

# Stability Analysis of Spatially Invariant Systems with Event-Triggered Communication

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**Abstract**—In this paper we analyze spatially invariant interconnections consisting of a (finite) number of subsystems that use packet-based communication networks for the exchange of information. An example of such an interconnected system is the platoon of vehicles that uses cooperative control to drive autonomously. By building upon a recently developed hybrid systems framework for the considered spatially invariant interconnections, we investigate the design of event-based triggering mechanisms that aim to reduce the amount of transmission times while still guaranteeing behavior in terms of uniform global asymptotic stability (UGAS) for the overall interconnected system. To obtain tractable design conditions, we exploit the spatially invariant property. As a result, we obtain conditions based on only the *local* information of *one* of the subsystems in the interconnection and the interconnection structure itself. A nonlinear example is used to illustrate the applications and benefits of the obtained modeling approach.

**Keywords**—Networked control systems, event-triggered control, spatially invariant systems, large-scale systems.

## I. INTRODUCTION

The control of large-scale systems consisting of a number of similar interconnected units or subsystems is gaining more and more attention. Examples of these “systems of systems” include satellite constellations [1], flocks of systems [2], and vehicle platoons, see, e.g., [3]. Typically, when many (or even an infinite number of) subsystems are interconnected, analysis based on global monolithic models encounter severe limitations due to the very high dimensionality. As a result, a considerable amount of research effort has been targeted on analysis and design methods that aim to guarantee global system properties based on local conditions on only one of the subsystems in the interconnection, see, e.g., [4]–[9].

In the works [4]–[9] the communication between the subsystems is assumed to be perfect, which is, in the context of large-scale systems, often not the case. For this reason, it is of interest to study systems consisting of an infinite number of spatially invariant subsystems that are interconnected through packet-based communication networks. In [10], [11], inspired by [12]–[17], each *identical* subsystem and its communication network are modeled as a *hybrid system*, resulting in an infinite-dimensional, spatially invariant interconnection of hybrid systems. This hybrid modeling framework facilitates to obtain *local* conditions that only involve the local information

of *one* of the hybrid subsystems in the interconnection and that result in uniform global asymptotic stability (UGAS) guarantees for the overall system.

In the above mentioned framework, similar to [12]–[17], it is assumed that the transmission instants for each local communication network are determined purely based on time and that the transmission intervals are upper bounded by a maximally allowable transmission interval (MATI). The advantage of these time-based schemes is that they are, in general, predictable and easy to implement. However, as the required time-based specifications are typically determined via worst-case estimates, approaches relying on a MATI bound often result in redundant transmission times. As such, it seems natural to search for so-called resource-aware control strategies that determine the actual need for a transmission based on (output) measurements of the system.

Therefore, in this paper, we consider an event-triggered approach to determine the transmission instants online [18]–[21]. In particular, we will show that the hybrid framework for spatially invariant time-triggered systems as obtained in [10], [11] can be extended with an event-triggering mechanism (ETM). Hereto, we consider an interconnection consisting of a *finite* number of identical so-called networked control systems (NCSs), systems that consist of a plant and a controller of which the sensor and actuation data is transmitted over a shared communication network, see, e.g., [12]–[15]. The reason why we focus here on finite interconnections rather than infinite ones is a result of the guarantees on the existence of solutions, as will be explained below. For this type of spatially invariant systems with event-triggered communication, we will derive *local* conditions that guarantee UGAS of the overall system and lead to a positive minimum inter-event time (MIET).

The remainder of this paper is organized as follows. After presenting the necessary preliminaries and notational conventions in Section II, the class of systems considered in this paper is described in Section III by means of introducing the system configuration being the interconnection of NCSs, elaborating on the ETM for each local communication network, and presenting the hybrid modeling framework for each subsystem in the interconnection. In Section IV we derive (local) conditions for the proposed even-triggering strategy such that UGAS is guaranteed for the overall interconnected system. Here, also some general design considerations are discussed. Finally, in Section V, a nonlinear numerical example illustrating the applications of our results is provided, and in Section VI some concluding remarks are given.

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## II. PRELIMINARIES

The set of non-negative integers is denoted by  $\mathbb{N}$ , the set of real numbers by  $\mathbb{R}$ , and the set of non-negative real numbers by  $\mathbb{R}_{\geq 0}$ . The notation  $v \in \mathbb{R}^\bullet$  will denote real valued, finite vectors whose size is either clear from context or not relevant to the discussion. For vectors  $v_1, v_2, \dots, v_n \in \mathbb{R}^\bullet$ , we denote by  $(v_1, v_2, \dots, v_n)$  the vector  $[v_1^\top \ v_2^\top \ \dots \ v_n^\top]^\top$ , and by  $|\cdot|$  and  $\langle \cdot, \cdot \rangle$  the Euclidean norm and the usual inner product, respectively. The space of real symmetric  $n$  by  $n$  matrices is denoted  $\mathbb{R}_S^{n \times n}$ . The  $n$  by  $n$  identity and zero matrices are denoted by  $I_n$  and  $0_n$ . When the dimensions are clear from the context, these notations are simplified to  $I$  and  $0$ .

A function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is of class  $\mathcal{K}$  if it is continuous, strictly increasing and  $\alpha(0) = 0$ . It is of class  $\mathcal{K}_\infty$  if it is of class  $\mathcal{K}$  and, in addition, it is unbounded. A function  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is of class  $\mathcal{KL}$  if for each fixed  $s$ , the mapping  $r \mapsto \beta(r, s)$  belongs to class  $\mathcal{K}$  and for each fixed  $r$ , the mapping  $s \mapsto \beta(r, s)$  is decreasing and  $\beta(r, s) \rightarrow 0$  when  $s \rightarrow \infty$ . A continuous function  $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is said to be of class  $\mathcal{KLL}$  if, for each  $r \geq 0$ , both  $\gamma(\cdot, \cdot, r)$  and  $\gamma(\cdot, r, \cdot)$  belong to class  $\mathcal{KL}$ . Given a Banach space  $X$ , a function  $f : X \rightarrow X$  is said to be locally Lipschitz continuous if for each  $x_0 \in X$  there exists constants  $\delta, L > 0$  such that for all  $x \in X$  we have that  $\|x - x_0\|_X \leq \delta \Rightarrow \|f(x) - f(x_0)\|_X \leq L \|x - x_0\|_X$ , where  $\|\cdot\|_X$  denotes the norm in  $X$  [22].

In [6] and [10], the state-space of the considered overall systems was infinite-dimensional. However, as will be explained below, in this paper we only consider spatially invariant interconnections consisting of a *finite* number of (hybrid) subsystems. As a result, we introduce the set  $\mathbb{D} := \{1, 2, \dots, N\}$  with  $N$  the number of subsystems in the spatial dimension. For ease of notation, using this set  $\mathbb{D}$ , similar to [6] and [10], we define the following space, a slightly adjusted variant of the  $\ell_2$  sequence space of [22].

**Definition 1.** The space  $\ell_2^n$  is the set of functions mapping  $\mathbb{D}$  to  $\mathbb{R}^n$  equipped with the inner product  $\langle \cdot, \cdot \rangle_{\ell_2}$  for  $x, y \in \ell_2^n$  defined as

$$\langle x, y \rangle_{\ell_2} := \sum_{s \in \mathbb{D}} x(s)^\top y(s),$$

and the corresponding norm as  $\|x\|_{\ell_2} := \sqrt{\langle x, x \rangle_{\ell_2}}$ .

When the dimension  $n$  is clear from context or not relevant, we sometimes write  $\ell_2^n$  as  $\ell_2$ . The (spatial) shift operator  $\mathbf{S}$ , acting on variables in  $\ell_2$ , is defined as

$$(\mathbf{S}\xi)(s) := \xi(s+1).$$

To describe the finite interconnections as considered in this paper, we use modular arithmetic for  $s \in \mathbb{D}$ , i.e.,  $N+1 := 1$ . Consider now a function  $r = (r_+, r_-) \in \ell_2^{m_+ + m_-}$  and define  $m := (m_+, m_-)$ . In a similar fashion as in [10], we introduce the structured operator  $\Delta_{\mathbf{S}, m} : \ell_2^{m_+ + m_-} \rightarrow \ell_2^{m_+ + m_-}$  that satisfies

$$(\Delta_{\mathbf{S}, m} r)(s) = \begin{bmatrix} [I_{m_+} & 0] (\mathbf{S}^{-1} r)(s) \\ [0 & I_{m_-}] (\mathbf{S} r)(s) \end{bmatrix} = \begin{bmatrix} r_+(s-1) \\ r_-(s+1) \end{bmatrix}. \quad (1)$$

**Remark 1.** As shown in [6], Definition 1 and the structured operator of (1) can easily be extended to cope with multiple spatial dimensions, however, in this paper we only consider systems in one spatial dimension.

In this paper, we will describe each spatially invariant NCS in terms of a hybrid model. Hereto, similar to [10], [11], [23], we consider the overall interconnected system  $\mathcal{H}$  to be an interconnection consisting of spatially invariant hybrid (sub)systems  $\mathcal{H}(s)$  that are all given for  $s \in \mathbb{D}$  by

$$\mathcal{H}(s) : \begin{cases} \dot{\xi}(s) = F(\xi)(s), & \xi(s) \in \mathcal{C}(s) \\ \xi^+(s) \in G(\xi(s)), & \xi(s) \in \mathcal{D}(s) \end{cases} \quad (2)$$

with  $\xi = (\xi_c, \xi_d)$  the state of the overall system where  $\xi_c \in \ell_2^{m_1}$  comprises the (internal) dynamical states of the system while  $\xi_d \in \ell_2^{m_2}$  comprises (auxiliary) states like timers and counters. In (2) we have that  $G : \mathbb{R}^{m_1 + m_2} \rightrightarrows \mathbb{R}^{m_1 + m_2}$  is a set-valued function,  $\mathcal{C}(s)$  and  $\mathcal{D}(s)$  are closed subsets of  $\mathbb{R}^{m_1 + m_2}$ , and that the function  $F : \ell_2^{m_1} \times \ell_2^{m_2} \rightarrow \ell_2^{m_1} \times \ell_2^{m_2}$  is locally Lipschitz in its first argument, see also [23]. Since we only consider a finite number of (hybrid) subsystems, based on (2), we can also describe the overall hybrid system  $\mathcal{H}$ , given by

$$\mathcal{H} : \begin{cases} \dot{\xi} = F(\xi), & \xi \in \tilde{\mathcal{C}} \\ \xi^+ \in \tilde{G}(\xi), & \xi \in \tilde{\mathcal{D}}, \end{cases} \quad (3)$$

where  $\tilde{G} : \ell_2^{m_1 + m_2} \rightrightarrows \ell_2^{m_1 + m_2}$  is given by  $\tilde{G}(\xi)(s) = \bigcup_{s \in \mathbb{D}} G^s(\xi)$  with

$$G^s(\xi) = \begin{cases} \{G_0^s(\xi)\}, & \text{when } \xi \in D(s) \\ \emptyset, & \text{when } \xi \notin D(s) \end{cases}$$

and  $G_0^s(\xi) = (\xi(1), \xi(2), \dots, \xi(s-1), G(\xi(s)), \xi(s+1), \dots, \xi(N))$ , and where the flow and jump sets are given by  $\tilde{\mathcal{C}} = \mathcal{C}(1) \times \mathcal{C}(2) \times \dots \times \mathcal{C}(N)$  and  $\tilde{\mathcal{D}} = \bigcup_{s \in \mathbb{D}} D(s)$  with  $D(s) = \{\xi \in \ell_2^{m_1 + m_2} \mid \xi(s) \in \mathcal{D}(s)\}$ , respectively.

We now recall some standard definitions on the solutions of such hybrid systems. For the motivation and more details on these definitions, the interested reader is referred to [24].

**Definition 2.** A compact hybrid time domain is a set  $E = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$  with  $J \in \mathbb{N}_{>0}$  and  $0 \leq t_0 \leq t_1 \leq \dots \leq t_J$ . A hybrid time domain is a set  $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$  such that  $E \cap ([0, T] \times \{0, 1, \dots, J\})$  is a compact hybrid time domain for each  $(T, J) \in E$ .

**Definition 3.** A hybrid trajectory is a pair  $(\text{dom } \xi, \xi)$  consisting of a hybrid time domain  $\text{dom } \xi$  and a function  $\xi$  defined on  $\text{dom } \xi$  that is absolutely continuous in  $t$  on  $(\text{dom } \xi) \cap (\mathbb{R}_{\geq 0} \times \{j\})$  for each  $j \in \mathbb{N}$ .

**Definition 4.** A hybrid trajectory  $(\text{dom } \xi, \xi)$  with  $\xi : \text{dom } \xi \rightarrow \ell_2^n$  is a solution to the overall hybrid system  $\mathcal{H}$  given by (3) and composed of the identical hybrid subsystems  $\mathcal{H}(s)$  given by (2),  $s \in \mathbb{D}$ , if

- 1) for all  $j \in \mathbb{N}$  and for almost all  $t \in I_j := \{t \mid (t, j) \in \text{dom } \xi\}$ , we have  $\xi((t, j), s) \in \mathcal{C}(s)$  for all  $s \in \mathbb{D}$  and  $\dot{\xi}(t, j) = F(\xi(t, j))$ .
- 2) for all  $(t, j) \in \text{dom } \xi$  such that  $(t, j+1) \in \text{dom } \xi$ , we have that  $\xi((t, j), s) \in \mathcal{D}(s)$  and  $\xi((t, j+1), s) \in G(\xi((t, j), s))$  for some  $s \in \mathbb{D}$ .

Insinuated by these definitions, in this paper, we will thus consider variables  $\xi \in \text{dom } \xi \times \mathbb{D} \rightarrow \mathbb{R}^n$  that are vector-valued functions indexed by three independent variables, i.e.,  $\xi = \xi((t, j), s)$ , where  $(t, j) \in \text{dom } \xi$  is the hybrid time domain and  $s \in \mathbb{D}$  the spatial variable. For fixed  $s \in \mathbb{D}$  and  $(t, j) \in \text{dom } \xi$ , a variable  $\xi(t, j)$  can be considered as an element of  $\ell_2^m$  and  $\xi((t, j), s)$  as an element of  $\mathbb{R}^n$ , i.e., a real-valued vector. We often do not mention  $\text{dom } \xi$  and  $(t, j)$  explicitly, but understand that with each hybrid trajectory  $\xi$  comes a hybrid time domain  $\text{dom } \xi$ . Note that in (2) we denoted  $\xi((t, j+1), s)$  as in Definition 4 item 2) as  $\xi^+(s)$ .

### III. SYSTEM DESCRIPTION

In this section, we introduce the modeling setup for the spatially invariant interconnected event-triggered NCSs.

#### A. Networked control configuration

We consider the overall system that consists of a finite number of spatially invariant systems, interconnected according to, for instance, the structures of Fig. 1, see also Remark 2. More precisely, we consider so-called networked control systems (NCSs), see, e.g., [12]–[15], which are physically interconnected according to the structure shown in Fig. 2.

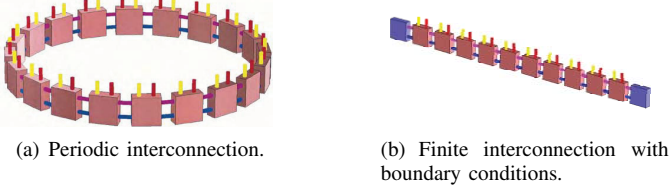


Fig. 1. Possible interconnection structures for a finite number of spatially invariant interconnected systems from [7].

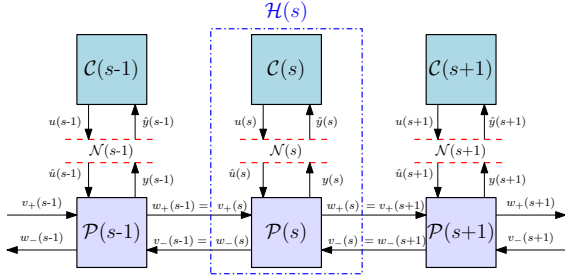


Fig. 2. Interconnection structure, where each subsystem  $\mathcal{P}(s)$  has its own network  $\mathcal{N}(s)$  to communicate with its controller  $\mathcal{C}(s)$ ,  $s \in \mathbb{D}$ . The “networked” (hybrid) subsystem  $\mathcal{H}(s)$  is the combination of subsystem  $\mathcal{P}(s)$ , its controller  $\mathcal{C}(s)$ , and its network  $\mathcal{N}(s)$ .

As illustrated, we consider a decentralized control configuration where each continuous-time plant  $\mathcal{P}(s)$  is given by

$$\mathcal{P}(s) : \begin{cases} \dot{x}_p(s) = f_p(x_p(s), v(s), \hat{u}(s)) \\ w(s) = g_p(x_p(s)) + A_{\text{SS}}v(s) \\ y(s) = h_p(x_p(s)) \end{cases} \quad (4)$$

with the initial condition  $x_p((0,0), s) = x_{0,p}(s) \in \mathbb{R}^{m_{x_p}}$ ,  $s \in \mathbb{D}$ , and  $x_{0,p} \in \ell_2^{m_{x_p}}$  and where  $x_p(s) \in \mathbb{R}^{m_{x_p}}$  denotes the (local) plant state,  $\hat{u}(s) \in \mathbb{R}^{m_u}$  the most recently received local control input, and  $y(s) \in \mathbb{R}^{m_y}$  the local output of the plant  $\mathcal{P}(s)$ ,  $s \in \mathbb{D}$ . Moreover, we have that

$$v(s) = \begin{bmatrix} v_+(s) \\ v_-(s) \end{bmatrix} \quad \text{and} \quad w(s) = \begin{bmatrix} w_+(s) \\ w_-(s) \end{bmatrix}$$

are used to describe the partitioning of the interconnection variables with  $v_+(s) \in \mathbb{R}^{m_+}$  and  $v_-(s) \in \mathbb{R}^{m_-}$  being the interconnected inputs, and  $w_+(s) \in \mathbb{R}^{m_+}$  and  $w_-(s) \in \mathbb{R}^{m_-}$  the interconnected outputs. The linear operator  $A_{\text{SS}}$  maps  $\mathbb{R}^{m_+ + m_-}$  to  $\mathbb{R}^{m_+ + m_-}$ . In addition, we have that  $f_p : \mathbb{R}^{m_{x_p}} \times \mathbb{R}^{m_+ + m_-} \times \mathbb{R}^{m_u} \rightarrow \mathbb{R}^{m_{x_p}}$ ,  $g_p : \mathbb{R}^{m_{x_p}} \rightarrow \mathbb{R}^{m_+ + m_-}$ ,  $h_p : \mathbb{R}^{m_{x_p}} \rightarrow \mathbb{R}^{m_y}$ , and  $q_p : \mathbb{R}^{m_{x_p}} \times \mathbb{R}^{m_+ + m_-} \rightarrow \mathbb{R}^{m_z}$  are nonlinear mappings, where it is assumed that  $f_p$  is sufficiently smooth, see also [23], and  $h_p$  is continuously differentiable. Note that here  $x_p \in \ell_2^{m_{x_p}}$  is used to denote the overall state of the interconnected plants  $\mathcal{P}(s)$ ,  $s \in \mathbb{D}$ , that  $v_+(s)$  and  $w_+(s)$  have the same size, and that  $v_-(s)$  and  $w_-(s)$  have the same size.

Each plant  $\mathcal{P}(s)$  is controlled by its own local controller  $\mathcal{C}(s)$ , which is connected to the plant via a local (independent) communication network  $\mathcal{N}(s)$  as shown in Fig. 2. The controller  $\mathcal{C}(s)$  is given by

$$\mathcal{C}(s) : \begin{cases} \dot{x}_c(s) = f_c(x_c(s), \hat{y}(s)) \\ u(s) = h_c(x_c(s)), \end{cases} \quad (5)$$

where  $x_c(s) \in \mathbb{R}^{m_c}$  denotes the controller state,  $\hat{y}(s) \in \mathbb{R}^{m_y}$  the most recently received output measurement of the plant, and  $u(s) \in \mathbb{R}^{m_u}$  the controller output for controller  $\mathcal{C}(s)$ ,  $s \in \mathbb{D}$ . Hence,  $f_c : \mathbb{R}^{m_{x_c}} \times \mathbb{R}^{m_y} \rightarrow \mathbb{R}^{m_{x_c}}$  and  $h_c : \mathbb{R}^{m_{x_c}} \rightarrow \mathbb{R}^{m_u}$  are also nonlinear mappings, where it is assumed that  $f_c$  is sufficiently smooth and  $h_c$  is continuously differentiable, see also [23].

Similar to [6], to compactly write the entire interconnected system as shown in Fig. 2, we define, based on the identical subsystems described by (4)-(5) and the network  $\mathcal{N}(s)$ ,  $v_+(s) := w_+(s-1)$  and  $v_-(s) := w_-(s+1)$ , which, by using (1), leads to the compact expression

$$v(s) = (\Delta_{\mathbf{S}, m} w)(s), \quad s \in \mathbb{D}. \quad (6)$$

Before we complete the mathematical model of the NCS setup, we first discuss how the communication networks  $\mathcal{N}(s)$  operate.

**Remark 2.** As a result of the modular arithmetic of  $s \in \mathbb{D}$ , we have that  $v_+(1) = w_+(N)$  and  $v_-(N) = w_-(1)$ . Hence, (6) can indeed be used to describe the periodic interconnection of Fig. 1(a), see also [6]. In addition, it can be proven that the overall system as in Fig. 2 can also represent the finite interconnection of Fig. 1(b), however, hereto one needs the assumption that the interconnection is so-called **M**-reversible. In addition, to define the overall system  $\mathcal{H}$ , also the boundary conditions itself need to be specified, i.e.,  $v_+(1) = \mathbf{M}w_+(1)$  and  $v_-(N) = \mathbf{M}^{-1}w_-(N)$  where the nonsingular matrix  $\mathbf{M}$  called the boundary conditions matrix needs to be defined. For more information and a detailed analysis concerning such interconnections with boundary conditions we refer to [8], [9].

#### B. Communication networks and protocols

In the networked control configuration of Fig. 2, all the local networks operate independently of each other in the sense that each network  $\mathcal{N}(s)$  has its own collection of transmission/sampling times  $t_{j^s}^s$ ,  $j^s \in \mathbb{N}$  satisfying  $0 \leq t_0^s < t_1^s < t_2^s \dots$ . At each of these transmission times  $t_{j^s}^s$  (parts of) the output  $y(s)$  and input  $u(s)$  are sampled and transmitted over the network  $\mathcal{N}(s)$  to the controller  $\mathcal{C}(s)$  and plant  $\mathcal{P}(s)$ , respectively. A scheduling protocol determines which of the actuator and/or sensor nodes in the network is granted access to the network. After a node is granted access to the network, it collects and transmits the values of the corresponding entries in  $y(t_{j^s}^s, s)$  and  $u(t_{j^s}^s, s)$ , resulting in an update according to

$$\begin{aligned} \hat{y} \left( (t_{j^s}^s)^+, s \right) &= y(t_{j^s}^s, s) + h_y(j^s, e(t_{j^s}^s)) \\ \hat{u} \left( (t_{j^s}^s)^+, s \right) &= u(t_{j^s}^s, s) + h_u(j^s, e(t_{j^s}^s)), \end{aligned}$$

where the function  $h := (h_y, h_u)$  (with slight abuse of notation) models the (local) scheduling protocol [12]–[15] and where  $e(s)$ ,  $s \in \mathbb{D}$ , denotes the network-induced error defined by

$$e(s) := \begin{bmatrix} e_y(s) \\ e_u(s) \end{bmatrix} = \begin{bmatrix} \hat{y}(s) - y(s) \\ \hat{u}(s) - u(s) \end{bmatrix}, \quad s \in \mathbb{D}.$$

$$\begin{aligned} f(x, e)(s) &= \begin{bmatrix} f_p(x_p(s), (\Delta_{\mathbf{S}, m} w)(s), h_c(x_c(s)) + e_u(s)) \\ f_c(x_c(s), h_p(x_p(s)) + e_y(s)) \end{bmatrix}, \\ g(x, e)(s) &= \begin{bmatrix} -\frac{\partial h_p(x_p(s))}{\partial x_p(s)} f_p(x_p(s), (\Delta_{\mathbf{S}, m} w)(s), h_c(x_c(s)) + e_u(s)) \\ -\frac{\partial h_c(x_c(s))}{\partial x_c(s)} f_c(x_c(s), h_p(x_p(s)) + e_y(s)) \end{bmatrix} \end{aligned} \quad (10)$$

For this configuration it is assumed that the networks operate in a ZOH fashion, in the sense that  $\hat{y}$  and  $\hat{u}$  do not change between transmissions, although this can easily be modified if desired, see [13].

While in [10], [11] the transmission instants were time-based and therefore straightforward, in this paper, the networks are considered to operate in an event-triggered way. Therefore, based on the proposed framework as in [21], the triggering condition is given by

$$t_0^s = 0, \quad t_{j^s+1}^s = \inf \{t > t_{j^s}^s + \tau_{miet} \mid \eta((t, j), s) \leq 0\} \quad (7)$$

where  $\tau_{miet} \in \mathbb{R}_{>0}$  is the minimum inter-event time (MIET), which by definition prevents (local) Zeno behavior [21], and  $\eta((t, j), s) \in \mathbb{R}_{\geq 0}$  is the solution to the hybrid system

$$\begin{cases} \dot{\eta}(s) = \Psi(x(s), e(s), \tau(s)) \\ \eta^+(s) = \eta_0(\kappa(s), e(s)) \end{cases} \quad (8)$$

for all  $s \in \mathbb{D}$  where the functions  $\Psi$  and  $\eta_0$  will be specified later. With now the triple  $(\mathcal{P}(s), \mathcal{C}(s), \mathcal{N}(s))$  described for all  $s \in \mathbb{D}$ , the NCS setup is complete. As such, based on the concepts described in [10]–[17], [21], it is possible to obtain a hybrid model for each individual subsystem.

### C. A hybrid modeling framework

By combining the previously obtained models for the plant  $\mathcal{P}(s)$ , the controller  $\mathcal{C}(s)$ , and the network  $\mathcal{N}(s)$ , a hybrid system  $\mathcal{H}(s)$  can be obtained for each of the “networked” subsystems by eliminating the interconnection variables  $w$ . Hence, based on the results of [10], [11], [21] and by introducing the timers  $\tau((t, j), s) \in \mathbb{R}_{\geq 0}$  and the counters  $\kappa((t, j), s) \in \mathbb{N}$  for every fixed  $s \in \mathbb{D}$ , each triple  $(\mathcal{P}(s), \mathcal{C}(s), \mathcal{N}(s))$  can be written as

$$\mathcal{H}(s) : \begin{cases} \begin{cases} \dot{x}(s) = f(x, e)(s) \\ \dot{e}(s) = g(x, e)(s) \\ \dot{\tau}(s) = 1 \\ \dot{\kappa}(s) = 0 \\ \dot{\eta}(s) = \Psi(x(s), e(s), \tau(s)) \end{cases} & \left. \begin{array}{l} \text{when} \\ \tau(s) \in [0, \tau_{miet}] \\ \vee \eta(s) \geq 0 \end{array} \right\} \\ \begin{cases} x^+(s) = x(s) \\ e^+(s) = h(\kappa(s), e(s)) \\ \tau^+(s) = 0 \\ \kappa^+(s) = \kappa(s) + 1 \\ \eta^+(s) = \eta_0(\kappa(s), e(s)) \end{cases} & \left. \begin{array}{l} \text{when} \\ \tau(s) \in [\tau_{miet}, \infty) \\ \wedge \eta(s) \leq 0 \end{array} \right\} \end{cases} \quad (9)$$

with the state of the subsystem  $\mathcal{H}(s)$  given by  $\xi(s) = (x(s), e(s), \tau(s), \kappa(s), \eta(s))$  with  $x(s) = (x_p(s), x_c(s)) \in \mathbb{R}^{m_x}$ ,  $m_x = m_{x_p} + m_{x_c}$ , and  $e(s) \in \mathbb{R}^{m_e}$ ,  $m_e = m_y + m_u$ . Moreover, the nonlinear mappings  $f(x, e)(s)$  and  $g(x, e)(s)$  are given in (10) for  $s \in \mathbb{D}$  with

$$w = (\tilde{I}_{m_+ + m_-} - \tilde{A}_{\text{SS}} \Delta_{\mathbf{S}, m})^{-1} \tilde{g}_p(x_p)$$

where  $\tilde{A}_{\text{SS}}$  is the  $\ell_2$ -operator which maps  $\ell_2^{m_+ + m_-}$  to  $\ell_2^{m_+ + m_-}$  according  $(\tilde{A}_{\text{SS}} v)(s) = A_{\text{SS}} v(s)$  for all  $s \in \mathbb{D}$ , and  $\tilde{I}_{m_+ + m_-}$

and  $\tilde{g}_p(x_p)$  are the “diagonal”  $\ell_2$ -operators defined by  $\tilde{g}_p(x_p)(s) := g_p(x_p(s))$  and  $\tilde{I}_{m_+ + m_-}(s) := I_{m_+ + m_-}$ , respectively, for all  $s \in \mathbb{D}$ . Hereby, it is assumed that the inverse of  $(\tilde{I}_{m_+ + m_-} - \tilde{A}_{\text{SS}} \Delta_{\mathbf{S}, m})$  exists, which, for instance, can be guaranteed by requiring that there are no direct-feed-through terms ( $A_{\text{SS}} = 0$ ). Note that (9) is indeed of the general form of (2) for  $\xi_c = (x, e, \eta)$  and  $\xi_d = (\tau, \kappa)$ . Moreover, based on (2)-(3), the overall interconnection as depicted in Fig. 2 is now described by the hybrid system  $\mathcal{H}$ , being the finite interconnection of subsystems  $\mathcal{H}(s)$  given by (9),  $s \in \mathbb{D}$ .

**Remark 3.** Note that, in contrast to [4], [6], [10], [11], [23], in this paper we do not consider an *infinite* number of interconnected systems, but rather a finite number of systems. This is because, if we would have considered an infinite number of interconnected (hybrid) subsystems, no solutions could be defined globally, i.e., for all time  $t \in [0, \infty)$ , using the solution concept as in Section II from [24] as Zeno points are (almost always) inevitable, see also the discussion in [23]. Moreover, while in [23] a solution concept has been developed that allows for the definition of solutions beyond Zeno points in the form of right-accumulation points, it too cannot be applied as Zeno points in the form of left-accumulation points are also possible in the considered setup. However, as shown in [9], see also Remark 2, we can still base our finite-dimensional analysis on the infinite-dimensional results of [6], [10], [11], [23], in the sense that local conditions based only on the dynamics of *one* of the subsystems in the overall interconnection can be derived by exploiting the specific interconnection structure of (6), as we will see below. This also implies that if solutions could be defined globally for an infinite number of interconnected hybrid subsystems with the (possible) admission of left-accumulation points, the analysis presented in this paper would still hold for the infinite-dimensional case.

## IV. STABILITY ANALYSIS

With the entire system  $\mathcal{H}$  of Fig. 2 now modeled, its stability can be analyzed. Hereto, we consider a set of initial states for the overall system  $\mathcal{H}$  composed of the identical subsystems  $\mathcal{H}(s)$  given by (9),  $s \in \mathbb{D}$ , which is specified by

$$\mathbb{X}_0 = \ell_2^{m_x + m_e} \times \mathbb{R}_{\geq 0}^N \times \mathbb{N}^N \times \ell_2^1. \quad (11)$$

**Definition 5.** For the overall system  $\mathcal{H}$  with associated set of initial states  $\mathbb{X}_0$  given by (11) composed of the identical subsystems  $\mathcal{H}(s)$  given by (9),  $s \in \mathbb{D}$ , the set

$$\mathcal{E} = \{\xi \in \mathbb{X}_0 \mid x = 0 \wedge e = 0 \wedge \eta = 0\} \quad (12)$$

is said to be uniformly globally asymptotically stable (UGAS) if there exists a function  $\beta \in \mathcal{KLL}$  such that for any initial condition  $\xi(0, 0) \in \mathbb{X}_0$ , all corresponding solutions  $\xi$  satisfy for all  $(t, j) \in \text{dom } \xi$

$$|x(t, j), e(t, j), \eta(t, j)| \leq \beta(|x(0, 0), e(0, 0), \eta(0, 0)|, t, j).$$

Moreover, the set  $\mathcal{E}$  of (12) is said to be uniform exponentially stable (UGES) if the function  $\beta$  can be taken of the form  $\beta(r, t, j) = Mr \exp(-\rho(t + j))$  for some  $M \geq 0$  and  $\rho > 0$ .

Using this definition, UGAS can be analyzed for the overall system. Hereto, we follow a similar approach as in [10], [11] in combination with the obtained results as in [21]. As such, we want to obtain conditions in terms of  $\tau_{miet}$ ,  $\Psi$ ,  $\eta_0$ , and the local dynamics of one of the subsystems in the interconnection such that UGAS of the set  $\mathcal{E}$  of (12) is guaranteed.

#### A. Conditions for UGAS (or UGES)

To obtain the conditions that guarantee UGAS, similar as to [10]–[16], [21], we first assume the following conditions to hold for the local scheduling protocols.

**Condition 1.** For the local scheduling protocol given by  $h$  there exist the functions  $\underline{\alpha}_W, \bar{\alpha}_W \in \mathcal{K}_\infty$  and  $W : \mathbb{N} \times \mathbb{R}^{m_e} \rightarrow \mathbb{R}_{\geq 0}$  that is locally Lipschitz in its second argument, and the constant  $\lambda \in (0, 1)$  such that for all  $\kappa(s) \in \mathbb{N}$  and  $e(s) \in \mathbb{R}^{m_e}$ ,  $s \in \mathbb{D}$ , such that

$$\underline{\alpha}_W(|e(s)|) \leq W(\kappa(s), e(s)) \leq \bar{\alpha}_W(|e(s)|) \quad (13a)$$

$$W(\kappa(s) + 1, h(\kappa(s), e(s))) \leq \lambda W(\kappa(s), e(s)). \quad (13b)$$

**Condition 2.** The function  $W$  as given in Condition 1 satisfies for almost all  $e(s) \in \mathbb{R}^{m_e}$  and all  $\kappa(s) \in \mathbb{N}$

$$\left\langle \frac{\partial W(\kappa(s), e(s))}{\partial e(s)}, g(x, e)(s) \right\rangle \leq \quad (14)$$

$$L_W W(\kappa(s), e(s)) + H(x(s), v(s))$$

for some constant  $L_W \in \mathbb{R}_{\geq 0}$  and continuous function  $H : \mathbb{R}^{m_x} \times \mathbb{R}^{m_+ + m_-} \rightarrow \mathbb{R}$ .

**Remark 4.** As shown in [12]–[14], several protocols, including the sampled-data (SD), round-robin (RR), and try-once-discard (TOD) protocol, satisfy the requirements of Condition 1. In addition, as the function  $g(x, e)(s)$  is known and only depends on local dynamics, it is easy to obtain  $L_W$  and  $H(x(s), v(s))$ .

Besides having conditions on the scheduling protocol, to obtain local conditions guaranteeing UGAS of the overall system, also conditions on the interconnection structure are needed. Hereto, similar as to [11], we propose to introduce a (local) supply function  $\mathcal{S}_i : \mathbb{R}^{m_+ + m_-} \times \mathbb{R}^{m_+ + m_-} \rightarrow \mathbb{R}$  for the interconnection variables, see [25] for the terminology. To define this (local) supply function  $\mathcal{S}_i$  we introduce “new” interconnection variables  $(q_+(s), q_-(s))$  given by  $q_+(s) := (w_+(s), v_-(s))$  and  $q_-(s) := (v_+(s), w_-(s))$ . Note that for the considered interconnection structures of Fig. 1 and 2, as shown in [7], [9], [11], we thus have that

$$q_+(s) = q_-(s + 1) \quad \text{for all } s \in \mathbb{D}, \quad (15)$$

or compactly,  $q_+ = \Delta_{\mathbf{S}, \hat{m}} q_-$ , as a result of (6), with  $\hat{m} = (0, m_+ + m_-)$ . For the supply function  $\mathcal{S}_i$  we now require that the interconnection structure is such that the following condition holds.

**Condition 3.** For the interconnection variables  $q_+$  and  $q_-$  the interconnection is neutral in the sense that

$$\sum_{s \in \mathbb{D}} \mathcal{S}_i(q_+(s), q_-(s)) = 0. \quad (16)$$

Since we have that (15) holds, as shown in [11] by introducing the matrix  $X_{\mathbf{S}} \in \mathbb{R}_{\mathbf{S}}^{m_+ + m_-}$ , we can define the local supply function  $\mathcal{S}_i$  as

$$\mathcal{S}_i(q_+(s), q_-(s)) := \begin{bmatrix} q_+(s) \\ q_-(s) \end{bmatrix}^T \begin{bmatrix} -X_{\mathbf{S}} & 0 \\ 0 & X_{\mathbf{S}} \end{bmatrix} \begin{bmatrix} q_+(s) \\ q_-(s) \end{bmatrix}, \quad (17)$$

such that (16) indeed holds for the interconnection structures as depicted in Fig. 1 and 2.

When all of the above is taken into consideration, the main result for stability of the overall system can be stated by combining the stability results of [11] and [21]. Hereto, we consider the function  $\phi : \mathbb{R}_{\geq 0} \times \mathbb{D} \rightarrow \mathbb{R}_{\geq 0}$ , which evolves for every fixed  $s \in \mathbb{D}$  according to

$$\frac{d\phi}{d\tau(s)}(\tau(s), s) = -2L_W \phi(\tau(s), s) - \gamma(\phi^2(\tau(s), s) + 1) \quad (18)$$

for  $\tau(s) \in [0, \tau_{miet}]$  and is given by  $\phi(\tau(s), s) = \lambda$  for  $\tau(s) > \tau_{miet}$  where we have the initial condition chosen to be  $\phi(0, s) = \lambda^{-1}$  with  $\lambda$  following from Condition 1, see [14],  $\gamma$  is still to be specified certain positive constant, and  $L_W$  follows from Condition 2. We can now compose the following theorem regarding stability of the overall system in the sense of UGAS of the set  $\mathcal{E}$  of (12).

**Theorem 1.** Consider the overall system  $\mathcal{H}$ , composed of the identical subsystems  $\mathcal{H}(s)$  of (9),  $s \in \mathbb{D}$ , with associated  $\mathbb{X}_0$  of (11), satisfying the Conditions 1, 2, and 3. Assume there exist a locally Lipschitz function  $\mathcal{V} : \mathbb{R}^{m_x} \rightarrow \mathbb{R}_{\geq 0}$ , the function  $\Psi : \mathbb{R}^{m_x} \times \mathbb{R}^{m_e} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  as in (9), the functions  $\underline{\alpha}_V, \bar{\alpha}_V \in \mathcal{K}_\infty$ , a positive semi-definite function  $\varrho : \mathbb{R}^{m_y} \times \mathbb{R}^{m_u} \rightarrow \mathbb{R}_{\geq 0}$ , and the constants  $0 < \varepsilon < \gamma$  such that

- for all  $x(s) \in \mathbb{R}^{m_x}$ ,  $s \in \mathbb{D}$ ,
$$\underline{\alpha}_V(|x(s)|) \leq \mathcal{V}(x(s)) \leq \bar{\alpha}_V(|x(s)|) \quad (19)$$

- for all  $e(s) \in \mathbb{R}^{m_e}$  and almost all  $x(s) \in \mathbb{R}^{m_x}$ 

$$\begin{aligned} \langle \nabla \mathcal{V}(x(s)), f(x, e)(s) \rangle &\leq -\varepsilon^2 |x(s)|^2 - \varrho(y(s), u(s)) \\ &+ [\gamma^2 - \varepsilon^2] W^2(\kappa(s), e(s)) - H^2(x(s), v(s)) \\ &+ \mathcal{S}_i(q_+(s), q_-(s)), \end{aligned} \quad (20)$$

where  $\mathcal{S}_i(q_+(s), q_-(s))$  is given by (17),

- and it holds that

$$\Psi(x(s), e(s), \tau(s)) \leq M(\xi(s)) - \delta_\eta(\eta(s)) \quad (21a)$$

$$\Psi(x(s), e(s), \tau(s)) \geq 0, \quad \text{for } 0 \leq \tau(s) \leq \tau_{miet}, \quad (21b)$$

where  $\delta_\eta \in \mathcal{K}_\infty$  is arbitrary and  $M(\xi(s))$  is given by

$$M(\xi(s)) = \begin{cases} M_1(\xi(s)), & \text{for } 0 \leq \tau(s) \leq \tau_{miet} \\ M_2(\xi(s)), & \text{for } \tau(s) > \tau_{miet} \end{cases}, \quad \text{with}$$

$$M_1(\xi(s)) = \varrho(y(s), u(s)) + (H(x(s), v(s)) - \gamma\phi(\tau(s), s)W(\kappa(s), e(s)))^2 \quad (22a)$$

$$\begin{aligned} M_2(\xi(s)) &= \varrho(y(s), u(s)) + H^2(x(s), v(s)) \\ &- 2\gamma\phi(\tau(s), s)W(\kappa(s), e(s))H(x(s), v(s)) \\ &- (\gamma^2 + 2\gamma\phi(\tau(s), s)L_W)W^2(\kappa(s), e(s)). \end{aligned} \quad (22b)$$

In addition, it is assumed that the function  $\eta_0(\kappa(s), e(s))$ , in (9) is given by

$$\begin{aligned} n_0(\kappa(s), e(s)) &= \gamma\phi(\tau(s), s)W^2(\kappa(s), e(s)) \\ &- \gamma\phi(0)W^2(\kappa(s) + 1, h(\kappa(s), e(s))). \end{aligned} \quad (23)$$

If now  $\tau_{miet}$  satisfies

$$\tau_{miet} \leq \begin{cases} \frac{1}{L_W r} \arctan\left(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L_W}\right)+1+\lambda}\right), & \gamma > L_W \\ \frac{1-\lambda}{L_W(1+\gamma)}, & \gamma = L_W \\ \frac{1}{L_W r} \operatorname{arctanh}\left(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L_W}\right)+1+\lambda}\right), & \gamma < L_W \end{cases}$$

where  $r = \sqrt{|\gamma/L_W|^2 - 1}$  then the set  $\mathcal{E}$  given by (12) is UGAS.

If, in addition, there exist strictly positive real numbers  $\underline{\alpha}_W^c, \bar{\alpha}_W^c, \underline{\alpha}_V^c$ , and  $\bar{\alpha}_V^c$  such that for all  $e(s) \in \mathbb{R}^{m_e}$   $\underline{\alpha}_W^c |e(s)| \leq W(\kappa(s), e(s)) \leq \bar{\alpha}_W^c |e(s)|$ , and for all  $x(s) \in \mathbb{R}^{m_x}$   $\underline{\alpha}_V^c |x(s)|^2 \leq \mathcal{V}(x(s)) \leq \bar{\alpha}_V^c |x(s)|^2$ , then the set  $\mathcal{E}$  is UGES.

The proof is given in the Appendix. Observe that the function  $\Psi$  only depends on the local (state) variables  $\xi(s)$  and not on the entire state  $\xi$ . Theorem 1 provides conditions such that stability of the overall system is guaranteed. However, it might not be straightforward to construct the appropriate functions. Therefore, some design considerations are discussed in the next subsection.

**Remark 5.** Concerning the stability analysis approach, Theorem 1 shows quite some similarities with [16, Theorem III.4] and [21, Theorem 1]. However, the major difference between these theorems and Theorem 1 is given by the in amount of information needed to guarantee UGAS for the overall interconnected system. More precisely, since [16, Theorem III.4] and [21, Theorem 1] are based on global monolithic models, they encounter severe (numerical) limitations when considering a large amount of interconnected subsystems, even when all of the subsystems are identical. In this work, because we exploit the (specific) interconnection structure, the ‘‘global’’ condition (16) is always satisfied, resulting in the stability analysis as in Theorem 1 being based on *local* conditions only, allowing us to consider considerably larger interconnections. This benefit will also be illustrated in the numerical example.

### B. General design considerations and implementation

In this subsection it is discussed how to construct the event generator as in (7)-(8) through the definition of the function  $\Psi$  satisfying (21a) and (21b) and the control objective of guaranteeing asymptotic stability, see also [21]. To do so, lower bounds for the functions  $M_1$  and  $M_2$  need to be derived, which only depend on local measurements. A lower bound for the function  $M_1(\xi(s))$  can easily be obtained from (22a), namely  $M_1(\xi(s)) \geq \varrho(y(s), u(s))$ . Consider now the following lemma.

**Lemma 1.** For  $a, b \in \mathbb{R}$ , it holds that  $2ab \leq a^2 + b^2$ .

By employing Lemma 1, also a lower bound for  $M_2(\xi(s))$  given in (22b) can be derived by means of

$$\begin{aligned} M_2(\xi(s)) &\geq \varrho(y(s), u(s)) + H^2(x(s), v(s)) \\ &\quad - H^2(x(s), v(s)) - (\gamma\phi(\tau(s), s)W(\kappa(s), e(s)))^2 \\ &\quad - (\gamma^2 + 2\gamma\phi(\tau(s), s)L_W)W^2(\kappa(s), e(s)) \\ &\geq \varrho(y(s), u(s)) - \gamma(2\phi(\tau(s), s)L_W \\ &\quad + \gamma(1 + \phi^2(\tau(s), s)))W^2(\kappa(s), e(s)). \end{aligned}$$

By combining (21b) and the above found bounds on  $M_1$  and  $M_2$ , the function  $\Psi$ , which satisfies (21a), can now be defined as

$$\Psi(x(s), e(s), \tau(s)) = \varrho(y(s), u(s)) - \delta_\eta(\eta(s)) - (1 - \omega(\tau(s)))\bar{\gamma}W^2(\kappa(s), e(s)) \quad (24)$$

where

$$\omega(\tau(s)) = \begin{cases} 1, & \text{for } 0 \leq \tau(s) \leq \tau_{miet} \\ 0, & \text{for } \tau(s) > \tau_{miet} \end{cases}$$

and  $\bar{\gamma} = \gamma(2\phi(\tau(s), s)L_W + \gamma(1 + \phi^2(\tau(s), s)))$ . Observe that, given (18),  $\bar{\gamma}$  is a constant for  $\tau(s) > \tau_{miet}$ .

When taking these design considerations into account, it is possible to systematically construct the functions and conditions satisfying the conditions as stated in Theorem 1. For a more detailed explanation concerning these design considerations, when they are applicable, and for more information on how to construct the several other functions as mentioned in Theorem 1, we refer to [11], [15], and [21].

## V. NUMERICAL EXAMPLE

To illustrate the application of our results, we consider a numerical example that has already been considered in literature before, see, e.g., [11] or [21]. As such, consider the control setup as in Fig. 2 given by

$$\mathcal{P}(s) : \begin{cases} \dot{x}(s) = x(s)^2 - x(s)^3 + [-I_{m_+} & I_{m_-}]v(s) + u(s) \\ w(s) = \begin{bmatrix} I_{m_+} \\ I_{m_-} \end{bmatrix}x(s) \\ y(s) = x(s) \end{cases}$$

with the controllers  $\mathcal{C}(s) : u(s) = -2\hat{y}(s)$  for all  $s \in \mathbb{D}$ , the interconnection given by (6), and where  $\hat{y}(s)$  denotes the most recently received measurement of the output  $y(s) = x(s)$ . Note that only the output to the controller  $y$  is transmitted over the communication network, and, hence,  $\hat{u}(s) = u(s)$ . By defining the networked-induced error as  $e(s) = \hat{x}(s) - x(s)$  for all  $s \in \mathbb{D}$ , we have that  $u(s) = -2(x(s) + e(s))$  and the closed-loop system can be described by

$$\begin{aligned} \dot{x}(s) &= x(s)^2 - x(s)^3 - v_+(s) + v_-(s) - 2(x(s) + e(s)) \\ \dot{e}(s) &= x(s)^3 - x(s)^2 + v_+(s) - v_-(s) + 2(x(s) + e(s)) \end{aligned} \quad (25)$$

For the scheduling protocol we now choose to use the SD protocol with  $W(\kappa(s), e(s)) = |e(s)|$  and  $\lambda = 1 \cdot 10^{-3}$ , see also Remark 4, in order to compare the results with the earlier literature works [21] and [11]. As a result, we find by means of (25) that in (14)  $L_W = 2$  and  $H(x(s), v(s)) = |2x(s) - x(s)^2 + x(s)^3 + v_+(s) - v_-(s)|$  for all  $s \in \mathbb{D}$ . Consider the (local) function

$$\mathcal{V}(x(s)) = \sigma^2 \left( \alpha \frac{x(s)^2}{2} + \beta \frac{x(s)^4}{4} \right),$$

where  $\alpha, \beta, \sigma \in \mathbb{R}_{\geq 0}$ . Using the same analysis as in [11] and [21], it can be verified that  $\mathcal{V}$  satisfies (20) with  $\gamma = \sigma\sqrt{\alpha^2 + \beta^2 + \sigma^{-2}\varepsilon^2}$ ,  $\varrho(y(s), u(s)) = \zeta x(s)^2$ , and (17), where  $[\alpha, \beta, \zeta, c, \varepsilon, \sigma] = [7.6532, 2.6513, 1.0, 12.256, 0.1, 1.78]$ ,  $q_+(s) = (x(s), x(s+1))$ ,  $q_-(s) = (x(s-1), x(s))$ , and

$$X_S = \begin{bmatrix} 4 + c & -\frac{1}{2}\sigma^2\alpha \\ -\frac{1}{2}\sigma^2\alpha & -(4 + c) \end{bmatrix}.$$

Hence, according to Theorem 1 UGES of the set  $\mathcal{E}$  of (12) is guaranteed when  $\tau_{miet} \leq 0.1001$ , which is a very similar value as was obtained in the numerical example of [21] ( $\tau_{miet} = 0.0995$ ). However, while the numerical result of [21] was obtained for an interconnection of 2 NCSSs, we can use the analysis as obtained in this paper for an arbitrary finite number of interconnected NCSSs. As such, using (24), we show by means of a simulation of 50 interconnected NCSSs, a number that far outreaches the numerical limitations of [21], that Theorem 1 indeed guarantees UGES for the overall interconnected system. In Fig. 3, the simulation results of a periodic interconnection consisting of 50 NCSSs with  $\tau_{miet} = 0.1001$  are shown, where the initial condition for each subsystem is randomly taken to have a value between  $-1.5$  and  $1.5$ .

It can be clearly seen that indeed UGES is guaranteed as all states converge to zero, see Fig. 3(a). Moreover, we see that the average inter-event times of each subsystem is larger than the obtained bound on the maximally allowable transmission interval (MATI) as obtained in the numerical example of [11] ( $\tau_{mati} = 0.10211$ ), and, hence, indeed the analysis as in this paper reduces the number of transmission times in comparison with the time-based triggering approach as in [10], [11].

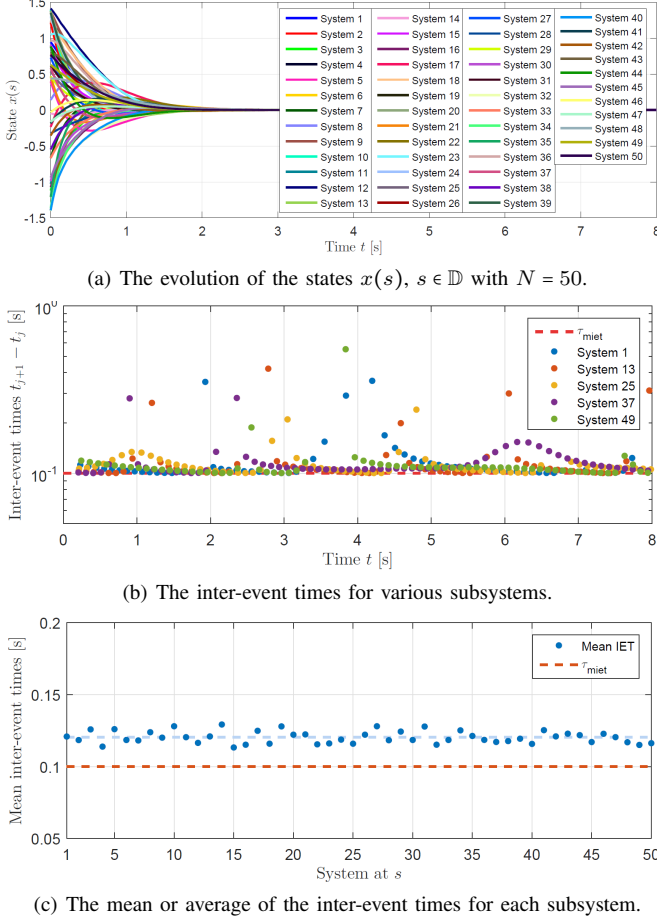


Fig. 3. Simulation results for the periodic interconnected overall system as in Fig. 1(a) consisting of 50 NCSs that are described by (25).

## VI. CONCLUSION

In this paper, global stability properties have been analyzed for a finite number of interconnected nonlinear NCSs that are spatially invariant. In particular, it has been shown that the developed general hybrid modeling framework for spatially invariant systems that communicate through packet-based communication networks from [11] can be extended with (dynamic) event-based triggering mechanisms as, for instance, proposed in [19]–[21]. Using a combination of the results of [11] and [21], sufficient Lyapunov-based conditions based only on *local* properties of *one* of the NCSs in the interconnection have been derived that guarantee UGAS (or UGES) for the overall interconnected system and a strictly positive lower bound on the inter-event times, which, in its turn, guarantees (local) Zeno-freeness. Finally, the application and benefits of the results have been shown by examining a nonlinear example.

*Proof of Theorem 1:* The proof is based on Lyapunov-based arguments for interconnected (infinite-dimensional) hybrid systems as in [23], extended using the concepts of [21]. Hence, let  $\xi$  be a solution to the overall interconnected system  $\mathcal{H}$  defined on the hybrid time domain  $\text{dom } \xi = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\}$  with  $J$  possibly  $\infty$  and  $t_J = \infty$  for any initial condition  $\xi(0, 0)$ . A sufficient condition under which UGAS of the set  $\mathcal{E}$  of (12) can be guaranteed is the existence of a function  $U : \ell_2^{m_x} \times \ell_2^{m_e} \times \mathbb{R}_{\geq 0}^N \times \mathbb{N}^N \times \ell_2^1 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  that is locally Lipschitz in its first and second argument, functions  $\underline{\alpha}_U, \bar{\alpha}_U \in \mathcal{K}_\infty$  and a positive definite function  $\varrho$  such that for all  $\xi \in \mathbb{X}_0$

$$\underline{\alpha}_U(\|\xi_c\|_{\ell_2}) \leq U(\xi) \leq \bar{\alpha}_U(\|\xi_c\|_{\ell_2}) \quad (26a)$$

$$U(\xi^+) - U(\xi) \leq 0 \quad (26b)$$

$$\text{when } \tau(s) \in [\tau_{miet}, \infty) \wedge \eta(\tau(s), s) \leq 0 \text{ for some } s \in \mathbb{D} \\ \langle \nabla U(\xi), F(\xi) \rangle_{\ell_2} \leq -\varrho(\|\xi_c\|_{\ell_2}) \quad (26c)$$

$$\text{when } \tau(s) \in [0, \tau_{miet}] \vee \eta(\tau(s), s) \geq 0 \text{ for all } s \in \mathbb{D}$$

with  $\xi_c = (x, e, \eta) \in \ell_2^{m_x} \times \ell_2^{m_e} \times \ell_2^1$ , and  $F(\xi) = (f(x, e), g(x, e), \mathbf{1}_N, \mathbf{0}_N, \Psi(x, e, \tau))$  where  $\Psi$  is the  $\ell_2$ -operator defined by  $\Psi(x, e, \tau)(s) = \Psi(x(s), e(s), \tau(s))$ . Note that non-strictness in the second (jump) condition is sufficient since all solutions are defined for all  $t \in \mathbb{R}_{\geq 0}$ . If in addition there exist  $\underline{\alpha}_U^c, \bar{\alpha}_U^c, \varepsilon \in \mathbb{R}_{\geq 0}$  such that  $\underline{\alpha}_U^c \|\xi_c\|_{\ell_2}^2 \leq U(\xi) \leq \bar{\alpha}_U^c \|\xi_c\|_{\ell_2}^2$ , and  $\varrho(r) \geq \varepsilon^2 r^2$ ,  $r \in \mathbb{R}_{\geq 0}$ , then the set  $\mathcal{E}$  of (12) is UGES.

To show that the conditions of Theorem 1 indeed can be used to prove that the set  $\mathcal{E}$  of (12) is UGAS, based on the results of [11], [21], using (18) we define the candidate Lyapunov function as

$$U(\xi) := V(x) + \sum_{s \in \mathbb{D}} \gamma \phi(\tau(s), s) W^2(\kappa(s), e(s)) + \eta(\tau(s), s),$$

where  $V(x) := \sum_{s \in \mathbb{D}} \mathcal{V}(x(s))$ . Given the fact that a reset can only occur when  $\tau(s) \geq \tau_{miet}$ , substitution of  $\phi(0, s) = \lambda^{-1}$  and  $\phi(\tau(s), s) = \lambda$  for all  $\tau(s) \geq \tau_{miet}$  in (23) yields

$$\eta_0(\kappa(s), e(s)) = \gamma \lambda W^2(\kappa(s), e(s)) \\ - \gamma \lambda^{-1} W^2(\kappa(s) + 1, h(\kappa(s), e(s))). \quad (27)$$

The triggering mechanism given by (8) ensures that  $\eta(s) \geq 0$  for all  $t \in \mathbb{R}_{\geq 0}$ , since from (13b) and (27) it follows that  $\eta_0 \geq 0$  at jumps. Moreover, when using the fact that  $\phi(\tau(s), s) \geq 0$  for all  $\tau(s) \in \mathbb{R}_{\geq 0}$  and for all  $s \in \mathbb{D}$ , and the radial unboundedness of the functions  $W$  and  $\mathcal{V}$  of (13a) and (19), respectively, for all  $s \in \mathbb{D}$ , it can be concluded that  $U$  is radially unbounded, i.e., there exist functions  $\underline{\alpha}_U, \bar{\alpha}_U \in \mathcal{K}_\infty$  such that (26a) holds.

To prove now that (26b) holds under the conditions proposed in Theorem 1, consider the situation that (only) the subsystem  $\mathcal{H}(\hat{s})$  for  $\hat{s} \in \mathbb{D}$  jumps, implying that  $\tau^+(\hat{s}) = 0$ . It then holds that

$$U(\xi^+) = V(x^+) + \sum_{s \in \mathbb{D}} \gamma \phi(\tau(s)^+, s) W^2(\kappa^+(s), e^+(s)) + \eta^+(s) \\ = V(x) + \sum_{s \in \mathbb{D}, s \neq \hat{s}} (\gamma \phi(\tau(s), s) W^2(\kappa(s), e(s)) + \eta(s)) \\ + \gamma \phi(0, \hat{s}) W^2(\kappa(\hat{s}) + 1, h(\kappa(\hat{s}), e(\hat{s}))) \\ + \eta_0(\kappa(\hat{s}), e(\hat{s})) \\ \leq V(x) + \sum_{s \in \mathbb{D}, s \neq \hat{s}} (\gamma \phi(\tau(s), s) W^2(\kappa(s), e(s)) + \eta(s)) \\ + \gamma \lambda^{-1} \lambda^2 W^2(\kappa(\hat{s}), e(\hat{s})) + \eta_0(\kappa(\hat{s}), e(\hat{s})) \leq U(\xi)$$

where for the latter inequality we used (13b), (27), and  $\phi(0, \hat{s}) = \lambda^{-1}$ . Obviously the same analysis holds when multiple subsystems jump at the same (continuous) event time, and hence, (26b) holds.

Now the attention is shifted to the last condition given by (26c) and is shown to hold too. Note that, as  $W$  is not differentiable with respect to  $\kappa$ , but as the component in  $F(\xi)$  corresponding to  $\kappa$  is zero,  $\langle \nabla U(\xi), F(\xi) \rangle$  can still be evaluated with a slight abuse of notation. For all  $(\tau, \kappa)$  and for almost all  $(x, e)$  it holds that

$$\begin{aligned}
& \langle \nabla U(\xi), F(\xi) \rangle_{\ell_2} = \langle \nabla V(x), f(x, e) \rangle_{\ell_2} \\
& + \sum_{s \in \mathbb{D}} \gamma \dot{\phi}(\tau(s), s) W^2(\kappa(s), e(s)) + \dot{\eta}(\tau(s), s) \\
& + \sum_{s \in \mathbb{D}} 2\gamma \phi(\tau(s), s) W(\kappa, e) \left\langle \frac{\partial W(\kappa(s), e(s))}{\partial e(s)}, g(x, e)(s) \right\rangle \\
& \stackrel{(14),(16),(20)}{\leq} \sum_{s \in \mathbb{D}} \left( -\varepsilon^2 |x(s)| - \rho(y(s), u(s)) - H^2(x(s), v(s)) \right. \\
& \quad \left. + (\gamma^2 - \varepsilon^2) W^2(\kappa(s), e(s)) + \Psi(x(s), e(s), \tau(s)) \right) \\
& - \sum_{s \in \mathbb{D}} \left( (2\gamma L_W \phi(\tau(s), s) + \gamma^2 (\phi^2(\tau(s), s) + 1)) W^2(\kappa(s), e(s)) \right) \\
& + \sum_{s \in \mathbb{D}} \left( W(\kappa(s), e(s)) (L_W W(\kappa(s), e(s)) + H(x(s), v(s))) \right. \\
& \quad \left. \cdot 2\gamma \phi(\tau(s), s) \right) \\
& = \sum_{s \in \mathbb{D}} \left( -\varepsilon^2 |x(s)| - \rho(y(s), u(s)) - \varepsilon^2 W^2(\kappa(s), e(s)) \right. \\
& \quad \left. + \Psi(x(s), e(s), \tau(s)) \right) \\
& - \sum_{s \in \mathbb{D}} \left( H^2(x(s), v(s)) + \gamma^2 \phi^2(\tau(s), s) W^2(\kappa(s), e(s)) \right. \\
& \quad \left. - 2\gamma \phi(\tau(s), s) W(\kappa(s), e(s)) H(x(s), v(s)) \right) \\
& \underbrace{\hspace{10em}} \\
& = - \sum_{s \in \mathbb{D}} \left( H(x(s), v(s)) - \gamma \phi(\tau(s), s) W(\kappa(s), e(s)) \right)^2 \\
& \stackrel{(22)}{\leq} \sum_{s \in \mathbb{D}} \left( -\varepsilon^2 |x(s)| - M(\xi(s)) - \varepsilon^2 W^2(\kappa(s), e(s)) \right. \\
& \quad \left. + \Psi(x(s), e(s), \tau(s)) \right).
\end{aligned}$$

As  $\Psi$  is upper bounded by  $M(\xi(s)) - \delta_\eta(\eta(s))$  according to (21a), it can be obtained that

$$\langle \nabla U(\xi), F(\xi) \rangle_{\ell_2} \leq - \sum_{s \in \mathbb{D}} \left( \varepsilon^2 |x(s)| + \delta_\eta(\eta(s)) + \varepsilon^2 W^2(\kappa(s), e(s)) \right)$$

which indeed gives (26c) and thereby completes the proof. ■

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