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Abstract—A popular design framework for networked control systems (NCSs) is the emulation-based approach combined with hybrid dynamical systems analysis techniques. In the rich literature regarding this framework, various bounds on the maximal allowable transmission interval (MATI) are provided to guarantee stability properties of the NCS using Lyapunov-based arguments for hybrid systems. In this work, we provide a generalization of these Lyapunov-based proofs, showing how the existing results for the MATI can be improved by only considering a different, more general hybrid Lyapunov function, while not altering the conditions in the analysis itself.

I. INTRODUCTION

Networked control systems (NCSs) are systems in which the sensors, controllers, and actuators of the plant are physically distributed and communicate via (packet-based) digital channels. This architecture is motivated by the many benefits it offers with respect to conventional (wired) control systems, including greater flexibility and low cost, see, e.g., [1]. On the other hand, when, for instance, the frequency of packet transmissions is insufficient, the NCS’s architecture can lead to instability. Moreover, as the communication network is often shared by multiple sensor, controller, and actuator nodes, there is a need for so-called scheduling protocols that govern the access of these nodes to the network.

To deal with these kind of network-induced phenomena, a popular two-step design framework for NCSs is the so-called emulation-based method as advocated in [2] combined with hybrid systems analysis tools, reflected in the works [3]–[9]. In this approach, first a (stabilizing) controller is designed for the plant while ignoring the communication constraints, i.e., ideal communication is assumed. In the second step, conditions on the network, e.g., bounds on transmission intervals and delays, are provided to guarantee closed-loop stability and performance of the NCS. To be more precise, as it was shown in [2], [3], the stability of NCSs is largely determined by the scheduling protocol used and the so-called maximal allowable transmission interval (MATI). Hence, the problem of characterizing the length of the MATI for a given protocol is an important concept in the analysis of NCSs.

Ideally, one would want to limit the amount of communication as much as possible, i.e., obtain a as high as possible bound on the MATI. In [3], the authors were able to improve the initial MATI bounds given in [2] by, among others, introducing the concept of UGES scheduling protocols, which effectively summarizes the properties of scheduling protocols. Subsequently, in [4], the bound on the MATI was even further improved by using a Lyapunov-based approach for hybrid systems to analyze the stability of the NCS. In fact, it turned out that the obtained results in [4] proved to be very powerful in the analysis of NCSs, see, e.g., [5]–[8]. However, recently we showed in [9] that guaranteed larger ‘stabilizing’ values for the MATI can be obtained when knowledge of positive lower bounds on the transmission interval, i.e., minimal allowable transmission intervals (MIATIs), is exploited. Instrumental in this work was the concept of (minimal) dwell-time, see, e.g., [10]–[12]. In particular, the analysis showed that, by adapting the (hybrid) Lyapunov function as used in the proof of [4, Th. 1], one can obtain improved bounds on the MATI to ensure a uniform global exponential stability property, under the same conditions as presented in [4, Th. 1]. This observation leads to the insight that exploiting different hybrid Lyapunov functions than what was used in [4] can directly lead to improved MATI bounds for NCSs.

Therefore, we investigate in this work if the MATI bounds can be even further improved by studying a generalization of the Lyapunov-based proofs for NCSs as in [4] and [9]. To this end, we provide a new, more general construction for the hybrid Lyapunov function and investigate how the Lyapunov function itself can be appropriately designed within this general construction such that stability of the NCS is guaranteed. This includes the possible designs for the Lyapunov function given in [4] and [9] as special cases. Along the way, we show how existing results for the MATI from [4] can be guaranteed improved by only considering this different, more general hybrid Lyapunov function while not altering the conditions in the analysis itself. Finally, we compare the results for all the different analyzes by means of a numerical example.

Notation: The set of real numbers is denoted by \( \mathbb{R} \) and the sets of non-negative real numbers and integers by \( \mathbb{R}_{\geq 0} \) and \( \mathbb{N} \), respectively. For vectors \( v_1, v_2, \ldots, v_n \in \mathbb{R}^n \), we denote by \( \langle v_1, v_2, \ldots, v_n \rangle \) the vector \( \begin{bmatrix} v_1^T & v_2^T & \cdots & v_n^T \end{bmatrix} \), and by \( |\cdot| \) and \( \langle \cdot , \cdot \rangle \) the Euclidean norm and the usual inner product, respectively. We use the notation \( r^*(t) = r(t^+) = \lim_{\tau \downarrow t} r(\tau) \) for \( r : \mathbb{R} \to \mathbb{R}_n \), provided the limit exists. The \( n \) by \( n \) identity and zero matrices are denoted by \( I_n \) and \( 0_n \), respectively. For a symmetric matrix \( A \in \mathbb{R}^{n \times n} \), \( \lambda_{\text{max}}(A) \) denotes the largest eigenvalue of \( A \).
II. SYSTEM DESCRIPTION

In this section, we introduce the NCS setup and a hybrid model describing the overall dynamics.

A. Networked control configuration

We consider the ‘standard’ NCS as shown in Fig. 1, where the continuous-time plant \( P \) communicates with the controller \( C \) via the network \( N \). The plant and controller dynamics are given by

\[
\begin{align*}
P: \begin{cases} 
\dot{x}_p &= f_p(x_p, u) \\
y &= g_p(x_p)
\end{cases} \quad \text{and} \quad C: \begin{cases} 
\dot{x}_c &= f_c(x_c, \hat{y}) \\
u &= g_c(x_c)
\end{cases},
\end{align*}
\]

where \( x_p \in \mathbb{R}^{m_p} \) and \( x_c \in \mathbb{R}^{m_c} \) denote the plant and controller state, respectively, \( u \in \mathbb{R}^{m_u} \) the control input and \( \hat{u} \in \mathbb{R}^{m_u} \) the most recently received control input by the plant, and \( y \in \mathbb{R}^{m_y} \) the output and \( \hat{y} \in \mathbb{R}^{m_y} \) the most recently received output of the plant. We assume that \( f_p \) and \( f_c \) are continuous, and \( g_p \) and \( g_c \) continuously differentiable.

![Fig. 1. The NCS setup as described in [2]–[4] and [9].](image)

For the network \( N \), we assume that it has a collection of transmission times \( t_j, j \in \mathbb{N} \), which satisfy

\[
\tau_{miati} \leq t_{j+1} - t_j \leq \tau_{mati}, \quad j \in \mathbb{N}, \tag{2}
\]

where \( \tau_{mati} \) denotes the maximal allowable transmission interval (MATI) and \( \tau_{miati} \) the minimal allowable transmission interval (MIATI), such that \( 0 < \tau_{miati} \leq \tau_{mati} \), see also [2]–[4] and [9]. The upper bound \( \tau_{mati} \) is used to guarantee stability properties of the NCS, see [2]–[4], while the lower bound on the transmission intervals \( \tau_{miati} > 0 \) represents physical hardware limitations, and prevents Zeno behavior. However, such a MIATI can actually be exploited in the stability analysis too, leading to higher MATIs, see Section IV-B below and [9].

In addition, the network \( N \) might also be subdivided in several (sensor and/or actuator) nodes, where each node corresponds to a subset of the entries \( y/\hat{y} \) and/or \( u/\hat{u} \). At a transmission time \( t_j \), (parts of) the output \( y \) and the input \( u \) are sampled and transmitted to the controller \( C \) and the plant \( P \), respectively, which results in an update of the networked values according to

\[
\begin{align*}
\hat{y}(t_j) &= y(t_j) + h_y(j, e(t_j)) \\
\hat{u}(t_j) &= u(t_j) + h_u(j, e(t_j)),
\end{align*}
\]

where the function \( h := (h_y, h_u) \) with \( h : \mathbb{N}_0 \times \mathbb{R}^{m_e} \to \mathbb{R}^{m_y} \) models the scheduling protocol which determines when (sensor and/or actuator) node is granted access to the network at time \( t_j \), see [2]–[4], and where \( e := (e_u, e_y) = (\hat{y} - y, \hat{u} - u) \) denotes the network-induced error. We assume that \( \hat{y} \) and \( \hat{u} \) are constant in between two successive transmissions (i.e., the network nodes operate in a similar manner to a zero-order-hold (ZOH)). However, this can easily be modified, if desired, see [3].

B. Hybrid model

The above NCS setup can be rewritten in the hybrid system formalism\(^1\) advocated in [3]. To do so, similar to [3] and [4], we introduce the timer \( \tau \in \mathbb{R}_{\geq 0} \), which keeps track of the time elapsed since the last transmission and resets to zero after a transmission has occurred, and the counter \( \kappa \in \mathbb{N} \), which keeps track of the number of transmissions. Using these auxiliary variables, the NCS can be expressed as

\[
\mathcal{H} : \begin{cases} 
\dot{\xi} &= F(\xi) \quad \text{when} \quad \tau \in [0, \tau_{mati}] \\
\tau^+ &= G(\xi) \quad \text{when} \quad \tau \in [\tau_{miati}, \tau_{mati}) \tag{4}
\end{cases}
\]

with the full state of the hybrid system \( \xi := ((x_p, x_c), e, \tau, \kappa) \in \mathbb{R}^{m_p} \times \mathbb{R}^{m_c} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \) and where \( F(\xi) := (f(x, e), g(x, e), 1, 0) \) and \( G(\xi) := (x, h(\kappa, e), 0, \kappa+1) \) with \( x := (x_p, x_c) \in \mathbb{R}^{m_p} \), and

\[
\begin{align*}
f(x, e) &= \left[ f_p(x_p, g_c(x_c) + e_u) \\
g(x, e) &= \left[ \frac{\partial g_p}{\partial x} f_p(x_p, g_c(x_c) + e_u) \right].
\end{align*}
\]

We are interested in the stability of this hybrid model (4).

**Definition 1:** For the system \( \mathcal{H} \) given by (4), the set

\[
\mathcal{E} := \{ \xi = (x, e, \tau, \kappa) \in \mathbb{X} : x = 0 \text{ and } e = 0 \}
\]

is said to be globally exponentially stable (UGES) if there exist constants \( K, c > 0 \) such that all maximal solutions \( \xi \) to \( \mathcal{H} \) are complete and satisfy

\[
\| \xi(t) \| \leq K \left[ \| (x(0,0), e(0,0)) \| e^{-c(t+j)} \right]
\]

for all \( (t, j) \in \text{dom} \xi \).

III. STABILITY ANALYSIS

In [4], an instrumental stability analysis has been presented for NCSs based on Lyapunov arguments for hybrid systems. To be more precise, the proof was based on the construction of a hybrid Lyapunov function \( U : \mathbb{X} \to \mathbb{R}_{\geq 0} \) that is locally Lipschitz in its first two arguments and satisfies for some constants \( a_U, \tau_U, \varepsilon_U > 0 \) for (almost) all \( \xi \in \mathbb{X} \)

\[
a_U \| \xi \|^2 \leq U(\xi) \leq \tau_U \| \xi \|^2, \quad (\nabla U(\xi), F(\xi)) \leq -\varepsilon_U \| \xi \|^2, \quad \text{when} \quad \tau \in [0, \tau_{mati}] \quad (6a), \]

\[
U(\mathcal{G}(\xi)) - U(\xi) \leq 0, \quad \text{when} \quad \tau \in [\tau_{miati}, \tau_{mati}) \quad (6c).
\]

with \( F(\xi) \) and \( G(\xi) \) as in (4) and \( \xi := (x, e) \). The conditions of (6) translate to the Lyapunov function \( U \) being radially unbounded, strictly decreasing during flows of the hybrid system (i.e., in between transmission times) and to be not increasing at jumps of the hybrid system (i.e., when an update of the networked values occurs).

As a result of this Lyapunov-based analysis, conditions were obtained on the MATI such that UGES of the set \( \mathcal{E} \) for the NCS is guaranteed. Subsequently, it was shown in [9] that these results for the MATI can be improved when knowledge about the MIATI is explicitly exploited. Instrumental in this analysis was the construction of a more general Lyapunov function based on the concept of dwell-time [10]–[12].

\(^1\)For details and terminology on hybrid systems of the form (4), see [13].
In this work, we investigate if the obtained results in [4] and [9] can be even further improved by considering an even more general Lyapunov function, which satisfies the conditions (6). To this end, in line with the works [3]–[7] and [9], we first make the following general assumption.

**Assumption 1:** There exist a function $W : \mathbb{N} \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, a locally Lipschitz function $V : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, a continuous function $H : \mathbb{R}^m \to \mathbb{R}$, and constants $\lambda > 0$, $\omega_W, \omega_V, \gamma > 0$, and $0 < e < 1$ such that the following hold:

1. For all $\kappa \in \mathbb{N}$ and $e \in \mathbb{R}^m$,
   \[
   \omega_W |e| \leq W(\kappa, e) \leq \bar{\omega}_W |e| \tag{7a}
   \]
   \[
   W(\kappa + 1, h(\kappa, e)) \leq \lambda W(\kappa, e). \tag{7b}
   \]

2. For all $\kappa \in \mathbb{N}$, $x \in \mathbb{R}^m$, and almost all $e \in \mathbb{R}^m$,
   \[
   \left( \frac{\partial W(\kappa, e)}{\partial e} \cdot g(x, e) \right) \leq L W(\kappa, e) + H(x). \tag{8}
   \]

3. For all $x \in \mathbb{R}^m$,
   \[
   \omega_V |x|^2 \leq V(x) \leq \bar{\omega}_V |x|^2. \tag{9}
   \]

4. For all $\kappa \in \mathbb{N}$, $e \in \mathbb{R}^m$, and almost all $x \in \mathbb{R}^m$,
   \[
   \langle \nabla V(x), f(x, e) \rangle \leq -e^2 |x|^2 + \omega_W^2 (\gamma^2 - e^2) |e|^2 - H^2(x). \tag{10}
   \]

This assumption is essentially the same as the main assumption [4, Assumption 1]. Moreover, these conditions (7)-(10) have been proven to be systematically checkable for various classes of NCSs, see also, e.g., [6]–[9].

In view of Assumption 1 and inspired by [4] and [9], we propose for (4) the hybrid Lyapunov function

\[
U(\xi) = \phi_V(\tau)V(x) + \gamma \phi_W(\tau)W^2(\kappa, e) \tag{11}
\]

where $\phi_V, \phi_W : [0, \tau_{mats}] \to \mathbb{R}_{\geq 0}$ are some almost everywhere differentiable functions which are yet to be designed. For this Lyapunov function, we can state the following result.

**Theorem 1:** Consider system (4) and suppose that Assumption 1 holds. For a given value of $\tau_{mats} > 0$, if the MATI $\tau_{mats} > \tau_{mati}$ is such that there exist constants $\bar{\phi}_V, \tilde{\phi}_V, \bar{\phi}_W, \tilde{\phi}_W \in \mathbb{R}_{\geq 0}$ and functions $\phi_V, \phi_W : [0, \tau_{mats}] \to \mathbb{R}_{\geq 0}$ satisfying for all $\tau \in [0, \tau_{mats}]$

\[
\tilde{\phi}_V \leq \phi_V(\tau) \leq \bar{\phi}_V \quad \text{and} \quad \tilde{\phi}_W \leq \phi_W(\tau) \leq \bar{\phi}_W, \tag{12a}
\]

\[
\frac{d}{d\tau} \phi_V(\tau) \leq e^2 \eta \bar{\omega}_W^{-1} \phi_V(\tau), \tag{12b}
\]

\[
\frac{d}{d\tau} \phi_W(\tau) \leq -2L \phi_W(\tau) - \gamma \left( \varepsilon_{\phi} \phi_V(\tau) + \frac{\phi_W^2(\tau)}{\phi_V(\tau)} \right) \tag{12c}
\]

and for all $\tau \in [\tau_{mati}, \tau_{mats}]$

\[
\phi_V(0) \leq \phi_V(\tau) \leq \frac{\lambda}{\varepsilon_{\phi}} \phi_W(\tau) \tag{12d}
\]

\[
\phi_W(0) \leq \frac{1}{\lambda^2} \phi_W(\tau) \tag{12e}
\]

with $\varepsilon_{\phi} := 1 - \gamma^2 e^2 \eta > 0$ for some constant $\eta \in [0,1)$, then the set $\mathcal{E}$ given by (5) is UGES.

The proof is given in Appendix I. Observe that, when $\phi_V$ is designed such that additionally

\[
\frac{d}{d\tau} \phi_V(\tau) \geq 0 \quad \text{for all} \quad \tau > \tau_{mats} \tag{13}
\]

holds and, as $\phi_W$ is always decreasing by virtue of (12c), it follows from (12e) that in this case we can determine the MATI as the point in time when $\phi_W(\tau_{mats}) = \lambda^2 \phi_W(0)$. In particular, when the conditions of Theorem 1 are verified with (13), the MATI can be computed as the amount of time it takes for $\phi_V$ to decrease from $\phi_V(0)$ to $\lambda^2 \phi_W(0)$.

We analyze in the next section for which appropriate designs of the functions $\phi_V$ and $\phi_W$ (and, implicitly, under which condition on $\tau_{mats}$) the conditions (12) are satisfied.

**Remark 1:** The construction for the Lyapunov function as given by (11) indeed generalizes prior constructions from [4] and [9] by means of the functions $\phi_V$ and $\phi_W$. In fact, by choosing $\phi_V = 1$ and $\phi_W$ as in [4, Claim 1], we recover the Lyapunov function as used in [4], while choosing for some constant $\eta \in [0,1)$

\[
\frac{d}{d\tau} \phi_V(\tau) = \varepsilon^2 \min \{1, \omega_W^2 \eta \phi_V(\tau), \max \{\bar{\omega}_V, \gamma \phi_W \} \phi_V(\tau), \tau \in [0, \tau_{mats}] \tag{14}
\]

and $\phi_W(\tau) = \phi_V(\tau) \phi_W(\tau) \phi^*_V(\tau)$ where $\phi^*_V$ is as in [4, Claim 1], we recover the result from [9].

**IV. DESIGNING THE FUNCTIONS $\phi_W$ AND $\phi_V$**

In general, we want to have the slowest decrease of $\phi_W$ as possible (as this results in larger MATI bounds), and, hence, $\frac{d}{d\tau} \phi_W(\tau)$ is often taken as its upper bound, i.e.,

\[
\frac{d}{d\tau} \phi_W(\tau) = -2L \phi_W(\tau) - \gamma \left( \varepsilon_{\phi} \phi_V(\tau) + \frac{\phi_W^2(\tau)}{\phi_V(\tau)} \right). \tag{14}
\]

In addition, we choose $\phi_W(0) = \lambda^{-1}$ (as was also done in [4]), implying that $\phi_W(\tau_{mats}) = \lambda$, although this can be modified based on the results in [9], see also Remark 2 below. This leaves us with only the freedom of designing $\phi_V$. In the following, we discuss various designs for the function $\phi_V$ based on (12b), (12d), and (13) as above.

**A. Taking $\phi_V$ as a constant**

The most simple choice for $\phi_V$ is to take it as a positive constant, i.e., $\phi_V(\tau) = \phi_V > 0$ for all $\tau \in [0, \tau_{mats}]$ (and, hence, $\frac{d}{d\tau} \phi_V(\tau) = 0$), which satisfies (12a), (12b) and (12d) and therefore guarantees UGES of the set $\mathcal{E}$. In this case, we almost recover the Lyapunov function as used in [4] (N.B.: when we take $\phi_V = 1$ and $\eta = 0$ we exactly recover the case of [4], see also Remark 1). To obtain now the highest possible value for $\tau_{mats}$, we have to choose $\phi_V$ appropriately. In particular, there is a trade-off, i.e., taking a high value for $\phi_V$ results in the term $\phi_W^2(\tau) \phi_V^{-1}$ being smaller, but $\varepsilon_{\phi} \phi_V$ being larger in (14) and vice versa when $\phi_V$ is taken small. Fortunately, as $\phi_V$ is a constant (cf. (13) holds), in a similar fashion as in [4] an explicit solution for $\phi_V$ and an expression for $\tau_{mats}$ can be computed, see Appendix II. As a result, we can also find the value of $\phi_V$ that leads to the maximum MATI. As such, we can obtain the following result.
Proposition 1: Under Assumption 1 and for a given value of $\tau_{miati} > 0$, if $\tau_{mati} > \tau_{miati}$ satisfies the bound

\[
\frac{1}{L_1} \arctan \left( \frac{\hat{f}(1-\lambda)}{2 \sqrt{\xi \eta}} \right) + \frac{1}{L_2} \arctanh \left( \frac{\hat{f}(1-\lambda)}{2 \sqrt{\xi \eta}} \right) + \frac{1}{L_3} \arctan \left( \frac{\hat{f}(1-\lambda)}{2 \sqrt{\xi \eta}} \right),
\]

with $\hat{f} := \sqrt{\left( \frac{2}{\xi} \right)^2 \phi - 1}$ and $\varepsilon = 1 - \gamma^2 \varepsilon^2 \eta > 0$ for some constant $\eta \in [0, 1)$, then the set $E$ in (5) is UGES for (4).

From the computation of (15) in Appendix II, it follows that choosing $\phi_V = 1/\sqrt{\sigma}$ results in the largest bound for the MATI. Observe now that we exactly recover the result from [4, Th. 1] when we choose $\eta = 0$ (as this results in $\varepsilon = 1$ and $\phi_V = 1$), see also the discussion above. This observation also implies that, similar to the result of [9, Th. 1], we always obtain a higher bound for the MATI than can be obtained using the result from [4] when we choose $\eta \in (0, 1)$, as in this case $\varepsilon < 1$. This observation is a direct result of the function $\phi_W$ decreasing slower with respect to the function as in [4, Claim 1]. In practice, we take $\eta$ as close as possible to 1 since this results in the highest value for $\tau_{mati}$.

Remark 2: Similar to [9], we can exploit information concerning the existence of the MATI to even further improve the bound (15). In particular, we can combine the results of Proposition 1 with the results from [9, Th. 1] to obtain even higher bounds for the MATI. That is, instead of assuming that $\phi_W(0) = \lambda^{-1}$ and $\phi_W(\tau_{mati}) = \lambda$, we can choose $\phi_W(0) = \sigma^{-1}$ and $\phi_W(\tau_{mati}) = \sigma$ where the value for $\sigma < \lambda$ is computed according to the same lines as in [9, Th. 1]. However, as in this case $\varepsilon_U$ is very small due to $\eta$ being chosen as close as possible to 1, see (21), the obtained extra improvement is in general negligible.

B. An increasing function $\phi_V$ and exploiting a MIATI

When we simulate the function $\phi_W$ with $\phi_V$ constant for various fixed values of $\phi_V$ for the example of Section V, we observe that the decrease rate of $\phi_W$ changes over time. In particular, choosing a higher value for $\phi_V$ results in a relative slower decrease rate for small values of $\tau$ but increases it when $\tau$ becomes larger. A smaller value for $\phi_V$ has the opposite effect. Based on this observation, one would like to have the dynamics of $\phi_V$ changing with time as this might result in an even further improvement of the MATI bound with respect to the results of Proposition 1. To this end, we investigate if we can obtain higher bounds on the MATI by considering various designs for $\phi_V$. Several choices can be envisioned as listed next.

1) Another simple choice is to take the dynamics of $\phi_V$ to be described by its upper bound, i.e., $\frac{d}{d\tau} \phi_V(\tau) = \varepsilon^2 \eta \varepsilon^3 \phi_V(\tau)$. Note that in this case $\phi_V$ is an increasing function, and, hence, it also satisfies (12d) automatically.

2) Since we know that the hybrid system (4) will not jump until $\tau_{miati}$ time units have passed, we can design $\phi_V$ such that it decreases first, as long as it increases in time again to satisfy the design constraints. In particular, we consider the function by

\[
\frac{d}{d\tau} \phi_V(\tau) = \begin{cases} -\varepsilon^2 \eta \varepsilon^3 \phi_V(\tau), & \text{when } \phi_V(\tau) \geq \phi_V(0) \\ -\varepsilon^2 \eta \varepsilon^3 \phi_V(\tau), & \text{otherwise} \end{cases}
\]

with $\beta_V(\tau) := \varepsilon^2 \eta \varepsilon^3 (\tau_{miati} - \tau)$. Hence, the larger $\tau_{miati}$ is, the more the function $\phi_V$ can decrease.

3) Based on the observations as discussed above, we might want the function $\phi_V$ to be increasing in the beginning, but decreasing when $\tau$ becomes larger. Therefore, we also analyze the function $\phi_V$ designed as

\[
\frac{d}{d\tau} \phi_V(\tau) = \begin{cases} -\varepsilon^2 \eta \varepsilon^3 \phi_V(\tau), & \text{when } \phi_V(\tau) \leq \phi_V(0) \\ -\varepsilon^2 \eta \varepsilon^3 \phi_V(\tau), & \text{when } \phi_V(\tau) > \phi_V(0) \\ 0, & \text{otherwise}. \end{cases}
\]

Note that, to satisfy (12d), the function $\phi_V$ cannot attain a lower value than $\phi_V(0)$ for all $\tau \geq \tau_{miati}$, explaining the constant value for $\phi_V$ for all $\tau \geq \tau_{miati}$.

4) For any $\tau \in \mathbb{R}_{\geq 0}$, we know that in (14) the term $\varepsilon \phi_V(\tau) + \phi_V(\tau) \phi_V(\tau)$ attains its minimal value when we design the function $\phi_V$ to be given by $\phi_V(\tau) = \frac{1}{\varepsilon \phi_W(\tau)}$. Hence, to obtain the highest bound on the MATI, $\phi_V$ should also be a decreasing function for all $\tau$ with a decrease rate in the order of the decrease rate of $\phi_W$. However, such a function would not satisfy (12d), implying that any decrease should be compensated by an increase of the function $\phi_V$. As such, we consider the function description for $\phi_V$ given by

\[
\frac{d}{d\tau} \phi_V(\tau) = \begin{cases} 1 \frac{d}{d\tau} \phi_V(\tau), & \text{when } \phi_V(\tau) \leq \phi_W(\tau) \\ \varepsilon^2 \eta \varepsilon^3 \phi_V(\tau), & \text{otherwise}. \end{cases}
\]

Observe that by means of the designs 2) – 4), similar to [9], we explicitly investigate if the presence of the MIATI can be exploited to improve the MATI.

Although (13) is satisfied for all the choices for $\phi_V$, as a result of the time-varying character of the function $\phi_V$ in all of the above situations, computing an analytical expression for the MATI is not an easy task. Therefore, we analyze whether or not choosing such a complex design for the function $\phi_V$ is justified by a significant improvement of the MATI by means of the numerical example in the next section.

V. COMPARISON OF RESULTS: NUMERICAL EXAMPLE

To make a comparison of our results with the ones from [4] and [9], we consider the same numerical example as in [9] of stabilizing an open-loop unstable plant $P$ with an output-feedback controller $C$ given by

\[
P : \dot{x}_p = A_p x_p + B_p u \text{ and } C : u = -K \hat{x}_p
\]

with $A_p = \frac{1}{2} \begin{pmatrix} -4 & 1 \\ -2 & 3 \end{pmatrix}$, $B_p = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, and $K = (-0.2 \ 0.5)$. 

TABLE I  
THE BOUND ON $\tau_{\text{mati}}$ FOR VARIOUS VALUES OF $\varepsilon$ WITH $\eta=0.9999$ AND $\phi_V(0)=1/\sqrt{\varepsilon}$. THE HIGHEST VALUES OBTAINED ARE MARKED GREEN.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$\tau^{[4]}_{\text{mati}}$</th>
<th>$\tau^{[9]}_{\text{mati}}$</th>
<th>$\tau^{\text{const}}_{\text{mati}}$</th>
<th>$\tau^{\text{inc}}_{\text{mati}}$</th>
<th>$\tau^{\text{miati,1}}_{\text{mati}}$</th>
<th>$\tau^{\text{miati,2}}_{\text{mati}}$</th>
<th>$\tau^{\text{miati,3}}_{\text{mati}}$</th>
<th>Improvement</th>
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<td>0.10816</td>
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<td>0.10932</td>
<td>0.10932</td>
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<tr>
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<td>0.13447</td>
<td>0.13447</td>
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<tr>
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<td>0.14017</td>
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<td>0.14017</td>
<td>0.14017</td>
<td>0.14017</td>
<td>0.168%</td>
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</table>

It follows that we assume that only the plant state $x_p$ is transmitted over the network, i.e., we have for the error that $e = \dot{x}_p - x_p$ with the number of nodes in the network $\ell := 2$. As a result, we have that the closed-loop dynamics are given by $f(x,e) = Ax + Ee$ and $g(x,e) = Cx + Fe$ with $A := A_P - B_P K$, $E := -B_P K$, $C := -A$, and $F := -E$.

Observe now that all the various MATI bounds as discussed in this work are computed based on the same conditions, i.e., the values for $\lambda \in (0,1)$, $L \geq 0$, $\omega_W = \omega_V > 0$ and $0 < \varepsilon < \gamma$ follow from Assumption 1 and are subsequently used in, for instance (15), to compute the various bounds for the MATI. Moreover, when we consider condition (8) and assume for almost all $e \in \mathbb{R}_m$, and all $k \in \mathbb{N}$ that $\frac{\partial V(x,e)}{\partial e} \leq M$ for some constant $M > 0$, conditions (8)-(10) can be summarized into a single LMI condition

$$
\begin{bmatrix}
\mathbf{A}^T X_T + X_T \mathbf{A} - 2\varepsilon I_m + M^2 C^T C & X_T E \\
E^T X_T & -\omega_W^2 [\gamma^2 - \varepsilon^2] I_m
\end{bmatrix} \preceq 0,
$$

where we have chosen $V(x) = x^T X_T x$ with $X_T$ being a symmetric positive definite matrix of size $m_x \times m_x$ such that $\lambda V = \lambda_{\text{max}}(X_T)$, $L = M\omega_W^{-1} |F|$ and $H(x) = M |Cx|$, see, for instance, [9, Section VI].

For a chosen scheduling protocol, in this example the try-once-discard (TOD) protocol with $\lambda = \sqrt{2 - \frac{1}{\ell}}$ and $\omega_W = \omega_V = M = 1$, see, e.g., [3], we can now use the LMI condition (20) to compute the optimal (i.e., minimal) value for $\gamma$ for a given value of $\varepsilon$ for the NCS of (19), leading to the various bounds for MATI. The results for various values of $\varepsilon$ can be seen in Table I, where a higher value of $\varepsilon$ corresponds to a faster convergence to the set $\mathcal{E}$, see (21). Here $\tau^{[4]}_{\text{mati}}$ represents the value obtained using the results from [4] (cf. computed using (15) with $\eta = 0$), $\tau^{[9]}_{\text{mati}}$ the value computed using [9, Th. 1]), and $\tau^{\text{const}}_{\text{mati}}$ the value computed using (15) with $\eta = 0.9999$. Moreover, based on numerical simulation of the function $\phi_V$, we also computed the MATI bounds for the various designs of the function $\phi_V$ as discussed in Section IV-B where $\tau^{\text{inc}}_{\text{mati}}$ represents the case of an increasing function $\phi_V$, $\tau^{\text{miati,1}}_{\text{mati}}$ the situation with (16), $\tau^{\text{miati,2}}_{\text{mati}}$ the situation with (17), and $\tau^{\text{miati,3}}_{\text{mati}}$ the situation with (18). It should be noted that we took here the value of $\tau^{\text{miati}}_{\text{mati}}$ equal to $\tau^{[4]}_{\text{mati}}$ and that choosing $\phi_V(0) = 1/\sqrt{\varepsilon}$ as initial condition resulted in the highest bounds for $\tau^{\text{miati}}_{\text{mati}}$. The improvement is computed with respect to the value for $\tau^{[4]}_{\text{mati}}$.

As shown in the table, and as proven in [9] and Section IV, respectively, $\tau^{[9]}_{\text{mati}}$ and $\tau^{\text{const}}_{\text{mati}}$ are always larger than $\tau^{[4]}_{\text{mati}}$. Hence, we indeed have a guaranteed improvement with respect to the results as obtained in [4]. In addition, it can be observed that $\tau^{\text{const}}_{\text{mati}}$ is actually in all cases also larger than $\tau^{[9]}_{\text{mati}}$, showing that the results for the MATI from [9, Th. 1] can be even further improved by considering a different Lyapunov function that resulted from the general Lyapunov function construction (11) and the design requirements from Theorem 1. Moreover, note that, in contrast to $\tau^{[9]}_{\text{mati}}$, we did not even exploit any information on a MIATI in the computation of $\tau^{\text{const}}_{\text{mati}}$, i.e., the MIATI can be taken arbitrarily small in Proposition 1. Hence, UGESs of the set $\mathcal{E}$ is guaranteed for a larger ‘range’ of transmission intervals in the case of $\tau^{\text{const}}_{\text{mati}}$, i.e., $t_{j+1} - t_j \in (0, \tau^{\text{const}}_{\text{mati}})$ when using Proposition 1 since $\tau^{\text{miati}}_{\text{mati}}$ can be chosen arbitrarily small, rather than $t_{j+1} - t_j \in 

$.

Considering the results for $\tau^{\text{inc}}_{\text{mati}}, \tau^{\text{miati,1}}_{\text{mati}}, \tau^{\text{miati,2}}_{\text{mati}}$, and $\tau^{\text{miati,3}}_{\text{mati}}$, we can conclude that choosing $\phi_V$ to be an increasing function or as (16) does not result in higher bounds for the MATI than what is obtained when using Proposition 1, at least for this example. On the other hand, for $\tau^{\text{miati,2}}_{\text{mati}}$ and $\tau^{\text{miati,3}}_{\text{mati}}$ we do obtain higher values when $\tau^{\text{miati}}_{\text{mati}}$ is chosen appropriately. However, note that in, for instance the case of $\tau^{\text{miati,2}}_{\text{mati}}$, a higher value for $\tau^{\text{miati}}_{\text{mati}}$ does not necessarily result in a higher value of $\tau^{\text{miati}}_{\text{mati}}$, making it difficult to design the ‘optimal’ function $\phi_V$, let alone compute an explicit expression for the MATI. Moreover, the improvement with respect to the result from Proposition 1 is marginal, while, similar to $\tau^{[9]}_{\text{mati}}$, the range of transmission intervals is shortened. This latter observation is probably related to the function $\phi_V$ being limited by its design considerations (12), and, therefore, by the rate of change it can attain. As such, the example suggests that considering (very) complex designs for $\phi_V$ might be redundant with respect to the obtained improvement.

VI. CONCLUSION

We have provided a generalized version of the Lyapunov-based proofs for NCSs from [4] and [9] and investigated whether or not this led to improved bounds on the MATI. Along the way, we have shown that, by exploiting the design freedom with respect to this new Lyapunov function construction, the MATI from [4] can be guaranteed improved. Moreover, a numerical example showed that we can also obtain higher values for the MATI than the ones resulting from [9, Th. 1], while also significantly extending the range of allowable transmission intervals with respect to [9] as the obtained result holds for a arbitrarily small MIATI. Although the improvements for the MATI turned out to be modest, we foresee in any case that this work opens up new insights and can possible inspire to obtain new analyzing techniques for NCSs and with it improve the results for the MATI even more.
APPENDIX I

PROOF OF THEOREM 1

Based on (7a), (9), and (12a), it directly follows that the Lyapunov function $U$ given by ...

References


