$ and $0 < \varepsilon < \gamma$ follow from Assumption 1 and are subsequently used in, for instance (15), to compute the various bounds for the MATI. Moreover, when we consider condition (8) and assume for almost all $e \in \mathbb{R}_{m_e}$ and all $\kappa \in \mathbb{N}$ that $\left|\frac{\partial W(\kappa,e)}{\partial e}\right| \leq M$ for some constant $M > 0$, conditions (8)-(10) can be summarized into a single LMI condition

$$\begin{bmatrix} \mathbf{A}^\top X_\mathbf{T} + X_\mathbf{T}\mathbf{A} + \varepsilon^2 I_{m_x} + M^2\mathbf{C}^\top\mathbf{C} & X_\mathbf{T}\mathbf{E} \\ \mathbf{E}^\top X_\mathbf{T} & -\underline{\alpha}_W^2[\gamma^2 - \varepsilon^2]I_{m_e} \end{bmatrix} \leq 0, \quad (20)$$

where we have chosen $V(x) = x^\top X_\mathbf{T} x$ with $X_\mathbf{T}$ being a symmetric positive definite matrix of size $m_x \times m_x$ such that $\overline{\alpha}_V = \lambda_{max}(X_\mathbf{T})$, $L = M\underline{\alpha}_W^{-1}|\mathbf{F}|$ and $H(x) = M|\mathbf{C}x|$, see, for instance, [9, Section V].

For a chosen scheduling protocol, in this example the try-once-discard (TOD) protocol with $\lambda = \sqrt{\ell - 1/\ell}$ and $\underline{\alpha}_W = \overline{\alpha}_W = M = 1$, see, e.g., [3], we can now use the LMI condition (20) to compute the optimal (i.e., minimal) value for γ for a given value of ε for the NCS of (19), leading to the various bounds for MATI. The results for various values of ε can be seen in Table I, where a higher value of ε corresponds to a faster convergence to the set $\mathcal{E}$, see (21). Here $\tau_{mati}^{[4]}$ represents the value obtained using the results from [4] (cf. computed using (15) with $\eta = 0$), $\tau_{mati}^{[9]}$ the value computed using [9, Th. 1]), and τ_{mati}^{const} the value computed using (15) with $\eta = 0.9999$. Moreover, based on numerical simulation of the function ϕ_W, we also computed the MATI bounds for the various designs of the function ϕ_V as discussed in Section IV-B where τ_{mati}^{inc} represents the case of an increasing function ϕ_V, $\tau_{mati}^{miati,1}$ the situation with (16), $\tau_{mati}^{miati,2}$ the situation with (17), and $\tau_{mati}^{miati,3}$ the situation with (18). It should be noted that we took here the value of τ_{miati} equal to $\tau_{mati}^{[4]}$ and that choosing $\phi_V(0) = 1/\sqrt{\varepsilon_\phi}$ as initial condition resulted in the highest bounds for τ_{mati}. The improvement is computed with respect to the value for $\tau_{mati}^{[4]}$.

As shown in the table, and as proven in [9] and Section IV, respectively, $\tau_{mati}^{[9]}$ and τ_{mati}^{const} are *always* larger than $\tau_{mati}^{[4]}$. Hence, we indeed have a guaranteed improvement with respect to the results as obtained in [4]. In addition, it can be observed that τ_{mati}^{const} is actually in all cases also larger than $\tau_{mati}^{[9]}$, showing that the results for the MATI from [9, Th. 1] can be even further improved by considering a different Lyapunov function that resulted from the general Lyapunov function construction (11) and the design requirements from Theorem 1. Moreover, note that, in contrast to $\tau_{mati}^{[9]}$, we did not even exploit any information on a MIATI in the computation of τ_{mati}^{const}, i.e., the MIATI can be taken arbitrarily small in Proposition 1. Hence, UGES of the set $\mathcal{E}$ is guaranteed for a larger 'range' of transmission intervals in the case of τ_{mati}^{const}, i.e., $t_{j+1} - t_j \in (0, \tau_{mati}^{const}]$ when using Proposition 1 since τ_{miati} can be chosen arbitrarily small, rather than $t_{j+1} - t_j \in [\tau_{mati}^{[4]}, \tau_{mati}^{[9]}]$ as is the case for the result from [9, Th. 1].

Considering the results for τ_{mati}^{inc}, $\tau_{mati}^{miati,1}$, $\tau_{mati}^{miati,2}$, and $\tau_{mati}^{miati,3}$, we can conclude that choosing ϕ_V to be an increasing function or as (16) does not result in higher bounds for the MATI than what is obtained when using Proposition 1, at least for this example. On the other hand, for $\tau_{mati}^{miati,2}$ and $\tau_{mati}^{miati,3}$ we do obtain higher values when τ_{miati} is chosen appropriately. However, note that in, for instance the case of $\tau_{mati}^{miati,2}$, a higher value for τ_{miati} does not necessarily result in a higher value of τ_{mati}, making it difficult to design the 'optimal' function ϕ_V, let alone compute an explicit expression for the MATI. Moreover, the improvement with respect to the result from Proposition 1 is marginal, while, similar to $\tau_{mati}^{[9]}$, the range of transmission intervals is shortened. This latter observation is probably related to the function ϕ_V being limited by its design considerations (12), and, therefore, by the rate of change it can attain. As such, the example suggests that considering (very) complex designs for ϕ_V might be redundant with respect to the obtained improvement.

VI. Conclusion

We have provided a generalization of the Lyapunov-based proofs for NCSs from [4] and [9] and investigated whether or not this led to improved bounds on the MATI. Along the way, we have showed that, by exploiting the design freedom with respect to this new Lyapunov function construction, the MATI from [4] can be guaranteed improved. Moreover, a numerical example showed that we can also obtain higher values for the MATI than the ones resulting from [9, Th. 1], while also significantly extending the range of allowable transmission intervals with respect to [9] as the obtained result holds for a arbitrarily small MIATI. Although the improvements for the MATI turned out to be modest, we foresee in any case that this work opens up new insights and can possible inspire to obtain new analyzing techniques for NCSs and with it improve the results for the MATI even more.

APPENDIX I
PROOF OF THEOREM 1

Based on (7a), (9), and (12a), it directly follows that the Lyapunov function U given by (11) is radially unbounded, and, hence, satisfies (6a). Moreover, when using (8) and (10), we obtain that, for all $(\tau,\kappa)\in[0,\tau_{mati}]\times\mathbb{N}$ and almost all $(x,e)\in\mathbb{R}^{m_x}\times\mathbb{R}^{m_e}$, the following holds

$$
\begin{aligned}
&\langle\nabla U(\xi),F(\xi)\rangle\\
&=\phi_V(\tau)\langle\nabla V(x),f(x,e)\rangle+\gamma\frac{\mathrm{d}}{\mathrm{d}t}\phi_W(\tau)W^2(\kappa,e)\\
&\quad+\frac{\mathrm{d}}{\mathrm{d}t}\phi_V(\tau)V(x)+2\gamma\phi_W(\tau)W(\kappa,e)\left\langle\frac{\partial W(\kappa,e)}{\partial e},g(x,e)\right\rangle\\
&\le\frac{\mathrm{d}}{\mathrm{d}t}\phi_V(\tau)V(x)-\varepsilon^2\phi_V(\tau)|x|^2-\phi_V(\tau)H^2(x)\\
&\quad+\underline{\alpha}_W^2(\gamma^2-\varepsilon^2)\phi_V(\tau)|e|^2+\gamma\frac{\mathrm{d}}{\mathrm{d}t}\phi_W W^2(\kappa,e)\\
&\quad+2\gamma L\phi_W(\tau)W^2(\kappa,e)+\underbrace{2\gamma\phi_W(\tau)H(x)W(\kappa,e)}_{\substack{=2\gamma\phi_W(\tau)\frac{1}{\sqrt{\phi_V(\tau)}}\sqrt{\phi_V(\tau)}H(x)W(\kappa,e)\\ \le\gamma^2\phi_W^2(\tau)\phi_V^{-1}(\tau)W^2(\kappa,e)+\phi_V(\tau)H^2(x)}}
\end{aligned}
$$

Using now the bounds (12b) and (12c) for the functions ϕ_V and ϕ_W, respectively, and (7a) and (9) for the functions W and V, respectively, the above can be rewritten as

$$
\begin{aligned}
&\langle\nabla U(\xi),F(\xi)\rangle\\
&\le\varepsilon^2\eta\phi_V(\tau)\overline{\alpha}_V^{-1}V(x)-\varepsilon^2\phi_V(\tau)|x|^2-\underline{\alpha}_W^2\varepsilon^2\phi_V(\tau)|e|^2\\
&\quad+\gamma\gamma^{-1}\eta\varepsilon^2\phi_V(\tau)W^2(\kappa,e) \qquad (21)\\
&\le-\varepsilon^2(1-\eta)\phi_V(\tau)\left(|x|^2+\underline{\alpha}_W^2|e|^2\right)
\end{aligned}
$$

and, hence, (6b) holds for $\varepsilon_U=\varepsilon^2(1-\eta)\underline{\phi}_V\min\{1,\underline{\alpha}_W^2\}$.

Shifting our attention now to condition (6c), we obtain by means of (7b) that during jumps of the hybrid system (4) $U(G(\xi))\le\phi_V(0)V(x)+\gamma\phi_W(0)\lambda^2W^2(\kappa,e)$. Hence, by virtue of (12d) and (12e), we can directly conclude that also (6c) is satisfied. Moreover, as the proof of [4, Th. 1] proves that the conditions (6) are sufficient for the set $\mathcal{E}$ to be UGES, this completes the proof. ■

APPENDIX II
COMPUTING THE MATI BOUND WHEN ϕ_V IS CONSTANT

By definition of (14) with $\phi_W(0)=\lambda^{-1}$ and $\phi_W(\tau^*)=\lambda$, we can write, see also the proof of [4, Lemma 2],$\tau^*=-\int_{\lambda^{-1}}^{\lambda}\left(\frac{1}{\gamma\phi_W^2\tilde{\phi}_V^{-1}+2L\phi_W+\gamma\varepsilon_\phi\tilde{\phi}_V}\right)\mathrm{d}\phi_W=-\frac{\sqrt{\tilde{\phi}_V}}{\gamma}\int_{\underline{s}}^{\overline{s}}\left(s^2-\operatorname{sign}\left(L^2-\gamma^2\varepsilon_\phi\right)\left(\frac{Lr_V}{\gamma}\right)^2\right)^{-1}\mathrm{d}s$ with $s:={}^{\phi_W}\!/\!\sqrt{\tilde{\phi}_V}+\sqrt{\tilde{\phi}_V}L/\gamma$, $\overline{s}:=s\frac{\lambda}{\sqrt{\tilde{\phi}_V}}+\sqrt{\tilde{\phi}_V}L/\gamma$, $\underline{s}:={}^1\!/\!\lambda\sqrt{\tilde{\phi}_V}+\sqrt{\tilde{\phi}_V}L/\gamma$, and $r_V:=\sqrt{\left|(\gamma/L)^2\varepsilon_\phi-1\right|\tilde{\phi}_V}$, and where $\operatorname{sign}(\cdot)$ is the sign function with $\operatorname{sign}(0)=0$. To obtain now the first formula in (15), when $\gamma^2\varepsilon_\phi>L^2$, we can use the fact that $-\frac{\sqrt{\tilde{\phi}_V}}{\gamma}\int_{\underline{s}}^{\overline{s}}\left(s^2+\left(\frac{Lr_V}{\gamma}\right)^2\right)^{-1}\mathrm{d}s=-\frac{\sqrt{\tilde{\phi}_V}}{Lr_V}\left(\arctan\left(\frac{\overline{s}\gamma}{Lr_V}\right)-\arctan\left(\frac{\underline{s}\gamma}{Lr_V}\right)\right)$ and that $\arctan(c_2)-\arctan(c_1)=\arctan\left(\frac{c_2-c_1}{1+c_1c_2}\right)$ for all $c_2\ge c_1\ge0$. This results in $\tau^*=\frac{1}{L\tilde{r}}\arctan\left(\frac{\tilde{r}(1-\lambda)}{\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}\left(\tilde{\phi}_V\varepsilon_\phi+\frac{1}{\tilde{\phi}_V}\right)-2\right)+1+\lambda}\right)$ with $\tilde{r}:={}^{r_V}\!/\!\sqrt{\tilde{\phi}_V}=\sqrt{\left|(\gamma/L)^2\varepsilon_\phi-1\right|}$. Observe now that the largest value of τ^* is obtained when the term $\tilde{\phi}_V\varepsilon_\phi+\frac{1}{\tilde{\phi}_V}$ is minimized, from which one can directly deduce that choosing $\tilde{\phi}_V=\frac{1}{\sqrt{\varepsilon_\phi}}$ maximizes the value for τ^*, resulting in $\tau^*=\frac{1}{L\tilde{r}}\arctan\left(\frac{\tilde{r}(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}\sqrt{\varepsilon_\phi}-1\right)+1+\lambda}\right)$. The second formula in (15), when $\gamma^2\varepsilon_\phi=L^2$, follows from the fact that $\tau^*=-\sqrt{\tilde{\phi}_V}/\gamma\int_{\underline{s}}^{\overline{s}}\frac{1}{s^2}\mathrm{d}s=\sqrt{\tilde{\phi}_V}/\gamma\left(\frac{1}{\overline{s}}-\frac{1}{\underline{s}}\right)$ and again the observation that the maximal value for τ^* is obtained when $\tilde{\phi}_V=\frac{1}{\sqrt{\varepsilon_\phi}}$. Following now the same analysis as in the first item, the third formula in (15), when $\gamma^2\varepsilon_\phi<L^2$, can be obtained from $-\sqrt{\tilde{\phi}_V}/Lr\int_{\underline{s}}^{\overline{s}}\left(s^2-\left(\frac{Lr}{\gamma}\right)^2\right)^{-1}\mathrm{d}s={}^1\!/\!L\tilde{r}\left(\operatorname{arctanh}\left(\frac{\overline{s}\gamma}{Lr}\right)-\operatorname{arctanh}\left(\frac{\underline{s}r}{Lr}\right)\right)$ and the identity $\operatorname{arctanh}(c_2)-\operatorname{arctanh}(c_1)=\operatorname{arctanh}\left(\frac{c_2-c_1}{1-c_1c_2}\right)$ for all $c_2\ge c_1\ge0$. Finally, the last formula in (15), when $L=0$, can be obtained by taking the limit case $L\to0$ of the first formula in (15).

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