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From Ideal to Packet-Based Communication for Spatially Invariant Systems with Various Interconnection Structures

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Abstract— **In this paper we consider systems consisting of a number of spatially invariant, i.e., identical, subsystems that use packet-based communication networks for the exchange of information. Recent literature has shown that for an *infinite* number of such interconnected networked subsystems, the overall system can be modeled as an *infinite* interconnection of identical hybrid (sub)systems. Based on this hybrid modeling perspective, conditions were derived guaranteeing uniform global exponentially stability (UGES) or $\mathcal{L}_p$-gain performance. These conditions were formulated *locally* in the sense that only the local dynamics of a single hybrid subsystem in the interconnection is needed in order to obtain a maximally allowable transmission interval (MATI) for all of the individual communication networks such that these global stability properties are guaranteed for the complete infinite-dimensional system. In this work we will connect these results to other known results in the literature concerning infinite-dimensional spatially invariant systems and extend them in various directions, thereby showing the generic nature of the obtained hybrid modeling framework. In particular, it is shown that the stability and performance conditions as derived for the perfect communication case of the infinite spatially invariant interconnection guarantee robustness of the stability/performance property in the packet-based communication case. In addition, extensions to periodic interconnections and finite interconnections with boundary conditions will be discussed explicitly, bringing the theory closer to practical applications, which will be epitomized in a two-sided vehicular platooning example concerning $\mathcal{L}_2$-stability.**

I. INTRODUCTION

Many systems consist of interconnections of similar units or subsystems that only interact with their nearest neighbors. Examples include airplane formation flight as described in [1], satellite constellations from [2], and most prominent vehicle platooning, see, e.g., [3] or [4]. Despite that these units often exhibit simple behavior and interact with their neighbors in a predictable fashion, the resulting overall system often shows rich and complex behavior. This becomes even more profound when the exchange of information between the subsystems does not occur via (ideal) dedicated point-to-point wired links but rather occurs via packed-based communication networks, inducing communication imperfections. Such networked control setup occurs in many applications, including vehicle platooning as discussed in [3] and [4], the case. As a result, stability analysis for these systems based on global monolithic models encounter severe limitations if many (or even an infinite number of) subsystems are interconnected due to the very high dimensionality of the system and large number of inputs and outputs.

Therefore, a considerable amount of research effort has been targeted on analysis methods that aim to guarantee *global* system properties based on *local* conditions on a subsystem and information about the interconnection structure. An interesting line of work in this direction, considering interconnections of an *infinite* number of subsystems, can be found in [5], [6], and more recently in [7]. Where the focus in [5] and [6] was on studying spatially invariant *linear* subsystems with *ideal* communication, in [7] *nonlinear* subsystems were taken into consideration that, in addition, used *packet-based communication networks* for the exchange of information. These networks transmit packets asynchronously and independently of each other and are equipped with scheduling protocols that determine which actuator, sensor or controller node is allowed access to the network at a certain time instant. It is shown in [7] that this infinite interconnection of "networked" subsystems can be modeled as an infinite interconnection of hybrid systems, using the modeling framework of, e.g., [8] (although a different solution concept had to be used in [7] than the one in [8] due to the inevitable presence of Zeno behavior as a result of the infinite-dimensional character of the system). This new hybrid modeling setup was subsequently used to derive *local* conditions, which only involved the local dynamics of a single hybrid subsystem in the interconnection, such that a bound on the maximally allowable transmission interval (MATI) was obtained for all the local communication networks guaranteeing uniform global exponential stability (UGES) or $\mathcal{L}_p$-stability.

In this paper, we build upon the developed modeling framework for an infinite interconnection of hybrid (sub)systems from [7] by recounting some results and illustrate them via various extensions and a new example. More precisely, we will take a look at to what extent the results from [7] are related to the original results from [6] (for the case of ideal communication), with the main focus on $\mathcal{L}_2$-stability. In particular, it is shown that the obtained performance analysis from [7] will lead to the insight that the condition for $\mathcal{L}_2$-stability for the ideal communication case as derived in [6] will already be sufficient to guarantee robustness of the $\mathcal{L}_2$-stability property for the non-idealness, packet-based communication case (i.e., for sufficient small MATI, the same $\mathcal{L}_2$-performance guarantees are obtained as guaranteed for the ideal communication case). In addition, extensions to periodic interconnections and finite

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interconnections with boundary conditions will be discussed for the packet-based communication case, showing that the properties of the infinite interconnection as discussed in [7] are inherited by them. This will underline the generic nature of the obtained hybrid modeling framework and the novel analysis from [7] and bring the theory closer to practical applications. Finally, the results as presented in this paper will be epitomized by the (linear) example concerning $\mathcal{L}_2$-stability of a two-sided platoon of vehicles, modeled as a finite interconnection with boundary conditions.

The remainder of the paper is organized as follows. First some notational conventions and preliminary definitions are presented in Section II. An overview of the class of systems considered in this paper and its corresponding stability analysis is given in Section III, revisiting some results from [7]. In Section IV and Section V the main results of this paper are presented. Here, the relation between the work of [6] and [7] is discussed and extensions to [7] are given. Finally, in Section VI, a linear numerical vehicle platooning example concerning $\mathcal{L}_2$-stability is provided, illustrating the applications of the main results of this paper, and in Section VII concluding remarks are given.

II. Preliminaries

The notation $v \in \mathbb{R}^\bullet$ will denote real-valued, finite vectors whose size is either clear from context or not relevant to the discussion. For vectors $v_1, v_2, \dots, v_n \in \mathbb{R}^\bullet$, we denote by $(v_1, v_2, \dots, v_n)$ the vector $\begin{bmatrix} v_1^\top & v_2^\top & \dots & v_n^\top \end{bmatrix}^\top$, and by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and the usual inner product, respectively. Moreover, we use the notation $r(t^+) = \lim_{\tau \downarrow t} r(\tau)$. The space of real symmetric n by n matrices is denoted $\mathbb{R}^{n \times n}_{\mathbf{S}}$. The n by n identity and zero matrices are denoted by I_n and 0_n, respectively. When the dimensions are clear from the context, these notations are simplified to I and 0. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$.

In [7], the state-space of the considered systems was infinite-dimensional in one spatial dimension. However, in this paper, we also consider finite-dimensional interconnections and multiple spatial dimensional systems, as we will see below. As a result, the set $\mathbb{D}_i$ is introduced in dimension i, being either the set of integers $\mathbb{Z}$ or the set $\{1, 2, \dots, N_i\}$ with N_i the number of subsystems in the i-th spatial dimension. Using this new set we can recall some definitions from [6].

Definition 1: The space $\ell^{L,n}$ is the set of functions mapping $\mathbb{D}_1 \times \dots \times \mathbb{D}_L$ to $\mathbb{R}^n$. The space $\ell_2^{L,n}$ is the set of functions $x \in \ell^{L,n}$ for which

$$\sum_{s_1 \in \mathbb{D}_1} \cdots \sum_{s_L \in \mathbb{D}_L} x(\mathbf{s})^\top x(\mathbf{s}) < \infty$$

holds equipped with the inner product $\langle \cdot, \cdot \rangle_{\ell_2}$ for $x, y \in \ell_2^{L,n}$ defined as

$$\langle x, y \rangle_{\ell_2} := \sum_{s_1 \in \mathbb{D}_1} \cdots \sum_{s_L \in \mathbb{D}_L} x(\mathbf{s})^\top y(\mathbf{s}),$$

and the corresponding norm as $\|x\|_{\ell_2} := \sqrt{\langle x, x \rangle_{\ell_2}}$. ■

When the dimensions L and n are clear from context or not relevant, we sometimes write $\ell_2^{L,n}$ as ℓ_2.

Definition 2: The space $\mathcal{L}_p$ for $p \in [1, \infty)$ is the set of functions ϕ mapping $\mathbb{R}_{\geq 0}$ to ℓ_2 for which $\int_0^\infty \|\phi(t)\|_{\ell_2}^p \, \mathrm{d}t < \infty$. For any $\phi \in \mathcal{L}_p$ we define the corresponding $\mathcal{L}_p$-norm as

$$\|\phi\|_{\mathcal{L}_p} := \left(\int_0^\infty \|\phi(t)\|_{\ell_2}^p \mathrm{d}t \right)^{1/p} < \infty. \quad ■$$

We will consider variables $d \in \mathcal{L}_p$ that are vector-valued functions indexed by $L + 1$ independent variables, i.e., $d = d(t, s_1, \dots, s_L)$, where $t \in \mathbb{R}_{\geq 0}$ is the (continuous) time and $s_1, s_2, \dots, s_L \in \mathbb{Z}$ are the spatial variables. The L-tuple $(s_1, s_2, \dots, s_L)$ is denoted by $\mathbf{s}$. For fixed $t \in \mathbb{R}_{\geq 0}$ and $\mathbf{s} \in \mathbb{D}_1 \times \dots \times \mathbb{D}_L$, a variable $d(t)$ can be considered as an element of $\ell^{L,n}$ or $\ell_2^{L,n}$ and $d(t, \mathbf{s})$ as an element of $\mathbb{R}^n$, i.e., a real-valued vector. For ease of notation, t is often omitted when considering such variables, however, from the context it will be clear which space is considered. The spatial shift operators $\mathbf{S}_i$, acting on functions in $\ell_2^{L,n}$, are now for $i = 1, 2, \dots, L$ defined as

$$(\mathbf{S}_i d)(\mathbf{s}) := d(s_1, \dots, s_{i-1}, s_i + 1, s_{i+1}, \dots, s_L).$$

In the case that $L = 1$, which we consider mainly in this paper, we denote $\mathbf{S}_1$ also as $\mathbf{S}$. Consider now a function $r = (r_+, r_-) \in \ell_2^{1,m_+} \times \ell_2^{1,m_-}$ and define $m := (m_+, m_-)$. We then introduce, in a similar fashion as in [6], the structured operator $\Delta_{\mathbf{S},m} : \ell_2^{1,m_+ + m_-} \to \ell_2^{1,m_+ + m_-}$ as

$$(\Delta_{\mathbf{S},m} r)(s) = \begin{bmatrix} \begin{bmatrix} I_{m_+} & 0 \end{bmatrix} (\mathbf{S}^{-1} r)(s) \\ \begin{bmatrix} 0 & I_{m_-} \end{bmatrix} (\mathbf{S} r)(s) \end{bmatrix} = \begin{bmatrix} r_+(s-1) \\ r_-(s+1) \end{bmatrix}. \quad (1)$$

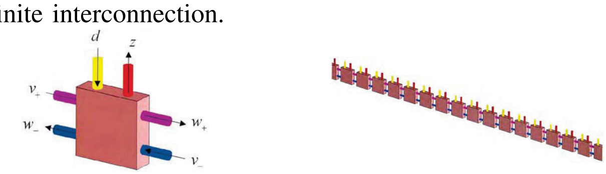
This structured operator can easily be expanded to cope with multiple spatial dimensions as shown in [6], [9].

III. System description & analysis

In this section, a brief overview of the considered class of systems and its corresponding analysis results is given. It is necessary to provide this background information in order to be able to explain the main results in this paper and their novelty. For more information and derivations, the interested reader is referred to [7].

A. Hybrid modeling framework

As mentioned before, the overall system we consider consists of a number of subsystems ("basic building blocks") that are all identical, see Fig. 1(a). In [6] these subsystems were interconnected according to several different structures including the one indicated in Fig. 1(b) to construct an infinite interconnection.



(a) Basic building block, one spatial dimension.

(b) Infinite interconnection.

Fig. 1. Spatially invariant interconnected systems from [6].

In this paper, we consider the case where each subsystem (or basic building block), denoted by $\mathcal{P}(s)$ and indexed by $s \in \mathbb{Z}$, is described by

$$\mathcal{P}(s): \quad \begin{bmatrix} \dot{x}(s) \\ w(s) \\ z(s) \end{bmatrix} = \begin{bmatrix} f_p(x(s), v(s), d(s)) \\ g_p(x(s)) \\ q_p(x(s), v(s), d(s)) \end{bmatrix} \quad (2)$$

with the initial condition $x(0,s) = x_0(s) \in \mathbb{R}^{m_0}$, $s \in \mathbb{Z}$, where $x_0 \in \ell_2^{1,m_0}$, and where $x(s) \in \mathbb{R}^{m_0}$ denotes the state, $v(s) = (v_+(s), v_-(s)) \in \mathbb{R}^{m_+ + m_-}$ the interconnected inputs, $w(s) = (w_+(s), w_-(s)) \in \mathbb{R}^{m_+ + m_-}$ the interconnected outputs, $d(s) \in \mathbb{R}^{m_d}$ the external (disturbance) input, and $z(s) \in \mathbb{R}^{m_z}$ the performance output of subsystem $\mathcal{P}(s)$, $s \in \mathbb{Z}$. Note that $x \in \ell_2^{1,m_0}$ denotes the state of the overall system, that $v_+(s)$ and $w_+(s)$ have the same size, and that $v_-(s)$ and $w_-(s)$ have the same size.

To complete the system description, we also consider that each subsystem $\mathcal{P}(s)$ communicates with its neighbors via a packet-based (wireless) communication network $\mathcal{N}(s)$, $s \in \mathbb{Z}$, see also [7]. This results in a system configuration as, for example, shown in Fig. 2.

Remark 1: In (2) we have that $f_p : \mathbb{R}^{m_0} \times \mathbb{R}^{m_+ + m_-} \times \mathbb{R}^{m_d} \to \mathbb{R}^{m_0}$, $g_p : \mathbb{R}^{m_0} \to \mathbb{R}^{m_+ + m_-}$, and $q_p : \mathbb{R}^{m_0} \times \mathbb{R}^{m_+ + m_-} \times \mathbb{R}^{m_d} \to \mathbb{R}^{m_z}$ are nonlinear mappings, where it is assumed that f_p is sufficiently smooth and that g_p is continuously differentiable, see [7] for a more detailed analysis.

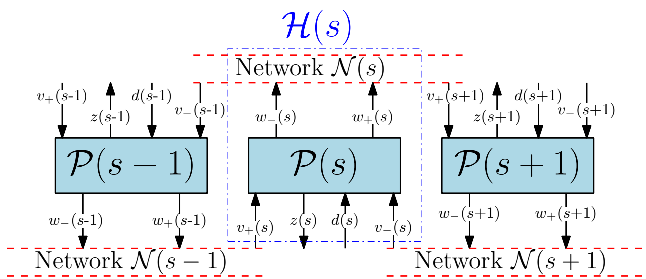


Fig. 2. Infinite networked interconnection, where each subsystem $\mathcal{P}(s)$ has its own communication network $\mathcal{N}(s)$ to communicate with its neighbors. The overall "networked" subsystem $\mathcal{H}(s)$ is the combination of subsystem $\mathcal{P}(s)$ and its network $\mathcal{N}(s)$.

In this particular configuration, the interconnection variables $w(s)$ are transmitted over the network $\mathcal{N}(s)$, resulting in the interconnection structure being described as

$$v(s) = \begin{bmatrix} v_+(s) \\ v_-(s) \end{bmatrix} = \begin{bmatrix} \hat{w}_+(s-1) \\ \hat{w}_-(s+1) \end{bmatrix} = (\Delta_{\mathbf{S},m} \hat{w})(s) \tag{3}$$

for every $s \in \mathbb{Z}$, where $\hat{w}(t,s)$ is typically not equal to $w(t,s)$, but is the "networked" or "latest broadcast" value of $w(t,s)$ at time $t \in \mathbb{R}_{\geq 0}$ for the subsystem at $s \in \mathbb{Z}$.

Each local communication network $\mathcal{N}(s)$, $s \in \mathbb{Z}$, has its own collection of transmission/sampling times t_j^s, $j \in \mathbb{N}$, at which (part of) the networked values are updated. This implies that they all operate asynchronously and independently of each other. As a result of this setup, we have a networked-induced error $e(t,s) \in \mathbb{R}^{m_e}$ for each pair $(\mathcal{P}(s), \mathcal{N}(s))$, defined as the difference between the networked values and the true values, i.e., $e(s) = \hat{w}(s) - w(s)$. In addition, each network is equipped with a scheduling protocol function $h : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}^{m_e}$ that determines which node is allowed to communicate and, hence, which entries of the error $e(s)$ are reset to zero. In fact, it holds that $e\left((t_j^s)^+\right) = h(j, e(t_j^s))$, $j \in \mathbb{N}$, see [10]–[12] for more information on the protocol (function).

Remark 2: For more information about the terminology and/or the described setup of the pair $(\mathcal{P}(s), \mathcal{N}(s))$, see, e.g., [7] or [10].

Now, in order to complete the system description, it is assumed that the transmission times satisfy $0 \leq t_0^s < t_1^s < \ldots$ and $\delta \leq t_{j+1}^s - t_j^s \leq \tau_{mati}$ for all $s \in \mathbb{Z}$ and $j \in \mathbb{N}$, where τ_{mati} denotes the MATI for all the networks $\mathcal{N}(s)$, $s \in \mathbb{Z}$. It should be noted that $\delta > 0$ can be taken arbitrarily small since it is only imposed to prevent Zeno behavior (at least locally). Next, by introducing the timers $\tau(s) \in \mathbb{R}_{\geq 0}$ and the counters $\kappa(s) \in \mathbb{N}$ for every fixed $s \in \mathbb{Z}$, it can be shown that each "networked" subsystem can be written in the format of a hybrid system, see [7], drawing inspiration from [10]. In fact, we obtain that each pair $(\mathcal{P}(s), \mathcal{N}(s))$ as in Fig. 2 can be combined as

$$\mathcal{H}(s) : \begin{cases} \left.\begin{aligned} \dot{x}(s) &= f(x,e,d)(s) \\ \dot{e}(s) &= g(x,e,d)(s) \\ \dot{\tau}(s) &= 1 \\ \dot{\kappa}(s) &= 0 \end{aligned}\right\} & \begin{aligned} &\text{when} \\ &\tau(s) \in [0, \tau_{mati}] \end{aligned} \\ \left.\begin{aligned} x^+(s) &= x(s) \\ e^+(s) &= h(\kappa(s), e(s)) \\ \tau^+(s) &= 0 \\ \kappa^+(s) &= \kappa(s) + 1 \end{aligned}\right\} & \begin{aligned} &\text{when} \\ &\tau(s) \in [\delta, \infty) \end{aligned} \end{cases} \tag{4}$$

with the new state of the subsystem indexed by $s \in \mathbb{Z}$ given by $\xi(s) = (x(s), e(s), \tau(s), \kappa(s))$ and the output equation

$$z(s) = q\left(x, e, d\right)(s), \tag{5}$$

where $f : \ell_2^{1,m_0} \times \ell_2^{1,m_e} \times \ell_2^{1,m_d} \to \ell_2^{1,m_0}$, $g : \ell_2^{1,m_0} \times \ell_2^{1,m_d} \to \ell_2^{1,m_e}$, $h : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}^{m_e}$, and $q : \ell_2^{1,m_0} \times \ell_2^{1,m_e} \times \ell_2^{1,m_d} \to \ell_2^{1,m_z}$ are appropriate functions with $m_e = m_+ + m_-$. The overall interconnection as depicted in Fig. 2 is now described by the hybrid system $\mathcal{H}$, being the infinite interconnection of subsystems $\mathcal{H}(s)$, $s \in \mathbb{Z}$.

Remark 3: As shown in [7], multiple system configurations can be taken into consideration, as long as they can all be captured in the general hybrid modeling framework of (4)-(5). However, in this paper the focus is on the configuration as in Fig. 2 in order to connect to the configuration of Fig. 1(b) as was considered in [6].

B. LMI condition for $\mathcal{L}_2$-stability

As a result of the obtained hybrid modeling framework (4)-(5), UGES or $\mathcal{L}_2$-stability of the overall interconnected system can now be analyzed by extending Lyapunov-based arguments for hybrid systems, as described in, e.g., [8], to the considered infinite-dimensional systems. By doing so, one can obtain *local* conditions that lead to a bound on the MATI which guarantees the considered global stability properties, see [7]. In addition, when the local subsystem $\mathcal{P}(s)$ is described by linear dynamical equations, these conditions can be stated as LMIs. In this section, an overview of this latter result will be presented in detail for the case of $\mathcal{L}_2$-stability. To this end, we define the performance of a system $\mathcal{H}$, being the level of input attenuation with respect to a certain external output variable $z \in \mathcal{L}_p$ by using the $\mathcal{L}_p$-induced gain with $p \in [1, \infty)$ as the performance criterion.

Definition 3: The overall system $\mathcal{H}$, composed of the identical subsystems $\mathcal{H}(s)$, $s \in \mathbb{Z}$, with associated set of initial states $\mathbb{X}_0 = \ell_2^{1,m_0+m_e} \times [0,\tau_{mati})^\infty \times \mathbb{N}^\infty \subseteq \ell_2 \times \ell$, is said to be $\mathcal{L}_p$-stable ($p < \infty$) from input d to output z with an $\mathcal{L}_p$-gain less than or equal to $\theta \geq 0$, if there exists a function $\beta \in \mathcal{K}$ such that for any exogenous input $d \in \mathcal{L}_p$ and any initial condition $\xi(0) \in \mathbb{X}_0$, all corresponding solutions[1] $\xi = (\xi_c, \xi_d) \in \mathbb{X}_0$ to $\mathcal{H}$ are complete, in the sense that they are defined globally, i.e., for all $t \in [0,\infty)$, and it holds that

$$\|z\|_{\mathcal{L}_p} \leq \beta\left(\|\xi_c(0)\|_{\ell_2}\right) + \theta \|d\|_{\mathcal{L}_p}. \quad ■$$

Consider now again the diagram of Fig. 2 together with the model of (2), however now for the case that the plant $\mathcal{P}(s)$ is governed by a linear time-invariant system, i.e., each subsystem $\mathcal{P}(s)$ of (2) is now for $s \in \mathbb{Z}$ expressed as

$$\mathcal{P}(s): \begin{bmatrix} \dot{x}(s) \\ w(s) \\ z(s) \end{bmatrix} = \begin{bmatrix} A_{\mathbf{TT}} & A_{\mathbf{TS}} & B_{\mathbf{T}} \\ A_{\mathbf{ST}} & 0 & 0 \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix} \begin{bmatrix} x(s) \\ v(s) \\ d(s) \end{bmatrix}. \quad (6)$$

Remark 4: Note that there are no direct feed-through terms in (6), in the sense that $w(s)$ does not (directly) depend on $v(s)$. This condition is used to prevent that the jump of the subsystem s directly triggers a jump in the networked-induced errors $e(s-1)$ and/or $e(s+1)$ and thus possibly jumps of the subsystems $s-1$ and/or $s+1$, respectively. By adopting this condition, a chain reaction in the jumps of the overall system is prevented, and, hence, we are able to consider complete solutions for the overall system $\mathcal{H}$. A similar condition was also adopted in the finite-dimensional case of [10]–[12]. Moreover, in (6), it is assumed that the external (disturbance) input $d(s)$ does not directly influence $w(s)$ to reduce the complexity of the networked setup.

To state the conditions that guarantee $\mathcal{L}_2$-stability of the overall system, we first state the following assumption on the local scheduling protocol, see also [10]–[12].

Assumption 1: For the local scheduling protocol h there exist a function $W : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, and constants $\underline{\alpha}_W^c, \overline{\alpha}_W^c \in \mathbb{R}_{\geq 0}$, $\lambda \in (0,1)$, and $M > 0$ such that for all $\kappa(s) \in \mathbb{N}$, and all $e(s) \in \mathbb{R}^{m_e}$, $s \in \mathbb{Z}$, it holds that

$$\underline{\alpha}_W^c |e(s)| \leq W(\kappa(s), e(s)) \leq \overline{\alpha}_W^c |e(s)|$$
$$W(\kappa(s)+1, h(\kappa(s), e(s))) \leq \lambda W(\kappa(s), e(s)),$$

and that for all $\kappa(s) \in \mathbb{N}$ and almost all $e(s) \in \mathbb{R}^{m_e}$, $s \in \mathbb{Z}$ it holds that $\left|\frac{\partial W(\kappa(s),e(s))}{\partial e(s)}\right| \leq M$. ■

[1] Solutions are interpreted in the sense of [7], in which proper definitions of complete solutions where established for the considered interconnection consisting of an *infinite* number of hybrid systems. A novel solution concept had to be developed as standard solution concepts as in, e.g., [8], do not apply since Zeno behavior (an infinite number of jumps in a finite time interval) is inevitable, see [7]. In particular, to define the solutions globally, an alternative, but natural solution concept was introduced that allows to define solutions beyond Zeno points in the form of right-accumulation points. Note however that, when one considers a periodic or finite interconnection with boundary conditions as described in Section V, the solution concept from [8] is sufficient to describe the overall interconnected system $\mathcal{H}$.

Remark 5: Various scheduling protocols exist which satisfy Assumption 1, including the try-once-discard (TOD), sampled-data (SD), and round-robin (RR) protocols as shown in [10].

Besides having this assumption on the local scheduling protocol, to obtain *local* conditions (which only involve the local dynamics of *one* of the subsystems in the interconnection) we also need to exploit the particular interconnection structure in the sense that we will aim for so-called neutral interconnections, see, e.g., [6], [7], [13].

Definition 4: When, for a certain (local) interconnecting supply function $\mathcal{S}_i : \mathbb{R}^{m_+ + m_-} \times \mathbb{R}^{m_+ + m_-} \to \mathbb{R}$ and the interconnection variables $q_+(s) := (\hat{w}_+(s), v_-(s))$ and $q_-(s) := (v_+(s), \hat{w}_-(s))$, it holds that

$$\sum_{s \in \mathbb{D}} \mathcal{S}_i(q_+(s), q_-(s)) = 0, \quad (7)$$

then the interconnection is said to be *neutral*. ■

Since for the spatially invariant infinite interconnection of Fig. 2 it holds that $q_+(s) = q_-(s+1)$ for all $s \in \mathbb{Z}$, or compactly, $q_+ = \Delta_{\mathbf{S},\hat{m}} q_-$ with $\hat{m} = (0, m_+ + m_-)$, as a result of (1), by introducing the matrix $X_{\mathbf{S}} \in \mathbb{R}_{\mathbf{S}}^{m_+ + m_-}$, we can define the local supply function $\mathcal{S}_i$ as

$$\mathcal{S}_i(q_+(s), q_-(s)) := \begin{bmatrix} q_+(s) \\ q_-(s) \end{bmatrix}^\top \begin{bmatrix} -X_{\mathbf{S}} & 0 \\ 0 & X_{\mathbf{S}} \end{bmatrix} \begin{bmatrix} q_+(s) \\ q_-(s) \end{bmatrix}, \quad (8)$$

such that (7) indeed holds for the infinite interconnection structure as depicted in Fig. 2.

Using this neutral interconnection result and Assumption 1, it is possible to compose the desired LMI condition that leads to a bound on τ_{mati} guaranteeing $\mathcal{L}_2$-stability. To this end, (6) needs to be further partitioned to reflect the structure of $\Delta_{\mathbf{S},m}$ of (1), i.e.,

$$A_{\mathbf{ST}} = \begin{bmatrix} A_{\mathbf{ST}}^{++} \\ A_{\mathbf{ST}}^{--} \end{bmatrix}, \quad \text{and} \quad A_{\mathbf{TS}} = \begin{bmatrix} A_{\mathbf{TS}}^{++} & A_{\mathbf{TS}}^{--} \end{bmatrix}.$$

Based on this partitioning, following [6], we introduce

$$A_{\mathbf{SS},v}^{+} = \begin{bmatrix} 0 & 0 \\ 0 & I_{m_-} \end{bmatrix} \quad A_{\mathbf{SS},e}^{+} = \begin{bmatrix} I_{m_+} & 0 \\ 0 & 0 \end{bmatrix} \quad A_{\mathbf{ST}}^{+} = \begin{bmatrix} A_{\mathbf{ST}}^{++} \\ 0 \end{bmatrix}$$
$$A_{\mathbf{SS},v}^{-} = \begin{bmatrix} I_{m_+} & 0 \\ 0 & 0 \end{bmatrix} \quad A_{\mathbf{SS},e}^{-} = \begin{bmatrix} 0 & 0 \\ 0 & I_{m_-} \end{bmatrix} \quad A_{\mathbf{ST}}^{-} = \begin{bmatrix} 0 \\ A_{\mathbf{ST}}^{--} \end{bmatrix}$$

and

$$A_{\mathbf{TS}}^{+} = \begin{bmatrix} A_{\mathbf{TS}}^{++} & 0 \end{bmatrix}, \quad A_{\mathbf{TS}}^{-} = \begin{bmatrix} 0 & A_{\mathbf{TS}}^{--} \end{bmatrix}.$$

Consider the set $\mathcal{X}_{\mathbf{T}} := \left\{ X_{\mathbf{T}} \in \mathbb{R}_{\mathbf{S}}^{m_0 \times m_0} \mid X_{\mathbf{T}} \succ 0 \right\}$ of positive definite and symmetric matrices. We are now able to state the following theorem regarding $\mathcal{L}_2$-stability of the overall system, see [7] for a proof.

Theorem 1: Consider the overall system $\mathcal{H}$, composed of the identical subsystems $\mathcal{H}(s)$ of (4)-(5), $s \in \mathbb{Z}$, with associated $\mathcal{P}(s)$ of (6) and $d \neq 0$. Assume there exist a function $W : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, a matrix $X_{\mathbf{T}} \in \mathcal{X}_{\mathbf{T}}$, a matrix $X_{\mathbf{S}} \in \mathbb{R}_{\mathbf{S}}^{(m_+ + m_-) \times (m_+ + m_-)}$, and constants $\underline{\alpha}_W^c, \overline{\alpha}_W^c, \theta \in \mathbb{R}_{\geq 0}$, $M, \gamma, \mu > 0$, $\lambda \in (0,1)$ such that Assumptions 1 is satisfied and the LMI $\mathcal{J} - J \preceq 0$ holds with the matrices J and $\mathcal{J}$ as in

$$J := -M^2 \left[\begin{array}{ccc|c} A_{\mathbf{TT}}^\top \widehat{A}_{\mathbf{ST}} A_{\mathbf{TT}} & A_{\mathbf{TT}}^\top \widehat{A}_{\mathbf{ST}} A_{\mathbf{TS}} & A_{\mathbf{TT}}^\top \widehat{A}_{\mathbf{ST}} B_{\mathbf{T}} & 0 \\ A_{\mathbf{TS}}^\top \widehat{A}_{\mathbf{ST}} A_{\mathbf{TT}} & A_{\mathbf{TS}}^\top \widehat{A}_{\mathbf{ST}} A_{\mathbf{TS}} & A_{\mathbf{TS}}^\top \widehat{A}_{\mathbf{ST}} B_{\mathbf{T}} & 0 \\ B_{\mathbf{T}}^\top \widehat{A}_{\mathbf{ST}} A_{\mathbf{TT}} & B_{\mathbf{T}}^\top \widehat{A}_{\mathbf{ST}} A_{\mathbf{TS}} & B_{\mathbf{T}}^\top \widehat{A}_{\mathbf{ST}} B_{\mathbf{T}} & 0 \\ \hline 0 & 0 & 0 & -\underline{\alpha}_W^{c\,2} M^{-2} \gamma^2 I_{m_+ + m_-} \end{array}\right] =: \left[\begin{array}{ccc|c} & J_{DD} & & \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \\ \hline 0 & 0 & 0 & \underline{\alpha}_W^{c\,2} \gamma^2 I_{m_+ + m_-} \end{array}\right] \tag{9}$$

$$\begin{aligned} \mathcal{J} &:= \left[\begin{array}{ccc|c} \widehat{A}_{\mathbf{TT}} + \mu C_{\mathbf{T}}^\top C_{\mathbf{T}} + A_{\mathbf{ST}}^{\Delta} & X_{\mathbf{T}} A_{\mathbf{TS}} + \mu C_{\mathbf{T}}^\top C_{\mathbf{S}} + A_{\mathbf{ST},\mathbf{SS},v}^{\Delta} & X_{\mathbf{T}} B_{\mathbf{T}} + \mu C_{\mathbf{T}}^\top D & A_{\mathbf{ST},\mathbf{SS},e}^{\Delta} \\ (A_{\mathbf{TS}})^\top X_{\mathbf{T}} + \mu C_{\mathbf{S}}^\top C_{\mathbf{T}} + \left(A_{\mathbf{ST},\mathbf{SS},v}^{\Delta}\right)^\top & \mu C_{\mathbf{S}}^\top C_{\mathbf{S}} + A_{\mathbf{SS},v}^{\Delta} & \mu C_{\mathbf{S}}^\top D & \left(A_{\mathbf{SS},ev}^{\Delta}\right)^\top \\ B_{\mathbf{T}}^\top X_{\mathbf{T}} + \mu D^\top C_{\mathbf{T}} & \mu D^\top C_{\mathbf{S}} & -\mu\theta^2 I_{m_d} + \mu D^\top D & 0 \\ \hline \left(A_{\mathbf{ST},\mathbf{SS},e}^{\Delta}\right)^\top & A_{\mathbf{SS},ev}^{\Delta} & 0 & A_{\mathbf{SS},e}^{\Delta} \end{array}\right] \\ &=: \left[\begin{array}{ccc|c} & \mathcal{J}_{DD} & & \begin{array}{c} A_{\mathbf{ST},\mathbf{SS},e}^{\Delta} \\ \left(A_{\mathbf{SS},ev}^{\Delta}\right)^\top \\ 0 \end{array} \\ \hline \left(A_{\mathbf{ST},\mathbf{SS},e}^{\Delta}\right)^\top & A_{\mathbf{SS},ev}^{\Delta} & 0 & A_{\mathbf{SS},e}^{\Delta} \end{array}\right] \end{aligned} \tag{10}$$

(9) and (10), respectively, with $\widehat{A}_{\mathbf{TT}} := A_{\mathbf{TT}}^\top X_{\mathbf{T}} + X_{\mathbf{T}} A_{\mathbf{TT}}$ and $\widehat{A}_{\mathbf{ST}} := A_{\mathbf{ST}}^\top A_{\mathbf{ST}}$, and with

$$\begin{aligned} A_{\mathbf{ST}}^{\Delta} &= (A_{\mathbf{ST}}^{+})^\top X_{\mathbf{S}} A_{\mathbf{ST}}^{+} - (A_{\mathbf{ST}}^{-})^\top X_{\mathbf{S}} A_{\mathbf{ST}}^{-} \\ A_{\mathbf{SS},ev}^{\Delta} &= \left(A_{\mathbf{SS},e}^{+}\right)^\top X_{\mathbf{S}} A_{\mathbf{SS},v}^{+} - \left(A_{\mathbf{SS},e}^{-}\right)^\top X_{\mathbf{S}} A_{\mathbf{SS},v}^{-} \\ A_{\mathbf{SS},v}^{\Delta} &= (A_{\mathbf{SS},v}^{+})^\top X_{\mathbf{S}} A_{\mathbf{SS},v}^{+} - (A_{\mathbf{SS},v}^{-})^\top X_{\mathbf{S}} A_{\mathbf{SS},v}^{-} \\ A_{\mathbf{SS},e}^{\Delta} &= (A_{\mathbf{SS},e}^{+})^\top X_{\mathbf{S}} A_{\mathbf{SS},e}^{+} - (A_{\mathbf{SS},e}^{-})^\top X_{\mathbf{S}} A_{\mathbf{SS},e}^{-} \\ A_{\mathbf{ST},\mathbf{SS},v}^{\Delta} &= (A_{\mathbf{ST}}^{+})^\top X_{\mathbf{S}} A_{\mathbf{SS},v}^{+} - (A_{\mathbf{ST}}^{-})^\top X_{\mathbf{S}} A_{\mathbf{SS},v}^{-} \\ A_{\mathbf{ST},\mathbf{SS},e}^{\Delta} &= (A_{\mathbf{ST}}^{+})^\top X_{\mathbf{S}} A_{\mathbf{SS},e}^{+} - (A_{\mathbf{ST}}^{-})^\top X_{\mathbf{S}} A_{\mathbf{SS},e}^{-}. \end{aligned}$$

If now τ_{mati} satisfies $\tau_{mati} \leq \frac{1}{\gamma} \arctan\left(\frac{(1+\lambda)(1-\lambda)}{2\lambda}\right)$, then the overall system $\mathcal{H}$ is guaranteed to be $\mathcal{L}_2$-stable from d to z with an $\mathcal{L}_2$-gain less than or equal to θ. ■

Since γ is the only free variable for the computation of the bound $\frac{1}{\gamma} \arctan\left(\frac{(1+\lambda)(1-\lambda)}{2\lambda}\right)$ for τ_{mati} as λ follows from the local scheduling protocol, τ_{mati} can be maximized by means of minimizing γ subject to $\gamma > 0$ and the LMI $\mathcal{J} - J \leq 0$ (for a fixed θ).

IV. From ideal to non-ideal communication

In this section we connect the results of Theorem 1 for the non-ideal communication case to the results presented in [6] for the ideal communication case. In particular, we will show that the condition as was obtained for performance of the system with perfect communication in [6] already guarantees $\mathcal{L}_2$-stability of the system *with* packet-based communication for a sufficiently small MATI (with the same guaranteed $\mathcal{L}_2$-gain). In other words, we will show that the condition as used in [6] already has inherent robustness guarantees of the stability/performance property with respect to the non-idealness in the packet-based communication.

Consider hereto again the matrices J of (9) and $\mathcal{J}$ of (10). Observe now that the upper-left matrix corner of the matrix $\mathcal{J}$ of (10), denoted by $\mathcal{J}_{DD}$, is exactly the same matrix J as in Theorem 1 from [6] for $\mu = \theta = 1$. In fact, in [6] it is shown that under perfect or ideal communication the system is $\mathcal{L}_2$-stable from d to z with an $\mathcal{L}_2$-gain less than or equal to $\theta = 1$ when

$$\mathcal{J}_{DD} \prec 0. \tag{11}$$

In addition, from Theorem 1 we also know that the overall spatially invariant interconnected system $\mathcal{H}$ *with* networked communication is $\mathcal{L}_2$-stable from d to z with an $\mathcal{L}_2$-gain less than or equal to $\theta = 1$ when $J - \mathcal{J} \geq 0$ (and MATI sufficiently small).

In order to connect these two stability/performance results, consider the Schur lemma, which states that a matrix X, given by

$$X = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}, \tag{12}$$

is positive semidefinite when $C \succ 0$ and $A - BC^{-1}B^\top \geq 0$. Based on the matrices $\mathcal{J}$ and J of (9)-(10), one can take the matrix C in (12) as

$$C = -A_{\mathbf{SS},e}^{\Delta} + \underline{\alpha}_W^{c\,2} \gamma^2 I_{m_+ + m_-}, \tag{13}$$

while the matrix A in (12) is being defined as $A := -\mathcal{J}_{DD} + J_{DD}$, where J_{DD} denotes the upper-left matrix corner of the matrix J of (9). It is clear that the matrix C of (13) satisfies $C \succ 0$ when the constant $\gamma > 0$ is taken large enough. Moreover, based on this matrix C, it also follows that there exists a large enough value for γ such that, when $A = -\mathcal{J}_{DD} + J_{DD} \succ 0$ holds, also $A - BC^{-1}B^\top \geq 0$ holds. Hence, from the Schur lemma it now directly follows that the overall system $\mathcal{H}$ is $\mathcal{L}_2$-stable from d to z with an $\mathcal{L}_2$-gain less than or equal to $\theta = 1$ when

$$\mathcal{J}_{DD} \prec J_{DD} \tag{14}$$

and γ large enough (or, thus, τ_{mati} small enough). In addition, from the structure of the matrix J_{DD} itself and using that $\hat{A}_{\mathbf{ST}} \geq 0$, we also have that $J_{DD} \leq 0$ for $M \geq 0$.

Now, if we could show that (11) (the stability/performance condition as in [6] for the ideal communication case) implies (14), then we showed the desired connection. To prove that is indeed the case, consider the LMI of (11) to be satisfied for certain matrices $X_{\mathbf{T}} \in \mathcal{X}_{\mathbf{T}}$ and $X_{\mathbf{S}} \in \mathbb{R}_{\mathbf{S}}^{m_e \times m_e}$, and constant $\mu > 0$. Note that we can scale these variables with a factor $r > 0$, which essentially scales the matrix $\mathcal{J}_{DD}$ by this same factor r while the matrix J_{DD} stays constant (because it does not depend on either $X_{\mathbf{T}}$, $X_{\mathbf{S}}$, nor μ). As such, we can thus make sure that (14) is also satisfied when (11) is satisfied by choosing r large enough. Note now that, scaling of $X_{\mathbf{S}}$ by a factor r also influences the matrix C of (13), however this can still be compensated by choosing γ large enough (and, thus, MATI small enough).

Hence, based on the above observations, we established the following theorem.

Theorem 2: Take $\theta = 1$ and let $\lambda \in (0,1)$ be given. If there exist matrices $X_{\mathbf{T}} \in \mathcal{X}_{\mathbf{T}}$ and $X_{\mathbf{S}} \in \mathbb{R}_{\mathbf{S}}^{m_e \times m_e}$, and constants $M, \mu > 0$ such that (11) holds, then there also exist matrices $X_{\mathbf{T}} \in \mathcal{X}_{\mathbf{T}}$ and $X_{\mathbf{S}} \in \mathbb{R}_{\mathbf{S}}^{m_e \times m_e}$, and constants $M, \mu, \gamma > 0$ such that $\mathcal{J} - J \leq 0$ holds. ■

We thus have indeed shown that the condition (11) as derived in [6] for the case of perfect communication also a *sufficient* condition to guarantee robustness of the $\mathcal{L}_2$-stability property with an $\mathcal{L}_2$-gain less than or equal to $\theta = 1$ for the case of imperfect (packet-based) communication, under the assumption of τ_{mati} small enough.

V. Further extensions

Next to showing how the novel theorem for the case of non-ideal communication is connected to the theorem concerning systems with ideal communication as in [6], in this section, we will also show that the results as presented in the paper [7] have many natural extensions and applications as a result of the general (hybrid) framework based on (4)-(5). In particular, we will mention possible extensions with respect to the communication phenomena and it is shown that Theorem 1 also holds for other interconnection structures, including periodic and finite interconnections with boundary conditions.

A. Other communication phenomena

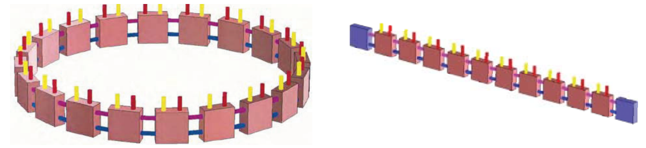
The framework of (4)-(5) enables us to apply known extensions from NCS-literature directly to our analysis. Hence, extensions including delays can be envisioned based on the framework developed in [12]. Moreover, the use of a "spatially invariant" scheduling protocol function h in (4) is not a necessary condition. Indeed, it can be shown that it is sufficient that all local scheduling protocol functions h^s, $s \in \mathbb{Z}$, are chosen such that Assumption 1 is satisfied for each of them with the same constants $\underline{\alpha}_W^c$, $\overline{\alpha}_W^c$, and M, $s \in \mathbb{Z}$. Hence, it is thus possible to obtain for every system a different λ^s and thus a different τ_{mati}^s, $s \in \mathbb{Z}$.

B. Different interconnection structures

Based on the results of [6], [9], [14], and [15], it can be shown that the Theorem 1 also applies to different interconnection types as long as they compose a *neutral interconnection* according to Definition 4. For example, besides the infinite interconnection of Fig. 1(b)\2 as was analyzed in [7], Definition 1 also allows us to consider periodic interconnections as depicted in Fig. 3(a). In particular, by expressing this type of interconnections by means of

$$\begin{aligned} v_+(s) &= \hat{w}_+(s-1), && \text{for} \quad 2 \leq s \leq N \\ v_-(s) &= \hat{w}_-(s+1), && \text{for} \quad 1 \leq s \leq N-1 \\ v_+(1) &= \hat{w}_+(N), && \text{and} \quad v_-(N) = \hat{w}_-(1), \end{aligned}$$

the periodic interconnection can again be captured by (3) and, hence, satisfies Definition 4 (for the same supply function $\mathcal{S}_i$ given by (8) as for the infinite interconnection case of Fig. 1(b)/2). This implies that the same analysis as for the infinite interconnection applies to the periodic interconnected system.



(a) Periodic interconnection. (b) Finite interconnection with boundary conditions.

Fig. 3. Extensions for spatially invariant interconnected systems from [9].

In addition to the periodic interconnection, it can be proven that the analysis for the infinite interconnection also holds for *finite* interconnections with boundary conditions as in Fig. 3(b). Indeed, similar to the periodic interconnection, we can describe each box of the finite interconnection of Fig. 3(b) as an instance of the basic building block from Fig. 1(a), except the two end ones, which require specific boundary conditions. In the case of zero boundary conditions, the interconnection can be described by

$$v_+(1) = 0 \quad \text{and} \quad v_+(s) = \hat{w}_+(s-1), \quad \text{for} \quad 2 \leq s \leq N,$$

where it should be noted that these types of interconnections are only defined when the interconnection is restricted to only the interconnection variables $v_+(s)$ and $w_+(s)$, see [9]. Such interconnections are sometimes referred to as "look-ahead" systems. It is obvious that the finite interconnection with zero boundary conditions also satisfies Definition 4 (for the same supply function $\mathcal{S}_i$ of (8)), and hence the same analysis can be applied as for the periodic and infinite interconnections.

However, when we are dealing with a finite interconnection with *non*zero boundary conditions, this is not so obvious anymore. In particular, consider the finite interconnections for which the interconnection relation between neighboring subsystems is described by

$$\begin{aligned} v_+(s) &= \hat{w}_+(s-1), && \text{for} \quad 2 \leq s \leq N \\ v_-(s) &= \hat{w}_-(s+1), && \text{for} \quad 1 \leq s \leq N-1 \\ v_+(1) &= \mathbf{M}\hat{w}_+(1), && \text{and} \quad v_-(N) = \mathbf{M}^{-1}\hat{w}_-(N), \end{aligned} \tag{15}$$

where $\mathbf{M}$ is a nonsingular matrix called the boundary conditions matrix.

Remark 6: This method for imposing the boundary conditions is motivated by the method of images and, in particulary, the concept of $\mathbf{M}$-reversibility. For the motivation and more details on these concepts, the interested reader is referred to [9], [14], [15].

As a result of the boundary conditions, the finite interconnection can no longer be classified as being "spatially invariant", however, they can still be analyzed in the same fashion as periodic/infinite interconnections. In particular, it has been proven in [15] that, if the basic building block is $\mathbf{M}$-reversible and the periodic/infinite interconnection is well-posed and stable, then the corresponding finite interconnection with boundary conditions matrix $\mathbf{M}$ is also stable. In other words, the conditions of Section III are also sufficient for the finite interconnection with boundary conditions under the assumption that the considered interconnection is $\mathbf{M}$-reversible.

To put this latter finite interconnection structure into more context, in the next section we will consider the practical example of a two-sided platoon of vehicles, which is such a spatially reversible system with boundary conditions.

Remark 7: The ℓ_2-space of Definition 1 also allows us to consider interconnected systems in higher dimensions. For a detailed analysis we refer to [6], [9], [15].

C. Future work

In this subsection we will very briefly outline several directions in which this research could be extended further. One of the obvious cases would be to consider heterogeneous or *spatially varying* systems. In other words, it would be no longer required for the subsystems to be identical. Using some LMI synthesis techniques for linear time varying systems and the obtained results as in [7], it might be possible to also obtain LMI synthesis conditions for spatially varying systems. Also controller synthesis as described in [6] could be incorporated in the hybrid modeling framework of (4)-(5). Finally, robust modeling techniques such as the linear fractional transformation approach in combination with small-gain type of arguments could be used to also take uncertainties into account, see also [9].

VI. Numerical example: $\mathcal{L}_2$-stability

To show the application of these new interconnection types, we will consider the numerical example of a two-sided platoon of cars, adapted from [15]. Consider a *finite* string of spatially invariant vehicles each with a length $L_v \in \mathbb{R}_{\geq 0}$. The absolute position of the rear bumper of the vehicle at position s is denoted by $r(s) \in \mathbb{R}$, from which it follows that the distance $d_v(s) \in \mathbb{R}_{\geq 0}$ between two vehicles can be defined by the difference between the position of the s^{th} vehicle and the middle of its closest neighbors, i.e.,

$$d_v(s) = -r(s) + \frac{1}{2}\left(r(s+1) + r(s-1)\right) - L_v.$$

For the (spatially invariant) vehicle model we use a double integrator system, given by

$$\begin{bmatrix} \dot{d}_v(s) \\ \dot{v}_v(s) \end{bmatrix} = \begin{bmatrix} -v_v(s) + {}^1\!/\!_2\left(v_v(s+1) + v_v(s-1)\right) \\ u(s) + d(s) \end{bmatrix}$$

with the performance output given by $z(s) = \varepsilon(s)$ and where $v_v(s)$, $u(s)$, and $d(s)$ are the vehicle's velocity, control input, and external disturbance input. Define now the distance error $\varepsilon(s) \in \mathbb{R}$ as

$$\varepsilon(s) = d_v(s) - d_r(s) = d_v(s) - (d_c + hv_v(s))$$

where $d_r(s)$ is the desired headway, given by $d_r(s) := d_c + hv_v(s)$, where d_c is the standstill distance and h the constant time headway, $s \in \mathbb{Z}$. The control input is chosen to be

$$\begin{aligned} u(s) &= k_s\left(r(s+1) - r(s) - d_r(s) - L_v\right) \\ &\qquad - k_s\left(r(s) - r(s-1) + d_r(s) + L_v\right) \\ &= 2k_s\varepsilon(s) \end{aligned}$$

for some constant $k_s > 0$, such that the dynamical model for each vehicle is now described by

$$\begin{bmatrix} \dot{\varepsilon}(s) \\ \\ \dot{v}_v(s) \end{bmatrix} = \begin{bmatrix} -v_v(s) + {}^1\!/\!_2\left(v_v(s+1) + v_v(s-1)\right) \\ - h\left(2k_s\varepsilon(s) + d(s)\right) \\ 2k_s\varepsilon(s) + d(s) \end{bmatrix}. \quad (16)$$

Hence, when we communicate the velocity over the communication networks from one vehicle to another as in Fig. 2, i.e., $w_+(s) = w_-(s) = v_v(s)$, $s \in \mathbb{Z}$, the model for each vehicle can be captured by (6) with $x(s) = (\varepsilon(s), v_v(s))$ and

$$A_{\mathbf{TT}} = \begin{bmatrix} -2hk_s & -1 \\ 2k_s & 0 \end{bmatrix}, \quad A_{\mathbf{TS}} = \begin{bmatrix} {}^1\!/\!_2 & {}^1\!/\!_2 \\ 0 & 0 \end{bmatrix}, \quad B_{\mathbf{T}} = \begin{bmatrix} -h \\ 1 \end{bmatrix}$$

$$A_{\mathbf{ST}} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad C_{\mathbf{T}} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_{\mathbf{S}} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad D = 0.$$

In addition, the finite two-sided platoon can be described as a spatially $\mathbf{M}$-reversible system, and therefore we can model the overall interconnection as a finite interconnection with boundary conditions as in Fig. 3(b), see also [14] or [15]. In particular, we consider a set of relevant boundary conditions that involves a virtual leader, located in front of the first vehicle of the platoon, and a virtual follower located behind the L-th vehicle of the platoon. These virtual leader and follower have the same velocity as the first and last vehicle of the platoon, respectively, which corresponds to the boundary conditions matrix $\mathbf{M} = 1$ in (15).

As a result, the analysis results as described in this paper can be used to verify if the system is $\mathcal{L}_2$-stable when non-ideal communication is used. In particular, it is easy to verify that, for the considered system of (16), indeed (11) holds for $h \geq 0.5$, $k_s = 5$, $\mu = 1$, and $\theta = 1$. This implies that there exists a τ_{mati} small enough such that $\mathcal{L}_2$-stability is guaranteed. Indeed, when Theorem 1 and the SD-protocol ($\underline{\alpha}_W^c = \overline{\alpha}_W^c = M = 1, \lambda = 1 \cdot 10^{-3}$) are used, we can obtain the values for τ_{mati} for which the platoon of vehicles is $\mathcal{L}_2$-stable with a gain less than or equal to $\theta = 1$, see Fig. 4.

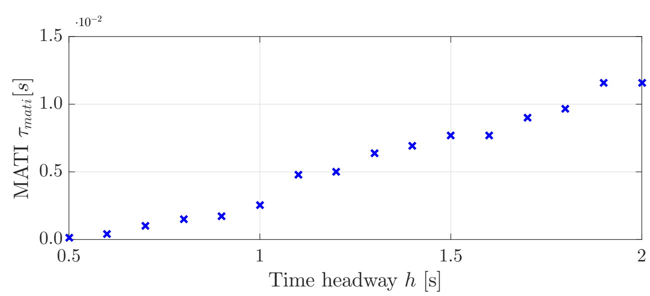


Fig. 4. Values of τ_{mati} guaranteeing $\mathcal{L}_2$-stability for the finite platoon of vehicles for various time headways h.

VII. Concluding remarks

In this paper we recounted and explored several potential extensions of the developed modeling framework for an infinite interconnection of hybrid systems to model so-called "networked" systems that use packet-based communication networks for the exchange of information. The main result in this context provided conditions that only involved the *local* information of *one* of the subsystems in the interconnection and the adopted scheduling protocol, and that lead to an explicit bound on the MATI for each individual communication network such that *global* stability and $\mathcal{L}_2$-performance properties are guaranteed. For linear systems these conditions could even be expressed as LMIs. In addition, we showed how the obtained results for this non-ideal communication case are connected to the results for the ideal communication case as obtained in [6]. This lead to the conclusion that the condition as derived in [6] for the perfect communication case is also a sufficient condition to guarantee robustness of the $\mathcal{L}_2$-stability property for the case of imperfect, packet-based communication (for sufficiently small MATI). Moreover, extensions regarding the framework of [7] have been discussed, in which we elaborated particularly on the inclusion of delays, the use of spatially varying scheduling protocols, and the applicability of Theorem 1 for periodic and finite interconnection with boundary conditions. Finally, the effectiveness of the results was shown by examining a finite platoon of vehicles with boundary conditions for which $\mathcal{L}_2$-stability could be guaranteed and corresponding time headways and MATIs could be computed explicitly.

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