Stability and Performance Analysis of Spatially Invariant Systems with Networked Communication

S. H. J. Heijmans, D. P. Borgers, *Student Member, IEEE*, and W. P. M. H. Heemels, *Fellow, IEEE*

***Abstract*—In this paper, tractable stability and performance conditions are presented for systems consisting of an infinite number of spatially invariant, i.e., identical subsystems that are described by (non)linear differential equations and interconnected (partly) through packet-based communication networks. These networks transmit packets asynchronously and independently of each other and are equipped with scheduling protocols that determine which actuator, sensor, or controller node is allowed access to the network. The overall system is modeled as an infinite interconnection of spatially invariant hybrid subsystems. To underline the relevance of this framework, it is shown how two well-known and natural system configurations can be captured in this hybrid modeling framework. Moreover, for the resulting overall infinite-dimensional hybrid system, a proper solution concept is introduced, which is necessary as many standard concepts do not apply as Zeno behavior is inevitable for the systems under study. Based on the proposed hybrid modeling framework, conditions leading to a maximally allowable transmission interval (MATI) for all of the individual communication networks are derived such that uniform global asymptotic stability (UGAS) or $\mathcal{L}_p$-stability of the overall system is guaranteed. Interestingly, by exploiting the interconnection structure, the conditions guaranteeing UGAS or $\mathcal{L}_p$-stability can be stated locally in the sense that they only involve the (local) dynamics of one subsystem in the interconnection and local conditions on the scheduling protocol. Finally, it is shown that in the linear case the derived conditions can even be stated in terms of "local" LMIs, making them amenable for computational verification.**

***Index Terms*—Hybrid systems, infinite-dimensionality, linear matrix inequalities, networked control systems (NCSs), spatial invariance.**

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The authors are with the Control Systems Technology Group, Department of Mechanical Engineering, Eindhoven University of Technology, Eindhoven 5600 MB, The Netherlands (e-mail: s.h.j.heijmans@tue.nl; d.p.borgers@tue.nl; m.heemels@tue.nl).

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I. Introduction

Many systems consist of interconnections of similar subsystems that only interact with their nearest neighbors. Examples of such "systems of systems" include airplane formation flight as described in [1], flocks of systems from [2], vehicle platooning, see, e.g., [3], [4], and so on. Despite that these units often exhibit simple behavior and interact with their neighbors in a predictable fashion, the resulting overall system often shows rich and complex behavior. Analysis and control design for these systems based on global monolithic models encounter severe limitations if many (or even an infinite number of) subsystems are interconnected due to the very high dimensionality of the system and large number of inputs and outputs.

Therefore, a considerable amount of research effort has been targeted on analysis methods that aim to guarantee *global* system properties based on *local* conditions on the subsystems and information about the interconnection structure. An interesting line of work in this direction, considering interconnections of an *infinite* number of subsystems, is given by [5] and [6]. There are two main reasons to consider infinite-dimensional systems. As was pointed out in [5], but also in [7], infinite approximations may be adequate to analyze interconnections consisting of a large number of subsystems. Another reason is that its properties are inherited by periodic or finite interconnections with boundary conditions [8]–[10]. The focus in [5] and [6] is on studying spatially invariant *linear* subsystems and deriving local LMI-based conditions that, together with specific interconnection structures, lead to uniform global exponential stability (UGES) and $\mathcal{L}_2$-stability guarantees for the overall interconnected system.

One of the main underlying assumptions in [5] and [6] is that the communication between the subsystems is perfect. However, in many applications, including autonomously driving platoons of vehicles in which the communication between the cars occurs over a wireless packet-based medium [3], this assumption does not hold. In systems with communication networks, network-induced artifacts such as time-varying transmission intervals (possibly due to packet losses or channel unavailability) and transmission delays are present next to scheduling protocols that determine which sensor, controller, or actuator node is allowed to communicate at a certain transmission time. Although a vast literature on such networked control systems (NCSs) exists [11], tractable analysis and design tools for interconnections of an extremely large or even an infinite number of (spatially invariant) systems that interact using packet-based communication networks, which operate independently and asynchronously, hardly exist.

In this paper, building upon our preliminary work [12], we study this particular problem, starting from a general setup consisting of an infinite number of identical subsystems described by (non)linear differential equations where the subsystems and/or controllers communicate with each other via (wireless) packet-based communication networks. These local communication networks are subject to time-varying transmission intervals and scheduling protocols, such as the well-known round-robin (RR), sampled-data (SD), and try-once-discard (TOD) protocols, to determine the network access. Inspired by the research line [13]–[16], the overall system is modeled as an interconnection of an infinite number of spatially invariant *hybrid* subsystems. For this type of infinite-dimensional hybrid systems, we introduce a proper solution concept, which is needed as many standard concepts do not apply due to the fact that Zeno behavior is inevitable because of the infinite dimensionality of the systems under study. To illustrate the type of systems considered, two well-known and natural system configurations are modeled according to this general hybrid systems framework. Based on this hybrid modeling setup, a maximally allowable transmission interval (MATI) for all of the individual communication networks is provided such that uniform global asymptotic stability (UGAS) or $\mathcal{L}_p$-stability of the overall system is guaranteed. By exploiting the interconnection structure, the conditions guaranteeing UGAS and $\mathcal{L}_p$-stability of the overall *infinite-dimensional* system can be stated *locally* in the sense that they only involve the (local) dynamics of *one* subsystem in the interconnection and local conditions on the protocol. Finally, we show that for the case of linear subsystems the derived conditions can be stated in terms of "local" LMIs making them amenable for computational verification.

The paper is organized as follows. First some notational conventions and preliminary definitions are presented in Section II. The class of systems considered in this paper is described in Section III by means of introducing two frequently occurring system configurations. In Section IV, (global) conditions guaranteeing stability and performance are presented, after which these conditions are reformulated in Section V in terms of local conditions. Moreover, by looking at the special case of linear subsystems, it is possible to reformulate the stability and performance conditions into LMIs, as shown in Section VI. Finally, in Section VII, some numerical examples illustrating the applications of our results are discussed, and in Section VIII some concluding remarks are given.

II. Preliminaries

The sets of integers and nonnegative integers are denoted by $\mathbb{Z}$ and $\mathbb{N}$, respectively. The set of real numbers is denoted by $\mathbb{R}$ and the set of nonnegative real numbers by $\mathbb{R}_{\geq 0}$. For vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$, we denote by $(v_1, v_2, \ldots, v_n)$ the vector $[v_1^\top, v_2^\top, \ldots, v_n^\top]^\top$. By $|\cdot|$ and $\langle\cdot,\cdot\rangle$, the Euclidean norm and the usual inner product are denoted in $\mathbb{R}^n$, respectively.

The space of real n by m matrices is denoted by $\mathbb{R}^{n\times m}$ and the space of real symmetric n by n matrices is denoted $\mathbb{R}^{n\times n}_{\mathbf{S}}$. The n by n identity and zero matrices are denoted by I_n and 0_n, respectively. When the dimensions are clear from the context, these notations are simplified to I and 0. A matrix $M \in \mathbb{R}^{n\times n}_{\mathbf{S}}$ is called positive (semi-)definite, denoted as $M \succ 0\ (\succeq 0)$, if $v^\top M v > 0\ (\geq 0)$ for all $v \neq 0$. Similar notations hold for the negative (semi-)definite property.

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and, in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each fixed s, the mapping $r \mapsto \beta(r, s)$ belongs to class $\mathcal{K}$ and for each fixed r, the mapping $s \mapsto \beta(r, s)$ is decreasing and $\beta(r, s) \to 0$ when $s \to \infty$. Moreover, the function β is said to be of class exp-$\mathcal{KL}$ if there exist $K, c > 0$, such that $\beta(r, s) = Kr\exp(-cs)$ for all $r, s \in \mathbb{R}_{\geq 0}$. Given a Banach space X, a function $f : X \to X$ is said to be locally Lipschitz continuous if for each $x_0 \in X$ there exists constants $\delta, L > 0$ such that for all $x \in X$ we have that $\|x - x_0\|_X \leq \delta \Rightarrow \|f(x) - f(x_0)\|_X \leq L\|x - x_0\|_X$, where $\|\cdot\|_X$ denotes the norm in X, see also [17].

In this paper, the state-space of the considered systems is infinite-dimensional, as we will see below. Therefore, we recall some definitions from [6].

Definition 1: The space $\ell^{L,n}$ is the set of functions mapping $\mathbb{Z}^L$ to $\mathbb{R}^n$. The space $\ell_2^{L,n}$ is the set of functions $x \in \ell^{L,n}$ for which it holds that

$$\sum_{s_1\in\mathbb{Z}} \cdots \sum_{s_L\in\mathbb{Z}} x(\mathbf{s})^\top x(\mathbf{s}) < \infty$$

equipped with the inner product $\langle\cdot,\rangle_{\ell_2}$ for $x, y \in \ell_2^{L,n}$ defined as

$$\langle x, y\rangle_{\ell_2} := \sum_{s_1\in\mathbb{Z}} \cdots \sum_{s_L\in\mathbb{Z}} x(\mathbf{s})^\top y(\mathbf{s})$$

and the corresponding norm as $\|x\|_{\ell_2} := \sqrt{\langle x, x\rangle_{\ell_2}}$.

When the dimensions L and n are clear from context or not relevant, we sometimes write $\ell_2^{L,n}$ as ℓ_2.

Definition 2: The space $\mathcal{L}_p$ for $1 \leq p < \infty$ is the set of functions ϕ mapping $\mathbb{R}_{\geq 0}$ to ℓ_2 for which $\int_0^\infty \|\phi(t)\|_{\ell_2}^p \,\mathrm{d}t < \infty$. Moreover, for any $\phi \in \mathcal{L}_p$, we define the corresponding $\mathcal{L}_p$-norm as

$$\|\phi\|_{\mathcal{L}_p} := \left(\int_0^\infty \|\phi(t)\|_{\ell_2}^p \,\mathrm{d}t\right)^{1/p} < \infty.$$

We will consider variables $d \in \mathcal{L}_p$ that are vector-valued functions indexed by $L + 1$ independent variables, i.e., $d = d(t, s_1, \ldots, s_L)$, where $t \in \mathbb{R}_{\geq 0}$ is the (continuous) time and $s_1, s_2, \ldots, s_L \in \mathbb{Z}$ are the spatial variables. The L-tuple $(s_1, s_2, \ldots, s_L)$ is denoted by $\mathbf{s}$. For fixed $t \in \mathbb{R}_{\geq 0}$ and $\mathbf{s} \in \mathbb{Z}^L$, a variable $d(t)$ can be considered as an element of $\ell^{L,n}$ or $\ell_2^{L,n}$ and $d(t, \mathbf{s})$ as an element of $\mathbb{R}^n$, i.e., a real-valued vector. For ease of notation, t is often omitted when considering such variables, however, from the context it will be clear which space is considered. The spatial shift operators $\mathbf{S}_i$, acting on functions in $\ell_2^{L,n}$, are now for $i = 1, 2, \ldots, L$ defined as

$$(\mathbf{S}_i d)(\mathbf{s}) := d(s_1, \ldots, s_{i-1}, s_i + 1, s_{i+1}, \ldots, s_L).$$

In the case that $L = 1$, which we consider mainly in this paper, we denote $\mathbf{S}_1$ also as $\mathbf{S}$.

III. System Description

In this section, the considered class of systems is introduced. The overall system consists of an infinite number of identical subsystems, i.e., "basic building blocks," see Fig. 1(a). These

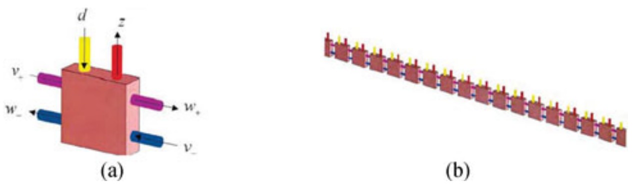

Fig. 1. Spatially invariant interconnected systems in one spatial dimension from [6]. (a) Basic building block. (b) Infinite interconnection.

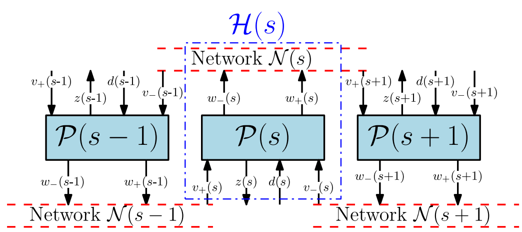

Fig. 2. Infinite networked interconnection, where each subsystem $\mathcal{P}(s)$, $s \in \mathbb{Z}$, has its own network $\mathcal{N}(s)$ to communicate with its neighbors. The overall "networked" (hybrid) subsystem $\mathcal{H}(s)$ is the combination of subsystem $\mathcal{P}(s)$ and its network $\mathcal{N}(s)$.

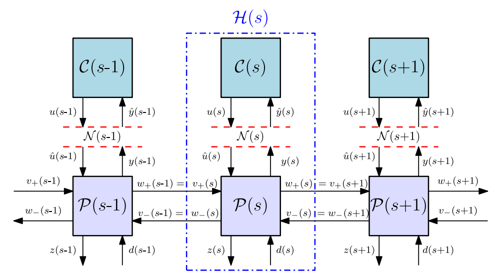

Fig. 3. Infinite interconnection, where each subsystem $\mathcal{P}(s)$ has its own network $\mathcal{N}(s)$ to communicate with its controller $\mathcal{C}(s)$, $s \in \mathbb{Z}$. The overall "networked" (hybrid) subsystem $\mathcal{H}(s)$ is the combination of subsystem $\mathcal{P}(s)$, its controller $\mathcal{C}(s)$, and its network $\mathcal{N}(s)$.

subsystems are interconnected according to a particular structure as indicated in Fig. 1(b), as was also considered in [6]. For simplicity of exposition, we focus in this paper on the interconnection structure as in Fig. 1(b), but other interconnections as in [6], [8]–[10] can be considered in a similar manner, see also Section VIII below.

Often when spatially interconnected systems are considered, the communication is assumed to be perfect and infinitely fast [4]–[6]. However, in many situations this assumption is not valid and the interconnection is based on packet-based communication networks, see, e.g., [13]–[16]. Below, we consider two system configurations, which are natural in many applications, and that will lead us to a unified hybrid system description. Indeed, our main results in Sections IV and V are based on this hybrid modeling setup, which applies for the two system configurations introduced now:

S1) *Spatially invariant interconnected systems with networked communication*, which is an extension of the configuration considered in [6] as it includes packet-based communication, see Fig. 2. Such a configuration is, for instance, applicable for platoons of vehicles.

S2) *Spatially invariant interconnected networked control systems*, which is an extension to the configurations considered in [6] and [18]. It extends [6] as, again, packet-based communication is used introducing network-induced imperfections. It extends [18] as it considers an infinite number of components and exploits spatial invariance, see Fig. 3. Large-scale systems can for instance be captured in this system configuration.

Note that in S1 the interconnection between the subsystems $\mathcal{P}(s)$, $s \in \mathbb{Z}$, always occurs via packet-based networks, while in S2 the interconnection of $\mathcal{P}(s)$, $s \in \mathbb{Z}$, is via physical couplings, only the communication with its controller is realized via a network. For both system configurations, we now show that they can be written in a unifying and general hybrid modeling framework.

A. Spatially Invariant Interconnected Systems With Networked Communication (S1)

In contrast to [6], here the communication between the subsystems is not continuous and perfect, but occurs via packet-based (wireless) communication networks that operate asynchronously and independently [13]–[16], [19], see Fig. 2.

To introduce the overall modeling setup, we start by providing the dynamical model describing a single subsystem $\mathcal{P}(s)$ in the interconnection indexed by $s \in \mathbb{Z}$. This is given by

$$\begin{bmatrix} \dot{x}(s) \\ w(s) \\ z(s) \end{bmatrix} = \begin{bmatrix} f_p(x(s), v(s), d(s)) \\ g_p(x(s)) \\ q_p(x(s), v(s), d(s)) \end{bmatrix} \tag{1}$$

with the initial condition $x(0,s) = x_0(s) \in \mathbb{R}^{m_0}$, $s \in \mathbb{Z}$, where $x_0 \in \ell_2^{1,m_0}$, and

$$v(s) = \begin{bmatrix} v_+(s) \\ v_-(s) \end{bmatrix} \quad \text{and} \quad w(s) = \begin{bmatrix} w_+(s) \\ w_-(s) \end{bmatrix} \tag{2}$$

where $x(s) \in \mathbb{R}^{m_0}$ denotes the state, $v_+(s) \in \mathbb{R}^{m_+}$ and $v_-(s) \in \mathbb{R}^{m_-}$ the interconnected inputs, $w_+(s) \in \mathbb{R}^{m_+}$ and $w_-(s) \in \mathbb{R}^{m_-}$ the interconnected outputs, $d(s) \in \mathbb{R}^{m_d}$ the external (disturbance) input, and $z(s) \in \mathbb{R}^{m_z}$ the performance output of subsystem $\mathcal{P}(s)$, $s \in \mathbb{Z}$. Hence, $f_p : \mathbb{R}^{m_0} \times \mathbb{R}^{m_+ + m_-} \times \mathbb{R}^{m_d} \to \mathbb{R}^{m_0}$, $g_p : \mathbb{R}^{m_0} \to \mathbb{R}^{m_+ + m_-}$, and $q_p : \mathbb{R}^{m_0} \times \mathbb{R}^{m_+ + m_-} \times \mathbb{R}^{m_d} \to \mathbb{R}^{m_z}$ are (non)linear mappings, where it is assumed that f_p is sufficiently smooth (which will be detailed further in Section IV-A), and that g_p is continuously differentiable. Note that $x \in \ell_2^{1,m_0}$ denotes the state of the overall system, that $v_+(s)$ and $w_+(s)$ have the same size, and that $v_-(s)$ and $w_-(s)$ have the same size.

Based on identical copies of the building block (1)–(2), the infinite interconnection of Fig. 1(b) (or Fig. 2) is obtained by defining

$$v_+(s) = \hat{w}_+(s-1) \quad \text{and} \quad v_-(s) = \hat{w}_-(s+1) \tag{3}$$

where, contrary to [6], $\hat{w}_+(s)$ and $\hat{w}_-(s)$ are typically *not* equal to $w_+(s)$ and $w_-(s)$, but are their networked values, i.e., the latest broadcast values of $w_+(s)$ and $w_-(s)$, respectively, as will be explained below in more detail.

Define $m := (m_+, m_-)$, and define in a similar fashion as in [6] the structured operator $\Delta_{\mathrm{S},m} : \ell_2^{1,m_+ + m_-} \to \ell_2^{1,m_+ + m_-}$ for

$r = (r_+, r_-) \in \ell_2^{1,m_+ + m_-}$ as

$$(\Delta_{\mathbf{S},m} r)(s) = \begin{bmatrix} \begin{bmatrix} I_{m_+} & 0 \end{bmatrix} (\mathbf{S}^{-1} r)(s) \\ \begin{bmatrix} 0 & I_{m_-} \end{bmatrix} (\mathbf{S} r)(s) \end{bmatrix} = \begin{bmatrix} r_+(s-1) \\ r_-(s+1) \end{bmatrix}. \quad (4)$$

By using this operator, the interconnection (3) can be compactly expressed for every $s \in \mathbb{Z}$ as

$$v(s) = (\Delta_{\mathbf{S},m} \hat{w})(s). \quad (5)$$

To complete the modeling framework, consider now a subsystem $\mathcal{P}(s)$ with its own local network $\mathcal{N}(s)$, $s \in \mathbb{Z}$, to communicate with its neighbors and its own collection of transmission/sampling times $t_{j^s}^s$, $j^s \in \mathbb{N}$, which satisfy $0 \le t_0^s < t_1^s < t_2^s < \ldots$. Note that all subsystems communicate asynchronously and independently, as each of them has its own sequence of transmission times. For the subsystem $\mathcal{P}(s)$, (parts of) the output $w(s)$ are sampled and transmitted over the network, at such a transmission time $t_{j^s}^s$. However, the (local) communication network $\mathcal{N}(s)$ is typically subdivided into several sensor, actuator and/or controller nodes, where each node corresponds to a subset of the entries in $w(s)$ (and thus $\hat{w}(s)$). Hence, a scheduling protocol that determines which of the nodes in the network is granted access to the network at a transmission time is needed, see also [14]–[16]. At transmission time $t_{j^s}^s$ for subsystem $\mathcal{P}(s)$, $s \in \mathbb{Z}$, the networked values are updated according to

$$\hat{w}\left(\left(t_{j^s}^s\right)^+, s\right) = w\left(t_{j^s}^s, s\right) + h\left(j^s, e\left(t_{j^s}^s, s\right)\right) \quad (6)$$

where $e(s)$ denotes the local network-induced error defined by

$$e(s) := \hat{w}(s) - w(s) = \begin{bmatrix} \hat{w}_+(s) - w_+(s) \\ \hat{w}_-(s) - w_-(s) \end{bmatrix} =: \begin{bmatrix} e_+(s) \\ e_-(s) \end{bmatrix} \quad (7)$$

where we have split the error $e(s)$ according to (2) into $e_+(s)$ and $e_-(s)$, $s \in \mathbb{Z}$. Note that we followed here the same method for modeling the scheduling protocols as described in [14]–[16] based on the protocol function $h : \mathbb{N} \times \mathbb{R}^{m_+ + m_-} \to \mathbb{R}^{m_+ + m_-}$, which can be used to describe, e.g., the RR and TOD protocol, and several others, see [14], [20]. Indeed, in (6) it is determined on the basis of the local transmission counter j^s and the local network error $e\left(t_{j^s}^s, s\right)$ which node is allowed to communicate and typically the corresponding entries in h are zero, see [14]–[16] for a detailed description. Note that we used here the notation $r(t^+) = \lim_{\tau \downarrow t} r(\tau)$ for $t \in \mathbb{R}_{\ge 0}$ and we assume the signals $\hat{w}$ to be left-continuous in the sense that for all $t \in \mathbb{R}_{>0}$ and all $s \in \mathbb{Z}$, $\hat{w}(t, s) = \lim_{\tau \uparrow t} \hat{w}(\tau, s)$ and $\hat{w}(0, s)$ is given as an initial condition.

The transmission times satisfy for all $j^s \in \mathbb{N}$, $s \in \mathbb{Z}$,

$$\delta \le t_{j^s+1}^s - t_{j^s}^s \le \tau_{\text{mati}} \quad (8)$$

where $\tau_{\text{mati}} \in \mathbb{R}_{\ge 0}$ denotes the MATI for the network corresponding to the subsystem at location $s \in \mathbb{Z}$ and $0 < \delta < \tau_{\text{mati}}$ is a positive time between two consecutive transmission times of a subsystem. It should be noted that δ can be taken arbitrarily small since it is only imposed to prevent Zeno behavior (at least locally) [21]. Due to hardware limitations in reality, such a lower bound δ on the transmission intervals always exists. Finally, it is assumed that each network operates in a zero-order-hold (ZOH) fashion, in the sense that $\hat{w}$ does not change between transmissions, although this can easily be modified if desired, see [14].

Based on the above setup, each pair $(\mathcal{P}(s), \mathcal{N}(s))$ can be rewritten into the hybrid system formalism advocated in [14]. To do so, the interconnection variables w need to be eliminated in the state and error dynamics. From (1), (3), and (7) it follows that $v(s) = (\Delta_{\mathbf{S},m} e)(s) + (\Delta_{\mathbf{S},m} w)(s)$ with $w(s) = g_p(x(s))$. Next, similar to [13]–[16], we introduce for every fixed $s \in \mathbb{Z}$ the timers $\tau(s) \in \mathbb{R}_{\ge 0}$, which keep track of the amount of time elapsed since the last transmission for each pair $(\mathcal{P}(s), \mathcal{N}(s))$ and are reset to zero after a new transmission corresponding to the pair $(\mathcal{P}(s), \mathcal{N}(s))$ has occurred, and the counters $\kappa(s) \in \mathbb{N}$, which keep track the number of transmissions for each pair $(\mathcal{P}(s), \mathcal{N}(s))$ and are needed to implement certain scheduling protocols. Now using these auxiliary variables, each "networked" subsystem can be expressed as a *hybrid system* [22] given by

$$\mathcal{H}(s): \begin{cases} \left.\begin{aligned} \dot{x}(s) &= f(x,e,d)(s) \\ \dot{e}(s) &= g(x,e,d)(s) \\ \dot{\tau}(s) &= 1 \\ \dot{\kappa}(s) &= 0 \end{aligned}\right\} & \text{when } \tau(s) \in [0, \tau_{\text{mati}}] \\ \left.\begin{aligned} x^+(s) &= x(s) \\ e^+(s) &= h(\kappa(s), e(s)) \\ \tau^+(s) &= 0 \\ \kappa^+(s) &= \kappa(s) + 1 \end{aligned}\right\} & \text{when } \tau(s) \in [\delta, \infty) \end{cases} \quad (9)$$

with the new state of the subsystem indexed by $s \in \mathbb{Z}$ given by $\xi(s) := (x(s), e(s), \tau(s), \kappa(s))$ and the output equation

$$z(s) = q(x, e, d)(s) \quad (10)$$

where we have that

$$\begin{aligned} f(x,e,d)(s) &= f_p\left(x(s), (\Delta_{\mathbf{S},m} e)(s) + (\Delta_{\mathbf{S},m} w)(s), d(s)\right) \\ g(x,e,d)(s) &= -\frac{\partial g_p(x(s))}{\partial x(s)} f_p(x(s), (\Delta_{\mathbf{S},m} e)(s) \\ &\quad + (\Delta_{\mathbf{S},m} w)(s), d(s)) \end{aligned} \quad (11)$$

and $q(x,e,d)(s) = q_p(x(s), (\Delta_{\mathbf{S},m} e)(s) + (\Delta_{\mathbf{S},m} w)(s), d(s))$ with $w(s) = g_p(x(s))$. We assume that f_p, g_p, and q_p are such that $f : \ell_2^{1,m_0} \times \ell_2^{1,m_e} \times \ell_2^{1,m_d} \to \ell_2^{1,m_0}$, $g : \ell_2^{1,m_0} \times \ell_2^{1,m_d} \to \ell_2^{1,m_e}$, and $q : \ell_2^{1,m_0} \times \ell_2^{1,m_e} \times \ell_2^{1,m_d} \to \ell_2^{1,m_z}$ with $m_e = m_+ + m_-$. The overall interconnection as depicted in Fig. 2 is now described by the hybrid system $\mathcal{H}$, being the infinite interconnection of subsystems $\mathcal{H}(s)$, $s \in \mathbb{Z}$. As such, we thus have that the overall hybrid system $\mathcal{H}$ is *infinite-dimensional*

Remark 1: Note that there are no direct-feedthrough terms in the networked interconnection of (1), meaning that $w(s)$ does not depend on $v(s)$. This condition is used to prevent that the jump of the subsystem s directly triggers a jump in the network-induced errors $e(s-1)$ and/or $e(s+1)$, and thus, possibly jumps of the subsystems $s-1$ and/or $s+1$, respectively. A similar condition was also adopted in the finite-dimensional case studied in, e.g., [14]–[16]. Moreover, we assume that the (disturbance) input $d(s)$ does not directly influence the interconnection variables $w(s)$ to reduce the complexity of the setup.

B. Spatially Invariant Interconnected NCSs (S2)

In this section, we introduce the modeling setup for the spatially invariant interconnected NCSs. Consider hereto again the overall system consisting of an infinite number of the spatially invariant NCSs, which are physically interconnected according to the structure shown in Fig. 3.

As illustrated, we consider a decentralized controller configuration where each continuous-time plant $\mathcal{P}(s)$ is given by

$$\mathcal{P}(s):\begin{cases}\dot{x}_p(s)=f_p\left(x_p(s),v(s),\hat{u}(s),d(s)\right)\\ w(s)=g_p\left(x_p(s),d(s)\right)+A_{\mathbf{SS}}v(s)\\ y(s)=h_p\left(x_p(s)\right)\\ z(s)=q_P\left(x_p(s),v(s),d(s)\right)\end{cases}\tag{12}$$

with the initial condition $x_p(0,s)=x_{0,p}(s)\in\mathbb{R}^{m_{x_p}}$, $s\in\mathbb{Z}$, and $x_{0,p}\in\ell_2^{1,m_{x_p}}$. The linear operator $A_{\mathbf{SS}}$ maps $\mathbb{R}^{m_++m_-}$ to $\mathbb{R}^{m_++m_-}$, and again (2) is used to describe the partitioning of the interconnection variables. In addition to the already introduced variables, we now have that $x_p(s)\in\mathbb{R}^{m_{x_p}}$ denotes the (local) plant state, $\hat{u}(s)\in\mathbb{R}^{m_u}$ the most recently received local control input, and $y(s)\in\mathbb{R}^{m_y}$ the local output of the plant $\mathcal{P}(s)$, $s\in\mathbb{Z}$. Hence, $f_p:\mathbb{R}^{m_{x_p}}\times\mathbb{R}^{m_++m_-}\times\mathbb{R}^{m_u}\times\mathbb{R}^{m_d}\to\mathbb{R}^{m_{x_p}}$, $g_p:\mathbb{R}^{m_{x_p}}\times\mathbb{R}^{m_d}\to\mathbb{R}^{m_++m_-}$, $h_p:\mathbb{R}^{m_{x_p}}\to\mathbb{R}^{m_y}$, and $q_p:\mathbb{R}^{m_{x_p}}\times\mathbb{R}^{m_++m_-}\times\mathbb{R}^{m_d}\to\mathbb{R}^{m_z}$ are again (non)linear mappings, where it is assumed that f_p is sufficiently smooth, see also Section IV-A below, and h_p is continuously differentiable. Note that here $x_p\in\ell_2^{1,m_{x_p}}$ is used to denote the overall state of the interconnected plants $\mathcal{P}(s)$, $s\in\mathbb{Z}$.

Each plant $\mathcal{P}(s)$ is controlled by its own local controller $\mathcal{C}(s)$, which is connected to the plant via a local (independent) communication network $\mathcal{N}(s)$ as shown in Fig. 3. The controller $\mathcal{C}(s)$ is given by

$$\mathcal{C}(s):\begin{cases}\dot{x}_c(s)=f_c\left(x_c(s),\hat{y}(s)\right)\\ u(s)=h_c\left(x_c(s)\right)\end{cases}\tag{13}$$

where $x_c(s)\in\mathbb{R}^{m_c}$ denotes the controller state, $\hat{y}(s)\in\mathbb{R}^{m_y}$ the most recently received output measurement of the plant, and $u(s)\in\mathbb{R}^{m_u}$ the controller output for controller $\mathcal{C}(s)$, $s\in\mathbb{Z}$. Hence, $f_c:\mathbb{R}^{m_{x_c}}\times\mathbb{R}^{m_y}\to\mathbb{R}^{m_{x_c}}$ and $h_c:\mathbb{R}^{m_{x_c}}\to\mathbb{R}^{m_u}$ are also (non)linear mappings, where it is assumed that f_c is sufficiently smooth, see Section IV-A, and h_c is continuously differentiable. Notice that there are no interconnections present between the local controllers and, as such, the control setup is indeed of a decentralized nature.

Similar to (3), based on the identical subsystems described by (12), (13), and the network $\mathcal{N}(s)$, the infinite interconnected system of Fig. 1(b) is obtained by defining

$$v_+(s)=w_+(s-1)\quad\text{and}\quad v_-(s)=w_-(s+1)\tag{14}$$

which, by using (4), leads to the compact expression

$$v(s)=\left(\Delta_{\mathbf{S},m}w\right)(s),\ s\in\mathbb{Z}.\tag{15}$$

To complete the NCS setup it has to be explained how the communication networks $\mathcal{N}(s)$ operate. Similar to the previously described configuration in Section III-A, these local networks all operate independently of each other, i.e., each network $\mathcal{N}(s)$ has its own collection of transmission/sampling times $t^s_{j^s}$, $j^s\in\mathbb{N}$. At each of these transmission times $t^s_{j^s}$ (parts of), the output $y(s)$ and input $u(s)$ are sampled and transmitted over the network $\mathcal{N}(s)$ to the controller $\mathcal{C}(s)$ and plant $\mathcal{P}(s)$, respectively. A scheduling protocol determines which of the nodes in the network is granted access to the network. After a node is granted access to the network, it collects and transmits the values of the corresponding entries in $y\left(t^s_{j^s},s\right)$ and $u\left(t^s_{j^s},s\right)$, which results in an update according to

$$\begin{aligned}\hat{y}\left(\left(t^s_{j^s}\right)^+,s\right)&=y\left(t^s_{j^s},s\right)+h_y\left(j^s,e\left(t^s_{j^s}\right)\right)\\ \hat{u}\left(\left(t^s_{j^s}\right)^+,s\right)&=u\left(t^s_{j^s},s\right)+h_u\left(j^s,e\left(t^s_{j^s}\right)\right)\end{aligned}$$

where the function $h:=(h_y,h_u)$ (with slight abuse of notation) models the (local) scheduling protocol and where $e(s)$, $s\in\mathbb{Z}$, denotes the network-induced error defined by

$$e(s):=\begin{bmatrix}e_y(s)\\ e_u(s)\end{bmatrix}=\begin{bmatrix}\hat{y}(s)-y(s)\\ \hat{u}(s)-u(s)\end{bmatrix},\ s\in\mathbb{Z}.\tag{16}$$

For this configuration, it is again assumed that the networks operate in a ZOH fashion, and the transmission times satisfy (8).

Now similar to the previous setup, each triple $(\mathcal{P}(s),\mathcal{C}(s),\mathcal{N}(s))$ can be rewritten in the format of a hybrid system as in (9)–(10) by eliminating the interconnection variables w. Based on (12), (13), (14), and (16) it follows that each "networked" subsystem takes the general form of (9)–(10), with the state of the subsystem $\mathcal{H}(s)$ given by $\xi(s)=\big(x(s),e(s),\tau(s),\kappa(s)\big)$ with $x(s)=(x_p(s),x_c(s))$, $m_0=m_{x_p}+m_{x_c}$, and $m_e=m_y+m_u$. Moreover, the (non)linear mappings $f(x,e,d)(s)$ and $g(x,e,d)(s)$ are given in (17) for $s\in\mathbb{Z}$, and $q(x,e,d)(s)=q_p(x(s),(\Delta_{\mathbf{S},m}w)(s),d(s))$ with $w=(\tilde{I}_{m_++m_-}-\tilde{A}_{\mathbf{SS}}\Delta_{\mathbf{S},m})^{-1}\tilde{g}_p(x_p,d)$ where $\tilde{A}_{\mathbf{SS}}$ is the operator which maps ℓ_2^{1,m_++m_-} to ℓ_2^{1,m_++m_-} according $(\tilde{A}_{\mathbf{SS}}v)(s)=A_{\mathbf{SS}}v(s)$ for all $s\in\mathbb{Z}$, and $\tilde{I}_{m_++m_-}$ and $\tilde{g}_p(x_p,d)$ are the "diagonal" operators mapping ℓ_2 to ℓ_2 defined by $\tilde{g}_p(x_p,d)(s):=g_p(x_p(s),d(s))$ and $\tilde{I}_{m_++m_-}(s):=I_{m_++m_-}$, respectively, for all $s\in\mathbb{Z}$. Moreover, it is assumed that the inverse of $(\tilde{I}_{m_++m_-}-\tilde{A}_{\mathbf{SS}}\Delta_{\mathbf{S},m})$ exists, which, for instance, can be guaranteed by requiring that there are no direct-feed-through terms ($A_{\mathbf{SS}}=0$). For more details, we refer to [6].

$$\begin{aligned}f(x,e,d)(s)&=\begin{bmatrix}f_p\left(x_p(s),\left(\Delta_{\mathbf{S},m}w\right)(s),h_c\left(x_c(s)\right)+e_u(s),d(s)\right)\\ f_c\left(x_c(s),h_p(x_p(s))+e_y(s)\right)\end{bmatrix}\\ g(x,e,d)(s)&=\begin{bmatrix}-\dfrac{\partial h_p\left(x_p(s)\right)}{\partial x_p(s)}f_p\left(x_p(s),\left(\Delta_{\mathbf{S},m}w\right)(s),h_c\left(x_c(s)\right)+e_u(s),d(s)\right)\\ -\dfrac{\partial h_c\left(x_c(s)\right)}{\partial x_c(s)}f_c\left(x_c(s),h_p(x_p(s))+e_y(s)\right)\end{bmatrix}.\end{aligned}\tag{17}$$

Remark 2: For simplicity of exposition, we focus here on the control configuration as in Fig. 3. However, the results presented in this paper also apply to other configurations such as the case of static state feedback control and the case where either $\hat{u}(s) = u(s)$ or $\hat{y}(s) = y(s)$, $s \in \mathbb{Z}$, meaning that the corresponding signals are not transmitted over a packet-based communication network, but are continuously available. Also the case where (parts of) $u(s)$ and $y(s)$ are transmitted over different networks as described in [19] fits our analysis methods. These configurations can all be captured in the same hybrid model (9), (10) on which our stability and performance results will be based.

IV. Stability and Performance

Using the hybrid modeling framework (9)–(10) for the spatially invariant interconnected networked systems, conditions for stability and performance are derived in this section. However, before these concepts can be defined, we first need to define the notion of solutions for the class of systems described by the interconnected hybrid systems $\mathcal{H}(s)$, $s \in \mathbb{Z}$, of (9).

A. Solution Concept

To define the notion of solutions for the overall system $\mathcal{H}$ composed of the hybrid subsystems given by (9), consider the (general) hybrid subsystem given for $s \in \mathbb{Z}$ by

$$\mathcal{H}(s) : \begin{cases} \dot{\xi}(s) = F(\xi, d)(s), & \xi(s) \in \mathcal{C} \\ \xi^+(s) \in G(\xi(s)), & \xi(s) \in \mathcal{D} \end{cases} \tag{18}$$

with $\xi = (\xi_c, \xi_d)$ the state of the overall system, where $\xi_c \in \ell_2^{1,m_1}$ comprises the (internal) dynamical states of the system like the plant and controller states and the networked-induced error while $\xi_d \in \ell^{1,m_2}$ comprises (auxiliary) states like the timers and counters, and where $d \in \ell_2^{1,m_d}$ denotes an external (disturbance) input. Note that ξ_d is typically not contained in ℓ_2 and therefore these states are separated from $\xi_c \in \ell_2^{1,m_1}$. In (18), it is assumed that $G : \mathbb{R}^{m_1+m_2} \rightrightarrows \mathbb{R}^{m_1+m_2}$ is a set-valued function with corresponding operator $\tilde{G} : \ell_2^{1,m_1} \times \ell^{1,m_2} \rightrightarrows \ell_2^{1,m_1} \times \ell^{1,m_2}$ defined as $\tilde{G}(\xi)(s) = G(\xi(s))$, $\mathcal{C}$ and $\mathcal{D}$ are closed subsets of $\mathbb{R}^{m_1+m_2}$, and that the function $F := (F_c, F_d)$ with $F_c : \ell_2^{1,m_1+m_d} \to \ell_2^{1,m_1}$ and $F_d : \ell^{1,m_2} \to \ell^{1,m_2}$. Moreover, we assume that F_c is sufficiently smooth such that the dynamical system $\dot{\xi}_c = F_c(\xi_c, d)$ gives rise to solutions ξ_c with values in the space ℓ_2^{1,m_1}. In particular, it is assumed that the function F_c is locally Lipschitz in its first argument and continuous in its second argument such that the existence and uniqueness of locally absolutely continuous solutions ξ_c are guaranteed for a given initial state $\xi_c(0) \in \ell_2^{1,m_1}$ and disturbance signals $d \in \mathcal{L}_p$ [17], [23], [24]. Note that (9) is indeed of the form (18). The overall system $\mathcal{H}$ is composed of the infinite number of subsystems $\mathcal{H}(s)$, all given by (18).

Now, inspired by [25], we are able to introduce a solution concept using the following definitions.

Definition 3: A point $\tau \in \mathcal{R} \subset \mathbb{R}$ is called a right-accumulation point of $\mathcal{R}$ if there exists a sequence $\{\tau_i\}_{i\in\mathbb{N}}$ such that $\tau_i \in \mathcal{R}$ and $\tau_i < \tau$ for all $i \in \mathbb{N}$ and, furthermore, $\lim_{i\to\infty} \tau_i = \tau$. A left-accumulation point is defined similarly by interchanging "$<$" with "$>$." A set $\mathcal{R} \subset \mathbb{R}$ is called right-isolated if it contains no left-accumulation points. Hence, we say that $\mathcal{R}$ is right-isolated if for all $\tau \in \mathcal{R}$ it holds that there is an $\varepsilon > 0$ such that $(\tau, \tau + \varepsilon) \cap \mathcal{R} = \emptyset$.

Definition 4: A pair $(\mathcal{R}, \xi)$, where $\mathcal{R}$ is a right-isolated closed subset of $[0, T)$,

$$\xi : [0, T)\backslash\mathcal{R} \to \ell_2^{1,m_1} \times \ell^{1,m_2}$$

is a solution to the overall system $\mathcal{H}$ composed of the subsystems $\mathcal{H}(s)$ given by (18), $s \in \mathbb{Z}$, on $[0, T)$ with $T > 0$ or $T = \infty$ for initial state $\xi_0 \in \ell_2^{1,m_1} \times \ell^{1,m_2}$ and external (disturbance) inputs $d \in \mathcal{L}_p$ if the following are satisfied:

1) $0 \in \mathcal{R}$.
2) For all $\tau \in \mathcal{R}$ and $s \in \mathbb{Z}$, the right-limit $\xi(\tau^+, s) := \lim_{t\downarrow\tau,\, t\notin\mathcal{R}} \xi(t, s)$ exists and for all $\tau \in \mathcal{R}\backslash\{0\}$ and $s \in \mathbb{Z}$ the left-limit $\xi(\tau^-, s) := \lim_{t\uparrow\tau,\, t\notin\mathcal{R}} \xi(t, s)$ exists. Moreover, for all $\tau \in \mathcal{R}$ and $s \in \mathbb{Z}$ it holds that
$$\xi(\tau^+, s) \in G\left(\xi\left(\tau^-, s\right)\right) \quad \text{or} \quad \xi(\tau^+, s) = \xi\left(\tau^-, s\right)$$
when $\xi(\tau^-, s) \in \mathcal{D}$, while for $\xi(\tau^-, s) \notin \mathcal{D}$ it holds that
$$\xi(\tau^+, s) = \xi\left(\tau^-, s\right)$$
where $\xi\left(\tau^-, s\right) := \xi_0(s)$, $s \in \mathbb{Z}$, when $\tau = 0$.
3) For all intervals (τ, τ^*) with $\tau \in \mathcal{R}$ and
$$\tau^* := \inf\{\theta > \tau \mid \theta \in \mathcal{R} \cup \{T\}\}$$
it holds that $\xi : (\tau, \tau^*) \to \ell_2^{1,m_1} \times \ell^{1,m_2}$ is absolutely continuous, $\dot{\xi}(t) = F(\xi(t), d(t))$ for almost all $t \in (\tau, \tau^*)$, and that $\xi(t, s) \in \mathcal{C}$ for all $t \in (\tau, \tau^*)$ and $s \in \mathbb{Z}$.

Definition 5: A solution $(\mathcal{R}, \xi)$ to the overall system $\mathcal{H}$ composed of the subsystems $\mathcal{H}(s)$ given by (18), $s \in \mathbb{Z}$, on $[0, T)$ is called maximal if there does not exist a $T' > T$ and a solution $(\mathcal{R}', \xi')$ to the overall system $\mathcal{H}$ on $[0, T')$ for which it holds that

1) $\mathcal{R}' \cap [0, T) = \mathcal{R}$.
2) $\xi'(t) = \xi(t)$ for all $t \in [0, T)\backslash\mathcal{R}$.

Definition 6: A solution $(\mathcal{R}, \xi)$ to the overall system $\mathcal{H}$ composed out of the subsystems $\mathcal{H}(s)$ given by (18), $s \in \mathbb{Z}$, on $[0, T)$ is called complete if $T = \infty$.

Note that complete solutions are always maximal. The set $\mathcal{R}$ contains the jump times, i.e., the times at which there is a jump in one of the subsystems $\mathcal{H}(s)$. Between the successive jump times τ and τ^*, ξ denotes the trajectories in the flow phases of the system (as imposed by item 3 of Definition 4 above). Item 2 of Definition 4 connects the flow phases at the jump times and specifies also the initial conditions.

Remark 3: Establishing proper definitions of solutions for interconnected hybrid systems is not an easy task as evidenced by the studies in [26] and [27], which is in our case further complicated by the infinite dimensionality of the system. Hybrid solution concepts as in the literature, see, e.g., [22], often assume that only a finite number of jumps exists in every finite time interval. Solutions with this property are sometimes called *non-Zeno* solutions to be consistent with the literature on hybrid systems [28]. However, these standard solution concepts do not apply to the systems described in Section III, as Zeno behavior (an infinite number of jumps in a finite time interval) is inevitable due to the infinite number of subsystems with an upper bound on the time elapse in the flow phases. Clearly, we would like to define the solutions beyond such Zeno points, which was not done in [22]. Therefore, inspired by [25], Definition 4 is introduced providing an alternative, but natural solution concept in this context. Right-accumulation points of event times are included and solutions can be defined beyond Zeno points in the sense that despite the occurrence of right-accumulation points or even an infinite number of jumps at one (continuous) time instant, solutions can still be defined globally on $\mathbb{R}_{\geq 0}$ (i.e., for $T = \infty$).

B. Stability and Performance Concepts

With a solution concept in place, stability (in absence of disturbances) and performance (in presence of disturbances) of the overall system can be defined. However, as a result of the specific structure of the networked systems considered in this paper, we are only interested in a relevant set of initial states specified by $\mathbb{X}_0 \subseteq \ell_2^{1,m_1} \times \ell^{1,m_2}$ for the overall system $\mathcal{H}$ in (18). As an example, for the networked systems composed of the identical subsystems $\mathcal{H}(s)$, $s \in \mathbb{Z}$, given by (9) this set is specified by

$$\mathbb{X}_0 = \ell_2^{1,m_0+m_e} \times [0,\tau_{\text{mati}})^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}} \tag{19}$$

where we used the notation $\mathbb{N}^{\mathbb{Z}}$ and $[0,\tau_{\text{mati}})^{\mathbb{Z}}$ as the set of functions mapping $\mathbb{Z}$ to $\mathbb{N}$ and $\mathbb{Z}$ to $[0,\tau_{\text{mati}})$, respectively.

We now define UGAS notion for the overall system.

Definition 7: For the overall system $\mathcal{H}$ with associated set of initial states $\mathbb{X}_0 \subseteq \ell_2^{1,m_1} \times \ell^{1,m_2}$, composed of the subsystems $\mathcal{H}(s)$, $s \in \mathbb{Z}$, given by (18), the set

$$\mathcal{E} = \left\{\xi = (\xi_c, \xi_d) \in \ell_2^{1,m_1} \times \ell^{1,m_2} \mid \xi_c = 0\right\} \tag{20}$$

is UGAS if there exists a function $\beta \in \mathcal{KL}$ such that for any initial condition $\xi(0) \in \mathbb{X}_0$, there exists at least one solution on $[0,T)$ (with $T > 0$), all corresponding maximal solutions $(\mathcal{R},\xi)$ to $\mathcal{H}$ with $d = 0$ are complete, and for all $t \in [0,\infty)\backslash\mathcal{R}$

$$\left\| \xi_c(t) \right\|_{\ell_2} \leq \beta\left(\left\| \xi_c(0) \right\|_{\ell_2}, t\right).$$

Moreover, if β is an exp-$\mathcal{KL}$ function, the set $\mathcal{E}$ is UGES.

The performance of the hybrid system $\mathcal{H}$ is defined as the level of input attenuation with respect to the external output variable

$$z = Q(\xi_c, d) \tag{21}$$

where $Q : \ell_2^{1,m_1+m_d} \to \ell_2^{1,m_z}$ by using the $\mathcal{L}_p$-induced gain with $p \in [0,\infty)$ as the performance criterion.

Definition 8: The overall system $\mathcal{H}$ with associated set of initial states $\mathbb{X}_0 \subseteq \ell_2^{1,m_1} \times \ell^{1,m_2}$, composed of the subsystems $\mathcal{H}(s)$, $s \in \mathbb{Z}$, as in (18) with (21), is said to be $\mathcal{L}_p$-stable ($p < \infty$) with an $\mathcal{L}_p$-gain less than or equal to $\theta \geq 0$ from input d to output z, if there exists a function $\beta \in \mathcal{K}$ such that for any exogenous input $d \in \mathcal{L}_p$ and any initial condition $\xi(0) \in \mathbb{X}_0$, there exists at least one solution on $[0,T)$ (with $T > 0$), all corresponding maximal solutions $(\mathcal{R},\xi)$ to $\mathcal{H}$ are complete, and it holds that

$$\left\| z \right\|_{\mathcal{L}_p} \leq \beta\left(\left\| \xi_c(0) \right\|_{\ell_2}\right) + \theta \left\| d \right\|_{\mathcal{L}_p}. \tag{22}$$

Next we will provide conditions that can be used to derive a bound on the MATI τ_{mati} in (9) guaranteeing UGAS (and sometimes even UGES) of the set $\mathcal{E}$ of (20) and/or $\mathcal{L}_p$-stability with an $\mathcal{L}_p$-gain less than or equal to $\theta \geq 0$.

C. Conditions for UGAS (or UGES)

Inspired by the stability results for finite-dimensional NCSs, see, e.g., [14] and [15], novel conditions under which the set $\mathcal{E}$ is UGAS (or UGES) can be obtained. For notational convenience, we write $f(x,e,0)$ as $f(x,e)$ and $g(x,e,0)$ as $g(x,e)$, since $d = 0$ throughout this section.

Theorem 1: Consider the overall system $\mathcal{H}$ composed of the identical subsystems $\mathcal{H}(s)$ given by (9), $s \in \mathbb{Z}$, with f and g given by (11) or (17) and with associated $\mathbb{X}_0$ of (19) and $d = 0$. Assume there exist a function $W : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, a locally Lipschitz function $V : \ell_2^{1,m_0} \to \mathbb{R}_{\geq 0}$, a continuous function $H : \mathbb{R}^{m_0} \times \mathbb{R}^{m_+ + m_-} \to \mathbb{R}$, real numbers $\lambda \in (0,1)$, $L \geq 0$, $\gamma > 0$, functions $\underline{\alpha}_W, \overline{\alpha}_W, \underline{\alpha}_V, \overline{\alpha}_V \in \mathcal{K}_\infty$, and a continuous, positive definite function ϱ such that

1) for all $\bar{\kappa} \in \mathbb{N}$ and $\bar{e} \in \mathbb{R}^{m_e}$

$$\underline{\alpha}_W(|\bar{e}|) \leq W(\bar{\kappa},\bar{e}) \leq \overline{\alpha}_W(|\bar{e}|) \tag{23a}$$

$$W(\bar{\kappa}+1, h(\bar{\kappa},\bar{e})) \leq \lambda W(\bar{\kappa},\bar{e}), \tag{23b}$$

2) for all $\kappa \in \mathbb{N}^{\mathbb{Z}}$, $x \in \ell_2^{1,m_0}$, $v \in \ell_2^{1,m_+ + m_-}$, and almost all $e \in \ell_2^{1,m_e}$ it holds for some $s \in \mathbb{Z}$ that

$$\left\langle \frac{\partial W(\kappa(s),e(s))}{\partial e(s)}, g(x,e)(s) \right\rangle \leq LW(\kappa(s),e(s)) + H(x(s),v(s)), \tag{24}$$

3) for all $x \in \ell_2^{1,m_0}$

$$\underline{\alpha}_V\left(\left\| x \right\|_{\ell_2}\right) \leq V(x) \leq \overline{\alpha}_V\left(\left\| x \right\|_{\ell_2}\right), \tag{25}$$

4) and for all $\kappa \in \mathbb{N}^{\mathbb{Z}}$, $e \in \ell_2^{1,m_e}$, $v \in \ell_2^{1,m_+ + m_-}$, and almost all $x \in \ell_2^{1,m_0}$

$$\begin{aligned}\langle \nabla V(x), f(x,e)\rangle_{\ell_2} &\leq -\varrho(\left\| x \right\|_{\ell_2}) \\ &+ \sum_{s\in\mathbb{Z}} \big(-\varrho\left(W(\kappa(s),e(s))\right) - H^2(x(s),v(s)) \\ &+ \gamma^2 W^2(\kappa(s),e(s))\big).\end{aligned} \tag{26}$$

If τ_{mati} satisfies

$$\tau_{\text{mati}} \leq \begin{cases} \dfrac{1}{Lr}\arctan\left(\dfrac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}\right)+1+\lambda}\right), & \gamma > L \\ \dfrac{1-\lambda}{L(1+\gamma)}, & \gamma = L \\ \dfrac{1}{Lr}\operatorname{arctanh}\left(\dfrac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}\right)+1+\lambda}\right), & \gamma < L \\ \dfrac{1}{\gamma}\arctan\left(\dfrac{(1+\lambda)(1-\lambda)}{2\lambda}\right), & L = 0 \end{cases} \tag{27}$$

where $r = \sqrt{\left|(\gamma/L)^2 - 1\right|}$ then the set $\mathcal{E}$ given in (20) is UGAS.

If, in addition, there exist strictly positive real numbers $\underline{\alpha}_W^c$, $\overline{\alpha}_W^c$, $\underline{\alpha}_V^c$, $\overline{\alpha}_V^c$, and ε such that for all $\bar{e} \in \mathbb{R}^{m_e}$ and $\bar{\kappa} \in \mathbb{N}$

$\underline{\alpha}_W^c\,|\bar{e}| \leq W(\bar{\kappa},\bar{e}) \leq \overline{\alpha}_W^c\,|\bar{e}|$, and for all $x \in \ell_2^{1,m_0}$ $\underline{\alpha}_V^c \,\|\, x \,\|_{\ell_2}^2 \leq V(x) \leq \overline{\alpha}_V^c \,\|\, x \,\|_{\ell_2}^2$, and $\varrho(r) \geq \varepsilon^2 r^2$ for $r \in \mathbb{R}_{>0}$, then the set $\mathcal{E}$ is UGES.

The proof is given in the Appendix.

Remark 4: Note that if condition (24) holds for a *single* $s \in \mathbb{Z}$, then it also holds for all $s \in \mathbb{Z}$ as a result of the spatial invariance of the subsystems in the configurations S1 and S2. As such, we classify (24) as a *local* condition, in the sense that we only need to check it for *one* subsystem in the interconnection.

Remark 5: Various scheduling protocols exist that satisfy (23) including the TOD, SD, and RR protocols, see [14].

D. Conditions for $\mathcal{L}_p$-Stability

For the case that there are external (disturbance) inputs present in the system, i.e., $d \neq 0$, based on the results of [16], [29], the following theorem is now composed of $\mathcal{L}_p$-stability.

Theorem 2: Consider the overall system $\mathcal{H}$ composed of the identical subsystems $\mathcal{H}(s)$ given by (9)–(10), $s \in \mathbb{Z}$, with f and g given by (11) or (17) and with associated $\mathbb{X}_0$ of (19) and $d \in \mathcal{L}_p$. Assume there exist a function $W : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, a locally Lipschitz function $V : \ell_2^{1,m_0} \to \mathbb{R}_{\geq 0}$, a continuous function $H : \mathbb{R}^{m_0} \times \mathbb{R}^{m_+ + m_-} \times \mathbb{R}^{m_d} \to \mathbb{R}$, real numbers $\lambda \in (0,1)$, $L \geq 0$, $\gamma > 0$, functions $\underline{\alpha}_W, \overline{\alpha}_W, \underline{\alpha}_V, \overline{\alpha}_V \in \mathcal{K}_\infty$, and a continuous, positive definite function ϱ such that

1) for all $\bar{\kappa} \in \mathbb{N}$ and $\bar{e} \in \mathbb{R}^{m_e}$ (23) holds,
2) for all $\kappa \in \mathbb{N}^{\mathbb{Z}}$, $x \in \ell_2^{1,m_0}$, $v \in \ell_2^{1,m_+ + m_-}$, $d \in \ell_2^{1,m_d}$, and almost all $e \in \ell^{1,m_e}$ it holds for some $s \in \mathbb{Z}$ that

$$\left\langle \frac{\partial W(\kappa(s), e(s))}{\partial e(s)}, g(x,e,d)(s) \right\rangle \leq LW(\kappa(s), e(s)) + H\left(x(s), v(s), d(s)\right), \quad (28)$$

3) for all $x \in \ell_2^{1,m_0}$ (25) holds and for all $e \in \ell_2^{1,m_e}$, $\kappa \in \mathbb{N}^{\mathbb{Z}}$, $d \in \ell_2^{1,m_d}$, $v \in \ell_2^{1,m_+ + m_-}$, and almost all $x \in \ell_2^{1,m_0}$

$$\langle \nabla V(x), f(x,e,d)\rangle_{\ell_2} \leq \mu\left(\theta^p \,\|\, d \,\|_{\ell_2}^p - \|\, q(x,e,d) \,\|_{\ell_2}^p\right) + \sum_{s\in\mathbb{Z}} \left(-H^2(x(s), v(s), d(s)) + \gamma^2 W^2(\kappa(s), e(s))\right) \quad (29)$$

holds for some $\mu > 0$ and $\theta \geq 0$.

If τ_{mati} satisfies (27), then the overall system $\mathcal{H}$ is $\mathcal{L}_p$-stable from d to z with a gain less than or equal to θ.

The proof is given in the Appendix. Theorems 1 and 2 are general theorems to guarantee stability of the overall system $\mathcal{H}$. However, it should be noted that $V : \ell_2^{1,m_0} \to \mathbb{R}_{\geq 0}$ is a *global* Lyapunov function, making it, due to the infinite-dimensional character of $\mathcal{H}$, extremely hard and often intractable to construct such a function. Therefore, it is of importance to exploit the interconnection structure to obtain local conditions, possibly based on a local Lyapunov function or storage function. This is the subject of the next section.

V. Local Conditions for UGES and $\mathcal{L}_p$-Stability

To systematically verify the conditions of Theorems 1 and 2, we provide here reformulations, such that, UGES and $\mathcal{L}_p$-stability can be guaranteed merely based on the local dynamics of the spatially invariant subsystems, the interconnection structure, and the local scheduling protocol, thereby making the conditions tractable.

A. Local Conditions for UGES

To guarantee UGES of the set $\mathcal{E}$ given by (20) according to Theorem 1 for the case $d = 0$, we take (23) as a starting point, where we assume that (a stronger version of) (23a) is satisfied in the sense that there exist strictly positive constants $\underline{\alpha}_W^c$ and $\overline{\alpha}_W^c$ such that

$$\underline{\alpha}_W^c\,|\bar{e}| \leq W(\bar{\kappa},\bar{e}) \leq \overline{\alpha}_W^c\,|\bar{e}| \quad (30)$$

for all $\bar{\kappa} \in \mathbb{N}$ and $\bar{e} \in \mathbb{R}^{m_e}$. Various protocols exist with this property, including the ones mentioned in Remark 5.

Let us now consider the condition (24). We assume that for almost all $\bar{e} \in \mathbb{R}^{m_e}$ and all $\bar{\kappa} \in \mathbb{N}$ it holds that

$$\left| \frac{\partial W(\bar{\kappa},\bar{e})}{\partial \bar{e}} \right| \leq M \quad (31)$$

for some constant $M > 0$. For all the considered protocols in [13]–[15], [20] such a constant exists. More precisely, for the considered protocols in Remark 5, it holds that condition (31) is fulfilled for $M_{RR} = \sqrt{l}$ and $M_{TOD} = M_{SD} = 1$, respectively, as shown in [16]. Using now the Cauchy–Schwarz inequality together with (30) and (31), it is directly concluded that, for the case $d = 0$, condition (24) can be guaranteed for some $s \in \mathbb{Z}$ by obtaining that

$$M\,\big|\,g(x,e,0)(s)\,\big| \leq L\underline{\alpha}_W^c\,\big|\,e(s)\,\big| + H(x(s), v(s)). \quad (32)$$

Since for the considered system configuration of (9)–(10) the function $g(x,e,d)(s)$ is known and assumed to only depend on local dynamics as a result of the interconnection structure as in Fig. 1, it is possible to obtain L and $H(x(s), v(s))$ as in (24). For instance, for the system configuration S1 of Fig. 2 it holds for some $s \in \mathbb{Z}$ that

$$\dot{e}(s) = g(x,e,0)(s) = -\frac{\partial g_p(x(s))}{\partial x(s)} f_p(x(s), v(s), d(s)) \quad (33)$$

from which we directly obtain that (24) (and thus also (32)) holds with $L = 0$ and $H\left(x(s), v(s)\right)$ given by

$$H\left(x(s), v(s)\right) = M \left| \frac{\partial g_p(x(s))}{\partial x(s)} f_p(x(s), v(s), 0) \right|. \quad (34)$$

Now by using the obtained function $H(x(s), v(s))$ from (32) and by taking $\varrho(r) \geq \varepsilon^2 r^2$, $r \in \mathbb{R}_{>0}$, with $0 < \varepsilon < \gamma$, we obtain by direct substitution in the right-hand side of (26) that

$$\begin{aligned} &-\varrho(\|\, x \,\|_{\ell_2}) + \sum_{s\in\mathbb{Z}} \big(-\varrho\left(W(\kappa(s), e(s))\right) \\ &\qquad - H^2(x(s), v(s)) + \gamma^2 W^2(\kappa(s), e(s))\big) \quad (35) \\ &\geq \sum_{s\in\mathbb{Z}} \Big({\underline{\alpha}_W^c}^2 [\gamma^2 - \varepsilon^2] \big|e(s)\big|^2 - \varepsilon^2 \big|x(s)\big|^2 - H^2(x(s), v(s)) \Big). \end{aligned}$$

To arrive at a local condition which guarantees UGES of the set $\mathcal{E}$ of (20), also the left-hand side of (26) needs to be evaluated, for which we will make use of the concepts from the theory of dissipative systems initiated in [30].

For a subsystem $\mathcal{H}(s)$ given by (9) for the case $d=0$, we propose to introduce a (local) storage function $\mathcal{V}:\mathbb{R}^{m_0}\to\mathbb{R}_{\geq 0}$ and a (local) supply function $\mathcal{S}_i:\mathbb{R}^{m_+ + m_-}\times\mathbb{R}^{m_+ + m_-}\to\mathbb{R}$ for the interconnection variables, see [30] for the terminology. To define this (local) supply function $\mathcal{S}_i$, we regroup the interconnection variables as $(q_+(s), q_-(s))$, where $q_+(s)$ are the "networked" interconnection variables of the subsystem at $s\in\mathbb{Z}$ with its successor at $s+1$ and $q_-(s)$ are the "networked" interconnection variables with its predecessor at $s-1$, i.e.,

$$q_+(s) := \begin{bmatrix} \begin{bmatrix} I_{m_+} & 0 \end{bmatrix} (\mathbf{S}v)(s) \\ \begin{bmatrix} 0 & I_{m_-} \end{bmatrix} v(s) \end{bmatrix} = \begin{bmatrix} v_+(s+1) \\ v_-(s) \end{bmatrix}$$

$$q_-(s) := \begin{bmatrix} \begin{bmatrix} I_{m_+} & 0 \end{bmatrix} v(s) \\ \begin{bmatrix} 0 & I_{m_-} \end{bmatrix} (\mathbf{S}^{-1}v)(s) \end{bmatrix} = \begin{bmatrix} v_+(s) \\ v_-(s-1) \end{bmatrix}.$$

Hence, for the system configuration S1 it holds that

$$q_+(s) = \begin{bmatrix} \hat{w}_+(s) \\ v_-(s) \end{bmatrix} = \begin{bmatrix} w_+(s)+e_+(s) \\ v_-(s) \end{bmatrix} \tag{36a}$$

and

$$q_-(s) = \begin{bmatrix} v_+(s) \\ \hat{w}_-(s) \end{bmatrix} = \begin{bmatrix} v_+(s) \\ w_-(s)+e_-(s) \end{bmatrix} \tag{36b}$$

while for the system configuration S2 we have that

$$q_+(s) = \begin{bmatrix} w_+(s) \\ v_-(s) \end{bmatrix} \quad \text{and} \quad q_-(s) = \begin{bmatrix} v_+(s) \\ w_-(s) \end{bmatrix}.$$

Note that it thus holds that

$$q_+(s) = q_-(s+1) \quad \text{for all} \quad s\in\mathbb{Z} \tag{37}$$

or compactly, $q_+ = \Delta_{\mathbf{S},\hat{m}} q_-$ with $\hat{m}=(0, m_+ + m_-)$. Based on these new interconnection variables $q_+(s)$ and $q_-(s)$, we define $\mathcal{S}_i$ as

$$\mathcal{S}_i(q_+(s), q_-(s)) := \begin{bmatrix} q_+(s) \\ q_-(s) \end{bmatrix}^\top \begin{bmatrix} -X_{\mathbf{S}} & 0 \\ 0 & X_{\mathbf{S}} \end{bmatrix} \begin{bmatrix} q_+(s) \\ q_-(s) \end{bmatrix} \tag{38}$$

with $X_{\mathbf{S}}\in\mathbb{R}_{\mathbf{S}}^{m_+ + m_-}$. The choice for (q_+, q_-) as interconnection variables along with the structure of $\mathcal{S}_i$ becomes now quickly apparent as it results in a so-called *neutral* interconnection with respect to the supply rate $\mathcal{S}_i$, see [30], for the overall system, as formalized by the following lemma.

Lemma 1: For the interconnection variables q_+ and q_-, the interconnection is neutral in the sense that

$$\sum_{s\in\mathbb{Z}} \mathcal{S}_i(q_+(s), q_-(s)) = \sum_{s\in\mathbb{Z}} \left(-q_+^\top(s) X_{\mathbf{S}} q_+(s) + q_-^\top(s) X_{\mathbf{S}} q_-(s)\right). = 0. \tag{39}$$

Proof of Lemma 1: As a direct result of (37), we obtain that indeed (39) follows, see also [6, Proposition 7]. ■

Now taking this neutral interconnection into account, Theorem 1 can be rewritten such that only local conditions (see Remark 4) are needed to guarantee UGES of the set $\mathcal{E}$ of (20).

Theorem 3: Consider the overall system $\mathcal{H}$, composed of the identical subsystems $\mathcal{H}(s)$ of (9), $s\in\mathbb{Z}$, with f and g given by (11) or (17) and with associated $\mathbb{X}_0$ of (19) and $d=0$. Assume there exist a function $W:\mathbb{N}\times\mathbb{R}^{m_e}\to\mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, a locally Lipschitz function $\mathcal{V}:\mathbb{R}^{m_0}\to\mathbb{R}_{\geq 0}$, a continuous function $H:\mathbb{R}^{m_0}\times\mathbb{R}^{m_+ + m_-}\to\mathbb{R}$, and the constants $\underline{\alpha}_W^c, \overline{\alpha}_W^c, \underline{\alpha}_V^c, \overline{\alpha}_V^c, L\in\mathbb{R}_{\geq 0}$, $M>0$, $0<\varepsilon<\gamma$, and $\lambda\in(0,1)$ such that

1) for all $\bar{\kappa}\in\mathbb{N}$ and $\bar{e}\in\mathbb{R}^{m_e}$ (23) holds with $\underline{\alpha}_W(r) = \underline{\alpha}_W^c r$ and $\overline{\alpha}_W(r) = \overline{\alpha}_W^c r$,
2) for all $\bar{\kappa}\in\mathbb{N}$ and almost all $\bar{e}\in\mathbb{R}^{m_e}$ (31) holds,
3) for all $\kappa\in\mathbb{N}^{\mathbb{Z}}$, $x\in\ell_2^{1,m_0}$, $v\in\ell_2^{1,m_+ + m_-}$, and almost all $e\in\ell_2^{1,m_e}$ (32) holds for some $s\in\mathbb{Z}$
4) for all $\bar{x}\in\mathbb{R}^{m_0}$
$$\underline{\alpha}_V^c |\bar{x}|^2 \leq \mathcal{V}(\bar{x}) \leq \overline{\alpha}_V^c |\bar{x}|^2 \tag{40}$$
5) and for all $\kappa\in\mathbb{N}^{\mathbb{Z}}$, $e\in\ell_2^{1,m_e}$, $v\in\ell_2^{1,m_+ + m_-}$ and almost all $x\in\ell_2^{1,m_0}$ it holds for some $s\in\mathbb{Z}$ that
$$\begin{aligned} \langle \nabla\mathcal{V}(x(s)), f(x,e)(s)\rangle &\leq -\varepsilon^2 \left[x(s)\right]^2 \\ &+ \underline{\alpha}_W^{c\,2}[\gamma^2-\varepsilon^2]\left|e(s)\right|^2 - H^2(x(s), v(s)) \\ &+ \mathcal{S}_i(q_+(s), q_-(s)) \end{aligned} \tag{41}$$

where $\mathcal{S}_i(q_+(s), q_-(s))$ is given by (38). If now τ_{mati} satisfies (27), then the set $\mathcal{E}$ of (20) is UGES.

Proof of Theorem 3: Summing (41) over $s\in\mathbb{Z}$, defining $V(x) := \sum_{s\in\mathbb{Z}} \mathcal{V}(x(s))$, and using (35) and (39), it can be directly seen that indeed (26) holds. ■

B. Local Conditions for $\mathcal{L}_p$-Stability

In a similar approach as described for the case where $d=0$, also the conditions as presented in Theorem 2 can be rewritten as local conditions for the case $d\neq 0$. Consider therefore again the newly imposed conditions (30) and (31). It now directly follows that condition (28) is satisfied when there exist a constant L and a function $H(x(s), v(s), d(s))$ such that

$$M\left|g(x,e,d)(s)\right| \leq L\underline{\alpha}_W^c \left|e(s)\right| + H(x(s), v(s), d(s)) \tag{42}$$

holds for some $s\in\mathbb{Z}$. Using the function $H(x(s), v(s), d(s))$ and by again taking $\varrho(r)\geq\varepsilon^2 r^2$, $r\in\mathbb{R}_{>0}$, with $0<\varepsilon<\gamma$, we obtain by direct substitution that

$$\begin{aligned} &\sum_{s\in\mathbb{Z}} \gamma^2 W^2(\kappa(s), e(s)) - H^2(x(s), v(s), d(s)) \\ &\quad \geq \sum_{s\in\mathbb{Z}} \underline{\alpha}_W^{c\,2}\gamma^2 |e(s)|^2 - H^2(x(s), v(s), d(s)). \end{aligned} \tag{43}$$

To evaluate the condition of (29), consider again a subsystem $\mathcal{H}(s)$ given by the general framework of (9) for the case $d\neq 0$. We propose, similar to the case where $d=0$, to introduce a (local) storage function $\mathcal{V}:\mathbb{R}^{m_0}\to\mathbb{R}_{\geq 0}$. However now, as $d\neq 0$, we introduce the (local) supply function $\mathcal{S}:\mathbb{R}^{m_d}\times\mathbb{R}^{m_z}\times\mathbb{R}^{m_+ + m_-}\times\mathbb{R}^{m_+ + m_-}\to\mathbb{R}$ as

$$\begin{aligned} \mathcal{S}(d(s), z(s), q_+(s), q_-(s)) &:= \mathcal{S}_p(d(s), z(s)) \\ &\quad + \mathcal{S}_i(q_+(s), q_-(s)) \end{aligned} \tag{44}$$

where $\mathcal{S}_i$ is given by (38) and

$$\mathcal{S}_p(d(s), z(s)) := \mu\left(\theta^p \left|d(s)\right|^p - \left|z(s)\right|^p\right)$$

with $z(s) = q(x,e,d)(s)$ given by (10). When again Lemma 1 is taken into account, Theorem 2 can be rewritten such that also only local conditions are needed to guarantee $\mathcal{L}_p$-stability.

Theorem 4: Consider the overall system $\mathcal{H}$, composed of the identical subsystems $\mathcal{H}(s)$ of (9)–(10), $s \in \mathbb{Z}$, with f and g given by (11) or (17) and with associated $\mathbb{X}_0$ of (19) and $d \in \mathcal{L}_p$. Assume there exist a function $W : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, a locally Lipschitz function $\mathcal{V} : \mathbb{R}^{m_0} \to \mathbb{R}_{\geq 0}$, a continuous function $H : \mathbb{R}^{m_0} \times \mathbb{R}^{m_+ + m_-} \to \mathbb{R}$, and the constants $\underline{\alpha}_W^c, \overline{\alpha}_W^c, \underline{\alpha}_V^c, \overline{\alpha}_V^c, L \in \mathbb{R}_{\geq 0}$, $M > 0$, $\gamma > 0$, and $\lambda \in (0, 1)$ such that

1) for all $\bar{\kappa} \in \mathbb{N}$ and $\bar{e} \in \mathbb{R}^{m_e}$ (23) holds with $\underline{\alpha}_W(r) = \underline{\alpha}_W^c r$ and $\overline{\alpha}_W(r) = \overline{\alpha}_W^c r$,
2) for all $\bar{\kappa} \in \mathbb{N}$ and almost all $\bar{e} \in \mathbb{R}^{m_e}$ (31) holds,
3) for all $\kappa \in \mathbb{N}^{\mathbb{Z}}$, $x \in \ell_2^{1,m_0}$, $v \in \ell_2^{1,m_+ + m_-}$, $d \in \ell_2^{1,m_d}$, and almost all $e \in \ell_2^{1,m_e}$ (42) holds for some $s \in \mathbb{Z}$
4) for all $\bar{x} \in \mathbb{R}^{m_0}$ (40) holds,
5) and for all $\kappa \in \mathbb{N}^{\mathbb{Z}}$, $e \in \ell_2^{1,m_e}$, $v \in \ell_2^{1,m_+ + m_-}$, $d \in \ell_2^{1,m_d}$ and almost all $x \in \ell_2^{1,m_0}$ it holds for some $s \in \mathbb{Z}$ that

$$\langle \nabla \mathcal{V}(x(s)), f(x, e, d)(s) \rangle \leq -H^2\left(x(s), v(s), d(s)\right) + \mathcal{S}\left(d(s), z(s), q_+(s), q_-(s)\right) + \underline{\alpha}_W^{c\,2} \gamma^2 \left| e(s) \right|^2 \tag{45}$$

for some $\mu > 0$ and $\theta \geq 0$,

with $\mathcal{S}\big(d(s), z(s), q_+(s), q_-(s)\big)$ given by (44). If now τ_{mati} satisfies (27), then the overall system $\mathcal{H}$ is $\mathcal{L}_p$-stable form d to z with a gain less than or equal to θ.

Proof of Theorem 4: Summing (45) over $s \in \mathbb{Z}$, defining $V(x) := \sum_{s \in \mathbb{Z}} \mathcal{V}(x(s))$, and using (39) and (43), it can be directly seen that indeed (29) holds. ∎

Both Theorem 3 as well as Theorem 4 provide tractable local conditions to guarantee UGES or $\mathcal{L}_p$-stability properties of the overall infinite-dimensional system. Moreover, in the linear case, the conditions stated by (41) and (45) result in LMIs, which we show in the next section.

VI. Linear Case

In this section, we discuss how to systematically construct the functions and constants satisfying the local conditions presented in Section V in the case that the general hybrid framework of (9)–(10) is composed of linear flow equations. For the sake of brevity, we only consider the configuration S1, although a similar analysis applies to the configuration S2.

A. System Description

Consider again the diagram of Fig. 2 together with the model of (1), however now for the case that the plant $\mathcal{P}(s)$ is governed by a linear time-invariant system, as studied in [6] for the case without communication networks. In this linear case, the dynamic model for each subsystem (1) is for every fixed $s \in \mathbb{Z}$ described by

$$\begin{bmatrix} \dot{x}(s) \\ w(s) \\ z(s) \end{bmatrix} = \begin{bmatrix} A_{\mathbf{TT}} & A_{\mathbf{TS}} & B_{\mathbf{T}} \\ A_{\mathbf{ST}} & 0 & 0 \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix} \begin{bmatrix} x(s) \\ v(s) \\ d(s) \end{bmatrix} \tag{46}$$

with the initial conditions $x(0, s) = x_0(s) \in \mathbb{R}^{m_0}$, $x_0 \in \ell_2^{1,m_0}$. However now, when using the spatial shift operator of (4) and the interconnection condition $v(s) = \left(\Delta_{\mathbf{S},m} \hat{w}\right)(s)$, see (5), the interconnected system can be expressed as

$$\begin{bmatrix} \dot{x}(s) \\ w(s) \\ z(s) \end{bmatrix} = \begin{bmatrix} A_{\mathbf{TT}} & A_{\mathbf{TS}} & B_{\mathbf{T}} \\ A_{\mathbf{ST}} & 0 & 0 \\ C_{\mathbf{T}} & C_{\mathbf{S}} & D \end{bmatrix} \begin{bmatrix} x(s) \\ \left(\Delta_{\mathbf{S},m} \hat{w}\right)(s) \\ d(s) \end{bmatrix} \tag{47}$$

for every fixed $s \in \mathbb{Z}$. Based on the operators $A_{\mathbf{TT}}$, $A_{\mathbf{TS}}$, etc., of (47), which map $\mathbb{R}^n$ to $\mathbb{R}^n$, we also define the "diagonal" operators $\tilde{A}_{\mathbf{TT}}$, $\tilde{A}_{\mathbf{TS}}$, etc., which map ℓ_2 to ℓ_2, as $(\tilde{A}_{\mathbf{TT}} x)(s) = A_{\mathbf{TT}} x(s)$, $(\tilde{A}_{\mathbf{TS}} v)(s) = A_{\mathbf{TS}} v(s)$, etc., for $s \in \mathbb{Z}$.

Based on (3), (4), (7), and (47), it follows that for the interconnection variables we have that

$$w(s) = A_{\mathbf{ST}} x(s) \tag{48}$$

while for the state dynamics it now holds that

$$\begin{aligned} \dot{x}(s) &= A_{\mathbf{TT}} x(s) + A_{\mathbf{TS}}\left(\Delta_{\mathbf{S},m} \hat{w}\right)(s) + B_{\mathbf{T}} d(s) \\ &= A_{\mathbf{TT}} x(s) + A_{\mathbf{TS}}\left(\Delta_{\mathbf{S},m} w\right)(s) \\ &\quad + A_{\mathbf{TS}}\left(\Delta_{\mathbf{S},m} e\right)(s) + B_{\mathbf{T}} d(s) \end{aligned}$$

which by using (48) leads to

$$\dot{x}(s) = \left(\left(\tilde{A}_{\mathbf{TT}} + \tilde{A}_{\mathbf{TS}} \Delta_{\mathbf{S},m} \tilde{A}_{\mathbf{ST}}\right) x\right)(s) + \left(\tilde{B}_{\mathbf{T}} d\right)(s) + \left(\tilde{A}_{\mathbf{TS}} \Delta_{\mathbf{S},m} e\right)(s).$$

Moreover, an expression for the error dynamics with $e(s)$ as in (7) can be derived by directly using (47) and $(\Delta_{\mathbf{S},m} \dot{\hat{w}})(s) = 0$, which follows from the ZOH assumption. This yields

$$\dot{e}(s) = -A_{\mathbf{ST}} \dot{x}(s). \tag{49}$$

Hence, in both the state dynamics and the error dynamics, the interconnection variables $w(s)$ have now been eliminated, allowing us to compose a hybrid model $\mathcal{H}(s)$ in the format of (9)–(10) for every fixed $s \in \mathbb{Z}$, i.e.,

$$\mathcal{H}(s) : \begin{cases} \left.\begin{aligned} \dot{x}(s) &= (\mathbf{A}x)(s) + (\mathbf{B}d)(s) \\ &\quad + (\mathbf{E}e)(s) \\ \dot{e}(s) &= -A_{\mathbf{ST}} \dot{x}(s) - B_{\mathbf{S}} \dot{d}(s) \\ \dot{\tau}(s) &= 1 \\ \dot{\kappa}(s) &= 0 \end{aligned}\right\} \text{when } \tau(s) \in [0, \tau_{\text{mati}}] \\ \left.\begin{aligned} x^+(s) &= x(s) \\ e^+(s) &= h\left(\kappa(s), e(s)\right) \\ \tau^+(s) &= 0 \\ \kappa^+(s) &= \kappa(s) + 1 \end{aligned}\right\} \text{when } \tau(s) \in [\delta, \infty) \end{cases} \tag{50}$$

with the output equation

$$z(s) = (\mathbf{C}x)(s) + (\mathbf{D}d)(s) + (\mathbf{H}e)(s) \tag{51}$$

where the operators which map ℓ_2 to ℓ_2 are defined as

$$\mathbf{A} = \tilde{A}_{\mathbf{TT}} + \tilde{A}_{\mathbf{TS}} \Delta_{\mathbf{S},m} \tilde{A}_{\mathbf{ST}}, \quad \mathbf{B} = \tilde{B}_{\mathbf{T}}, \quad \mathbf{E} = \tilde{A}_{\mathbf{TS}} \Delta_{\mathbf{S},m}$$
$$\mathbf{C} = \tilde{C}_{\mathbf{T}} + \tilde{C}_{\mathbf{S}} \Delta_{\mathbf{S},m} \tilde{A}_{\mathbf{ST}}, \quad \mathbf{D} = \tilde{D}, \quad \mathbf{H} = \tilde{C}_{\mathbf{S}} \Delta_{\mathbf{S},m}.$$

Remark 6: When perfect communication is assumed, i.e., $\hat{w} = w$, we have that $e = 0$, in which case (50), (51) recovers the setup described in [6] for the case $A_{\mathbf{SS}} = 0$ and $B_{\mathbf{S}} = 0$.

B. LMI Condition for UGES

In order to make Theorem 3 numerically tractable for the system configuration S1, we now provide local LMI-based reformulations of the conditions in Theorem 3 by exploiting the linearity of the considered underlying systems. These reformulations have connections to the ideal communication case treated in [6], see also Remark 6. Here, we generalize these results to the case of nonideal packet-based communication.

As a result of (33), (34), and (49) we directly obtain that, for the case $d = 0$, (32) is satisfied for $L = 0$ and

$$H\left(x(s), v(s)\right) = M\left|A_{\mathbf{ST}}A_{\mathbf{TT}}x(s) + A_{\mathbf{ST}}A_{\mathbf{TS}}v(s)\right|.$$

Define now the (local) matrix $J \in \mathbb{R}^{m_0+m_e+m_++m_-}$ by

$$J := \begin{bmatrix} -\varepsilon^2 I_{m_0} - M^2 A_{\mathbf{TT}}^\top \widehat{A}_{\mathbf{ST}} A_{\mathbf{TT}} & 0 & -M^2 A_{\mathbf{TT}}^\top \widehat{A}_{\mathbf{ST}} A_{\mathbf{TS}} \\ 0 & \underline{\alpha}_W^c{}^2[\gamma^2 - \varepsilon^2]I & 0 \\ -M^2 A_{\mathbf{TS}}^\top \widehat{A}_{\mathbf{ST}} A_{\mathbf{TT}} & 0 & -M^2 A_{\mathbf{TS}}^\top \widehat{A}_{\mathbf{ST}} A_{\mathbf{TS}} \end{bmatrix}$$

with $\widehat{A}_{\mathbf{ST}} := A_{\mathbf{ST}}^\top A_{\mathbf{ST}}$, such that it can be concluded that for (part of) the right-hand side of (41) it holds that

$$\begin{aligned} &-\varepsilon^2\left|x(s)\right|^2 + \underline{\alpha}_W^c{}^2[\gamma^2 - \varepsilon^2]\left|e(s)\right|^2 - H^2\left(x(s), v(s)\right) \\ &= -\varepsilon^2\left|x(s)\right|^2 - M^2\left|A_{\mathbf{ST}}A_{\mathbf{TT}}x(s) + A_{\mathbf{ST}}A_{\mathbf{TS}}v(s)\right|^2 \\ &\quad + \underline{\alpha}_W^c{}^2[\gamma^2 - \varepsilon^2]\left|e(s)\right|^2 \\ &= \langle (x(s), e(s), v(s)), J(x(s), e(s), v(s))\rangle. \end{aligned}$$

For the (local) storage function $\mathcal{V}$ of (41) we now propose to use $\mathcal{V}(x(s)) = x^\top(s) X_{\mathbf{T}} x(s)$, where $X_{\mathbf{T}}$ is a symmetric positive definite matrix, i.e., $X_{\mathbf{T}} \in \mathcal{X}_{\mathbf{T}} := \left\{X_{\mathbf{T}} \in \mathbb{R}_{\mathbf{S}}^{m_0 \times m_0} \mid X_{\mathbf{T}} \succ 0\right\}$. To express $(q_+(s), q_-(s))$ in terms of $x(s)$, $e(s)$, and $v(s)$, (47) needs to be partitioned further to reflect the structure of $\Delta_{\mathbf{S},m}$ of (5), i.e.,

$$A_{\mathbf{ST}} = \begin{bmatrix} A_{\mathbf{ST}}^{++} \\ A_{\mathbf{ST}}^{--} \end{bmatrix}, \quad \text{and} \quad A_{\mathbf{TS}} = \begin{bmatrix} A_{\mathbf{TS}}^{++} & A_{\mathbf{TS}}^{--} \end{bmatrix}.$$

Based on this partitioning, analogous to [6], we introduce

$$A_{\mathbf{SS},v}^{+} = \begin{bmatrix} 0 & 0 \\ 0 & I_{m_-} \end{bmatrix} \; A_{\mathbf{SS},e}^{+} = \begin{bmatrix} I_{m_+} & 0 \\ 0 & 0 \end{bmatrix} \quad A_{\mathbf{ST}}^{+} = \begin{bmatrix} A_{\mathbf{ST}}^{++} \\ 0 \end{bmatrix}$$

$$A_{\mathbf{SS},v}^{-} = \begin{bmatrix} I_{m_+} & 0 \\ 0 & 0 \end{bmatrix} \; A_{\mathbf{SS},e}^{-} = \begin{bmatrix} 0 & 0 \\ 0 & I_{m_-} \end{bmatrix} \quad A_{\mathbf{ST}}^{-} = \begin{bmatrix} 0 \\ A_{\mathbf{ST}}^{--} \end{bmatrix}$$

and hence, (36a) and (36b) result in

$$\begin{aligned} q_+(s) &= \begin{bmatrix} A_{\mathbf{ST}}^{+} & A_{\mathbf{SS},e}^{+} & A_{\mathbf{SS},v}^{+} \end{bmatrix} (x(s), e(s), v(s)) \\ q_-(s) &= \begin{bmatrix} A_{\mathbf{ST}}^{-} & A_{\mathbf{SS},e}^{-} & A_{\mathbf{SS},v}^{-} \end{bmatrix} (x(s), e(s), v(s)). \end{aligned} \tag{52}$$

Moreover, with the defined matrices

$$A_{\mathbf{TS}}^{+} = \begin{bmatrix} A_{\mathbf{TS}}^{++} & 0 \end{bmatrix}, \quad \text{and} \quad A_{\mathbf{TS}}^{-} = \begin{bmatrix} 0 & A_{\mathbf{TS}}^{--} \end{bmatrix}$$

we have that

$$\begin{aligned} A_{\mathbf{TS}}v(s) &= A_{\mathbf{TS}}\begin{bmatrix} v_+(s) \\ v_-(s) \end{bmatrix} \\ &\overset{(36)}{=} \begin{bmatrix} A_{\mathbf{TS}}^{++} & A_{\mathbf{TS}}^{--} \end{bmatrix}\left(\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} q_+(s) + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} q_-(s)\right) \\ &= A_{\mathbf{TS}}^{-} q_+(s) + A_{\mathbf{TS}}^{+} q_-(s). \end{aligned} \tag{53}$$

The local condition (41) can now be rewritten into a local LMI condition. Direct substitution of $\mathcal{V}$ and $(q_+(s), q_-(s))$ in (41) and using (50) for the case $d = 0$ yields

$$\begin{aligned} &\langle \nabla\mathcal{V}(x(s)), f(x,e)(s)\rangle - \mathcal{S}_i\left(q_+(s), q_-(s)\right) \\ &\overset{(38),(46)}{=} \left(A_{\mathbf{TT}}x(s) + A_{\mathbf{TS}}v(s)\right)^\top X_{\mathbf{T}}x(s) + q_+^\top(s)X_{\mathbf{S}}q_+(s) \\ &\quad - q_-^\top(s)X_{\mathbf{S}}q_-(s) + x(s)^\top X_{\mathbf{T}}\left(A_{\mathbf{TT}}x(s) + A_{\mathbf{TS}}v(s)\right) \\ &\overset{(53)}{=} \begin{bmatrix} x(s) \\ q_+(s) \end{bmatrix}^\top \begin{bmatrix} 0 & X_{\mathbf{T}}A_{\mathbf{TS}}^{-} \\ \left(A_{\mathbf{TS}}^{-}\right)^\top X_{\mathbf{T}} & X_{\mathbf{S}} \end{bmatrix} \begin{bmatrix} x(s) \\ q_+(s) \end{bmatrix} \\ &\quad + \begin{bmatrix} x(s) \\ q_-(s) \end{bmatrix}^\top \begin{bmatrix} A_{\mathbf{TT}}^\top X_{\mathbf{T}} + X_{\mathbf{T}}A_{\mathbf{TT}} & X_{\mathbf{T}}A_{\mathbf{TS}}^{+} \\ \left(A_{\mathbf{TS}}^{+}\right)^\top X_{\mathbf{T}} & -X_{\mathbf{S}} \end{bmatrix} \begin{bmatrix} x(s) \\ q_-(s) \end{bmatrix} \\ &\overset{(52)}{=} \left\langle (x(s), e(s), v(s)), \mathcal{J}_+\left(x(s), e(s), v(s)\right)\right\rangle \\ &\quad + \left\langle (x(s), e(s), v(s)), \mathcal{J}_-\left(x(s), e(s), v(s)\right)\right\rangle \end{aligned} \tag{54}$$

with S_i given by (38) and $X_{\mathbf{T}}A_{\mathbf{TS}} = X_{\mathbf{T}}A_{\mathbf{TS}}^{-} + X_{\mathbf{T}}A_{\mathbf{TS}}^{+}$, and where, by using the coordinate transformation of (36), the matrices $\mathcal{J}_+$ and $\mathcal{J}_-$ are defined as

$$\begin{aligned} \mathcal{J}_+ := &\begin{bmatrix} 0 & 0 & X_{\mathbf{T}}A_{\mathbf{TS}}^{-} \\ 0 & 0 & 0 \\ \left(A_{\mathbf{TS}}^{-}\right)^\top X_{\mathbf{T}} & 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} \left(A_{\mathbf{ST}}^{+}\right)^\top X_{\mathbf{S}}A_{\mathbf{ST}}^{+} & \left(A_{\mathbf{ST}}^{+}\right)^\top X_{\mathbf{S}}A_{\mathbf{SS},e}^{+} & \left(A_{\mathbf{ST}}^{+}\right)^\top X_{\mathbf{S}}A_{\mathbf{SS},v}^{+} \\ \left(A_{\mathbf{SS},e}^{+}\right)^\top X_{\mathbf{S}}A_{\mathbf{ST}}^{+} & \left(A_{\mathbf{SS},e}^{+}\right)^\top X_{\mathbf{S}}A_{\mathbf{SS},e}^{+} & \left(A_{\mathbf{SS},e}^{+}\right)^\top X_{\mathbf{S}}A_{\mathbf{SS},v}^{+} \\ \left(A_{\mathbf{SS},v}^{+}\right)^\top X_{\mathbf{S}}A_{\mathbf{ST}}^{+} & \left(A_{\mathbf{SS},v}^{+}\right)^\top X_{\mathbf{S}}A_{\mathbf{SS},e}^{+} & \left(A_{\mathbf{SS},v}^{+}\right)^\top X_{\mathbf{S}}A_{\mathbf{SS},v}^{+} \end{bmatrix} \\ \mathcal{J}_- := &\begin{bmatrix} \widehat{A}_{\mathbf{TT}} & 0 & X_{\mathbf{T}}A_{\mathbf{TS}}^{+} \\ 0 & 0 & 0 \\ \left(A_{\mathbf{TS}}^{+}\right)^\top X_{\mathbf{T}} & 0 & 0 \end{bmatrix} \\ &- \begin{bmatrix} \left(A_{\mathbf{ST}}^{-}\right)^\top X_{\mathbf{S}}A_{\mathbf{ST}}^{-} & \left(A_{\mathbf{ST}}^{-}\right)^\top X_{\mathbf{S}}A_{\mathbf{SS},e}^{-} & \left(A_{\mathbf{ST}}^{-}\right)^\top X_{\mathbf{S}}A_{\mathbf{SS},v}^{-} \\ \left(A_{\mathbf{SS},e}^{-}\right)^\top X_{\mathbf{S}}A_{\mathbf{ST}}^{-} & \left(A_{\mathbf{SS},e}^{-}\right)^\top X_{\mathbf{S}}A_{\mathbf{SS},e}^{-} & \left(A_{\mathbf{SS},e}^{-}\right)^\top X_{\mathbf{S}}A_{\mathbf{SS},v}^{-} \\ \left(A_{\mathbf{SS},v}^{-}\right)^\top X_{\mathbf{S}}A_{\mathbf{ST}}^{-} & \left(A_{\mathbf{SS},v}^{-}\right)^\top X_{\mathbf{S}}A_{\mathbf{SS},e}^{-} & \left(A_{\mathbf{SS},v}^{-}\right)^\top X_{\mathbf{S}}A_{\mathbf{SS},v}^{-} \end{bmatrix} \end{aligned}$$

with $\widehat{A}_{\mathbf{TT}} := A_{\mathbf{TT}}^\top X_{\mathbf{T}} + X_{\mathbf{T}}A_{\mathbf{TT}}$, where it is used that

$$\begin{aligned} &A_{\mathbf{TS}}^{+}A_{\mathbf{ST}}^{-} = A_{\mathbf{TS}}^{-}A_{\mathbf{ST}}^{+} = A_{\mathbf{TS}}^{+}A_{\mathbf{SS},e}^{-} = A_{\mathbf{TS}}^{-}A_{\mathbf{SS},e}^{+} = 0 \\ &A_{\mathbf{TS}}^{+}A_{\mathbf{SS},v}^{-} = A_{\mathbf{TS}}^{+} \quad \text{and} \quad A_{\mathbf{TS}}^{-}A_{\mathbf{SS},v}^{+} = A_{\mathbf{TS}}^{-}. \end{aligned} \tag{55}$$

Remark 7: The matrices $\mathcal{J}_+$ and $\mathcal{J}_-$ combined recover in essence the matrix obtained in the main result of [6, Th. 1] with $A_{\mathbf{SS}} = 0$ and $B_{\mathbf{S}} = 0$, only with the additional entries related to the error e, and the omission of the performance related entries (as $d = 0$ in the considered situation).

By defining the matrix $\mathcal{J} := \mathcal{J}_+ + \mathcal{J}_-$ resulting in (56) with

$$\begin{aligned} A^{\Delta}_{\mathbf{ST}} &= \left(A^{+}_{\mathbf{ST}}\right)^{\top} X_{\mathbf{S}} A^{+}_{\mathbf{ST}} - \left(A^{-}_{\mathbf{ST}}\right)^{\top} X_{\mathbf{S}} A^{-}_{\mathbf{ST}} \\ A^{\Delta}_{\mathbf{SS},ev} &= \left(A^{+}_{\mathbf{SS},e}\right)^{\top} X_{\mathbf{S}} A^{+}_{\mathbf{SS},v} - \left(A^{-}_{\mathbf{SS},e}\right)^{\top} X_{\mathbf{S}} A^{-}_{\mathbf{SS},v} \\ A^{\Delta}_{\mathbf{SS},v} &= \left(A^{+}_{\mathbf{SS},v}\right)^{\top} X_{\mathbf{S}} A^{+}_{\mathbf{SS},v} - \left(A^{-}_{\mathbf{SS},v}\right)^{\top} X_{\mathbf{S}} A^{-}_{\mathbf{SS},v} \\ A^{\Delta}_{\mathbf{SS},e} &= \left(A^{+}_{\mathbf{SS},e}\right)^{\top} X_{\mathbf{S}} A^{+}_{\mathbf{SS},e} - \left(A^{-}_{\mathbf{SS},e}\right)^{\top} X_{\mathbf{S}} A^{-}_{\mathbf{SS},e} \qquad (57) \\ A^{\Delta}_{\mathbf{ST},\mathbf{SS},v} &= \left(A^{+}_{\mathbf{ST}}\right)^{\top} X_{\mathbf{S}} A^{+}_{\mathbf{SS},v} - \left(A^{-}_{\mathbf{ST}}\right)^{\top} X_{\mathbf{S}} A^{-}_{\mathbf{SS},v} \\ A^{\Delta}_{\mathbf{ST},\mathbf{SS},e} &= \left(A^{+}_{\mathbf{ST}}\right)^{\top} X_{\mathbf{S}} A^{+}_{\mathbf{SS},e} - \left(A^{-}_{\mathbf{ST}}\right)^{\top} X_{\mathbf{S}} A^{-}_{\mathbf{SS},e} \end{aligned}$$

and using (54), the local condition (41) reduces to the local LMI

$$\mathcal{J} - J \preceq 0 \qquad (58)$$

which guarantees stability of the infinite interconnected spatially invariant linear system with networked communication, as argued before.

Hence, we proved the following theorem.

Theorem 5: Consider the overall system $\mathcal{H}$, composed of the identical subsystems $\mathcal{H}(s)$ of (50), $s \in \mathbb{Z}$, with associated $\mathbb{X}_0$ of (19) and $d = 0$. Assume there exist a function $W : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, a matrix $X_{\mathbf{T}} \in \mathcal{X}_{\mathbf{T}}$ and a matrix $X_{\mathbf{S}} \in \mathbb{R}^{(m_+ + m_-)\times(m_+ + m_-)}_{\mathbf{S}}$, and constants $\underline{\alpha}^c_W, \overline{\alpha}^c_W \in \mathbb{R}_{\geq 0}$, $M > 0$, $0 < \varepsilon < \gamma$, and $\lambda \in (0, 1)$ such that

1) for all $\bar{\kappa} \in \mathbb{N}$ and $\bar{e} \in \mathbb{R}^{m_e}$ (23) holds with $\underline{\alpha}_W(r) = \underline{\alpha}^c_W r$ and $\overline{\alpha}_W(r) = \overline{\alpha}^c_W r$,
2) for all $\bar{\kappa} \in \mathbb{N}$ and almost all $\bar{e} \in \mathbb{R}^{m_e}$ (31) holds,
3) and the LMI (58) holds with $\widehat{A}_{\mathbf{ST}} := A^{\top}_{\mathbf{ST}} A_{\mathbf{ST}}$ and (57).

If now τ_{mati} satisfies $\tau_{\text{mati}} \leq \frac{1}{\gamma} \arctan\left(\frac{(1+\lambda)(1-\lambda)}{2\lambda}\right)$, then the set $\mathcal{E}$ of (20) is UGES.

Note that (58) is a *local* LMI that can easily be verified to conclude stability of the *infinite-dimensional* system $\mathcal{H}$.

Since γ is the only free variable for the computation of the bound $\frac{1}{\gamma} \arctan\left(\frac{(1+\lambda)(1-\lambda)}{2\lambda}\right)$ for τ_{mati} as λ follows from the local scheduling protocol, τ_{mati} can be maximized by means of minimizing γ subject to $0 < \varepsilon < \gamma$ and the LMI (58) for the case $d = 0$.

C. LMI Condition for $\mathcal{L}_2$-Stability

Next, for the system (50)–(51) we derive for the case $d \neq 0$, in a similar manner as in the previous subsection, local LMI conditions to guarantee $\mathcal{L}_2$-stability. Consider hereto again the result of (49). Similar to (34) we have that, for the case $d \neq 0$, (42) is satisfied for $L = 0$ and

$$\begin{aligned} H\left(x(s), v(s), d(s)\right) = M \,| A_{\mathbf{ST}} A_{\mathbf{TT}} x(s) &+ A_{\mathbf{ST}} A_{\mathbf{TS}} v(s) \\ &+ A_{\mathbf{ST}} B_{\mathbf{T}} d(s) | . \end{aligned} \qquad (59)$$

Hence, as a result of (59) and the linear system (46), for (part of) the right-hand side of (45) it holds that

$$\begin{aligned} &\underline{\alpha}^{c\,2}_W \gamma^2 \left| e(s) \right|^2 - H^2\left(x(s), v(s), d(s)\right) \\ &\quad = \underline{\alpha}^{c\,2}_W \gamma^2 \left| e(s) \right|^2 - M^2 \left| A_{\mathbf{ST}} A_{\mathbf{TT}} x(s) + A_{\mathbf{ST}} A_{\mathbf{TS}} v(s) \right. \\ &\qquad \left. + A_{\mathbf{ST}} B_{\mathbf{T}} d(s) \right|^2 \\ &\quad = \left\langle (x(s), e(s), v(s), d(s)), J^{\mathcal{L}_2} (x(s), e(s), v(s), d(s)) \right\rangle \end{aligned}$$

where the "local" matrix $J^{\mathcal{L}_2}$ is given by (60).

Based on the expression for w of (48) for $d \neq 0$ it now follows that $q_+(s)$ and $q_-(s)$ are given by

$$\begin{aligned} q_+(s) &= \begin{bmatrix} A^{+}_{\mathbf{ST}} & A^{+}_{\mathbf{SS},e} & A^{+}_{\mathbf{SS},v} & 0 \end{bmatrix} (x(s), e(s), v(s), d(s)) \\ q_-(s) &= \begin{bmatrix} A^{-}_{\mathbf{ST}} & A^{-}_{\mathbf{SS},e} & A^{-}_{\mathbf{SS},v} & 0 \end{bmatrix} (x(s), e(s), v(s), d(s)). \end{aligned} \qquad (61)$$

Using this result, again direct substitution of $\mathcal{V}(x(s)) = x^{\top}(s) X_{\mathbf{T}} x(s)$, $X_{\mathbf{T}} \in \mathcal{X}_{\mathbf{T}}$, $(q_+(s), q_-(s))$, and S_i and S_p of (38) and (44), respectively, in (45) and using (46) yields

$$\begin{aligned} &\langle \nabla \mathcal{V}(x(s)), f(x, e, d)(s) \rangle - \mathcal{S}_p(d(s), z(s)) - \mathcal{S}_i(q_+(s), q_-(s)) \\ &= \left(A_{\mathbf{TT}} x(s) + A_{\mathbf{TS}} v(s) + B_{\mathbf{T}} d(s)\right)^{\top} X_{\mathbf{T}} x(s) \\ &\quad + x(s)^{\top} X_{\mathbf{T}} \left(A_{\mathbf{TT}} x(s) + A_{\mathbf{TS}} v(s) + B_{\mathbf{T}} d(s)\right) \\ &\quad - \mu\theta^2 d^{\top}(s) d(s) + \mu z^{\top}(s) z(s) + q^{\top}_+(s) X_{\mathbf{S}} q_+(s) \\ &\quad - q^{\top}_-(s) X_{\mathbf{S}} q_-(s) \\ &= \begin{bmatrix} x(s) \\ q_+(s) \\ z(s) \end{bmatrix}^{\top} \begin{bmatrix} 0 & X_{\mathbf{T}} A^{-}_{\mathbf{TS}} & 0 \\ \left(A^{-}_{\mathbf{TS}}\right)^{\top} X_{\mathbf{T}} & X_{\mathbf{S}} & 0 \\ 0 & 0 & \mu I_{m_z} \end{bmatrix} \begin{bmatrix} x(s) \\ q_+(s) \\ z(s) \end{bmatrix} \\ &\quad + \begin{bmatrix} x(s) \\ q_-(s) \\ d(s) \end{bmatrix}^{\top} \begin{bmatrix} \widehat{A}_{\mathbf{TT}} & X_{\mathbf{T}} A^{+}_{\mathbf{TS}} & X_{\mathbf{T}} B_{\mathbf{T}} \\ \left(A^{+}_{\mathbf{TS}}\right)^{\top} X_{\mathbf{T}} & -X_{\mathbf{S}} & 0 \\ B^{\top}_{\mathbf{T}} X_{\mathbf{T}} & 0 & -\mu\theta^2 I_{m_d} \end{bmatrix} \begin{bmatrix} x(s) \\ q_-(s) \\ d(s) \end{bmatrix} \end{aligned}$$

$$\mathcal{J} = \begin{bmatrix} A^{\top}_{\mathbf{TT}} X_{\mathbf{T}} + X_{\mathbf{T}} A_{\mathbf{TT}} + A^{\Delta}_{\mathbf{ST}} & A^{\Delta}_{\mathbf{ST},\mathbf{SS},e} & X_{\mathbf{T}} A_{\mathbf{TS}} + A^{\Delta}_{\mathbf{ST},\mathbf{SS},v} \\ \left(A^{\Delta}_{\mathbf{ST},\mathbf{SS},e}\right)^{\top} & A^{\Delta}_{\mathbf{SS},e} & A^{\Delta}_{\mathbf{SS},ev} \\ (A_{\mathbf{TS}})^{\top} X_{\mathbf{T}} + \left(A^{\Delta}_{\mathbf{ST},\mathbf{SS},v}\right)^{\top} & \left(A^{\Delta}_{\mathbf{SS},ev}\right)^{\top} & A^{\Delta}_{\mathbf{SS},v} \end{bmatrix} \qquad (56)$$

$$J^{\mathcal{L}_2} := -M^2 \begin{bmatrix} A^{\top}_{\mathbf{TT}} \widehat{A}_{\mathbf{ST}} A_{\mathbf{TT}} & 0 & A^{\top}_{\mathbf{TT}} \widehat{A}_{\mathbf{ST}} A_{\mathbf{TS}} & A^{\top}_{\mathbf{TT}} \widehat{A}_{\mathbf{ST}} B_{\mathbf{T}} \\ 0 & -\underline{\alpha}^{c\,2}_W M^{-2} \gamma^2 I_{m_+ + m_-} & 0 & 0 \\ A^{\top}_{\mathbf{TS}} \widehat{A}_{\mathbf{ST}} A_{\mathbf{TT}} & 0 & A^{\top}_{\mathbf{TS}} \widehat{A}_{\mathbf{ST}} A_{\mathbf{TS}} & A^{\top}_{\mathbf{TS}} \widehat{A}_{\mathbf{ST}} B_{\mathbf{T}} \\ B^{\top}_{\mathbf{T}} \widehat{A}_{\mathbf{ST}} A_{\mathbf{TT}} & 0 & B^{\top}_{\mathbf{T}} \widehat{A}_{\mathbf{ST}} A_{\mathbf{TS}} & B^{\top}_{\mathbf{T}} \widehat{A}_{\mathbf{ST}} B_{\mathbf{T}} \end{bmatrix}. \qquad (60)$$

$$\mathcal{J}^{\mathcal{L}_2} = \begin{bmatrix} \widehat{A}_{\mathbf{TT}} + \mu C_{\mathbf{T}}^\top C_{\mathbf{T}} + A_{\mathbf{ST}}^\Delta & A_{\mathbf{ST},\mathbf{SS},e}^\Delta & X_{\mathbf{T}} A_{\mathbf{TS}} + \mu C_{\mathbf{T}}^\top C_{\mathbf{S}} + A_{\mathbf{ST},\mathbf{SS},v}^\Delta & X_{\mathbf{T}} B_{\mathbf{T}} + \mu C_{\mathbf{T}}^\top D \\ \left(A_{\mathbf{ST},\mathbf{SS},e}^\Delta\right)^\top & A_{\mathbf{SS},e}^\Delta & A_{\mathbf{SS},ev}^\Delta & 0 \\ \left(A_{\mathbf{ST}}\right)^\top X_{\mathbf{T}} + \mu C_{\mathbf{S}}^\top C_{\mathbf{T}} + \left(A_{\mathbf{ST},\mathbf{SS},v}^\Delta\right)^\top & \left(A_{\mathbf{SS},ev}^\Delta\right)^\top & \mu C_{\mathbf{S}}^\top C_{\mathbf{S}} + A_{\mathbf{SS},v}^\Delta & \mu C_{\mathbf{S}}^\top D \\ B_{\mathbf{T}}^\top X_{\mathbf{T}} + \mu D^\top C_{\mathbf{T}} & 0 & \mu D^\top C_{\mathbf{S}} & -\mu\theta^2 I_{m_d} + \mu D^\top D \end{bmatrix}. \tag{62}$$

$$= \left\langle (x(s), e(s), v(s), d(s)), \mathcal{J}_+^{\mathcal{L}_2}(x(s), e(s), v(s), d(s)) \right\rangle$$
$$+ \left\langle (x(s), e(s), v(s), d(s)), \mathcal{J}_-^{\mathcal{L}_2}(x(s), e(s), v(s), d(s)) \right\rangle$$

with $\widehat{A}_{\mathbf{TT}} = A_{\mathbf{TT}}^\top X_{\mathbf{T}} + X_{\mathbf{T}} A_{\mathbf{TT}}$ and where, by using the coordinate transformation of (61), the matrices $\mathcal{J}_+^{\mathcal{L}_2}$ and $\mathcal{J}_-^{\mathcal{L}_2}$ are defined as

$$\mathcal{J}_+^{\mathcal{L}_2} := \begin{bmatrix} I & 0 & 0 & 0 \\ A_{\mathbf{ST}}^+ & A_{\mathbf{SS},e}^+ & A_{\mathbf{SS},v}^+ & 0 \\ C_{\mathbf{T}} & 0 & C_{\mathbf{S}} & D \end{bmatrix}^\top \begin{bmatrix} 0 & X_{\mathbf{T}} A_{\mathbf{TS}}^- & 0 \\ (A_{\mathbf{TS}}^-)^\top X_{\mathbf{T}} & X_{\mathbf{S}} & 0 \\ 0 & 0 & \mu I \end{bmatrix}$$
$$\times \begin{bmatrix} I & 0 & 0 & 0 \\ A_{\mathbf{ST}}^+ & A_{\mathbf{SS},e}^+ & A_{\mathbf{SS},v}^+ & 0 \\ C_{\mathbf{T}} & 0 & C_{\mathbf{S}} & D \end{bmatrix}$$

$$\mathcal{J}_-^{\mathcal{L}_2} := \begin{bmatrix} I & 0 & 0 & 0 \\ A_{\mathbf{ST}}^- & A_{\mathbf{SS},e}^- & A_{\mathbf{SS},v}^- & 0 \\ 0 & 0 & 0 & I \end{bmatrix}^\top \begin{bmatrix} \widehat{A}_{\mathbf{TT}} & X_{\mathbf{T}} A_{\mathbf{TS}}^+ & X_{\mathbf{T}} B_{\mathbf{T}} \\ (A_{\mathbf{TS}}^+)^\top X_{\mathbf{T}} & -X_{\mathbf{S}} & 0 \\ B_{\mathbf{T}}^\top X_{\mathbf{T}} & 0 & -\mu\theta^2 I \end{bmatrix}$$
$$\times \begin{bmatrix} I & 0 & 0 & 0 \\ A_{\mathbf{ST}}^- & A_{\mathbf{SS},e}^- & A_{\mathbf{SS},v}^- & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Remark 8: The matrices $\mathcal{J}_+^{\mathcal{L}_2}$ and $\mathcal{J}_-^{\mathcal{L}_2}$ combined recover the matrix obtained in the main result of [6, Th. 1] for $\mu = \theta = 1$ (and $A_{\mathbf{SS}} = 0$ and $B_{\mathbf{S}} = 0$, see Remark 1), only with the additional entries related to the error.

By again using that (55) holds, we define the matrix $\mathcal{J}^{\mathcal{L}_2} := \mathcal{J}_+^{\mathcal{L}_2} + \mathcal{J}_-^{\mathcal{L}_2}$, which is given by (62) with (57), such that condition (45) can be reformulated into the LMI given by

$$\mathcal{J}^{\mathcal{L}_2} - J^{\mathcal{L}_2} \preceq 0. \tag{63}$$

Hence, Theorem 4 can also be reformulated into LMI conditions, resulting in the following new theorem.

Theorem 6: Consider the overall system $\mathcal{H}$, composed of the identical subsystems $\mathcal{H}(s)$ of (50)–(51) $s \in \mathbb{Z}$, with associated $\mathbb{X}_0$ of (19) and $d \in \mathcal{L}_2$. Assume there exist a function $W : \mathbb{N} \times \mathbb{R}^{m_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument, a matrix $X_{\mathbf{T}} \in \mathcal{X}_{\mathbf{T}}$, and a matrix $X_{\mathbf{S}} \in \mathbb{R}_{\mathbf{S}}^{(m_+ + m_-)\times(m_+ + m_-)}$, and constants $\underline{\alpha}_W^c, \overline{\alpha}_W^c, \theta \in \mathbb{R}_{\geq 0}$, $M, \gamma, \mu > 0$, $\lambda \in (0,1)$ such that

1) for all $\bar{\kappa} \in \mathbb{N}$ and $\bar{e} \in \mathbb{R}^{m_e}$ (23) holds with $\underline{\alpha}_W(r) = \underline{\alpha}_W^c r$ and $\overline{\alpha}_W(r) = \overline{\alpha}_W^c r$,
2) for all $\bar{\kappa} \in \mathbb{N}$ and almost all $\bar{e} \in \mathbb{R}^{m_e}$ (31) holds,
3) and the LMI (63) holds.

If now τ_{mati} satisfies $\tau_{\text{mati}} \leq \frac{1}{\gamma} \arctan\left(\frac{(1+\lambda)(1-\lambda)}{2\lambda}\right)$, then the overall system $\mathcal{H}$ is guaranteed to be $\mathcal{L}_2$-stable from d to z with a gain less than or equal to θ.

Similar to the situation where $d = 0$, the bound on τ_{mati} can be maximized by minimizing γ subject to the LMI (63).

VII. Examples

Due to space limitations we cannot provide full numerical examples here. However, in [12], we have already shown the effectiveness of our results in the linear case by considering the example of a platoon of vehicles, which can be captured in the system configuration S1 of Fig. 2. Based on only one vehicle, we analyzed UGES for the overall interconnected string of vehicles.

Moreover, additional examples concerning $\mathcal{L}_p$-stability and nonlinear dynamics using the framework developed in this paper can be found in [31], [32].

VIII. Concluding Remarks

In this paper, a novel approach was presented for the stability analysis of systems consisting of an infinite number of spatially invariant interconnected subsystems where packet-based (wireless) communication networks are part of the system configuration. Based on a model consisting of an infinite interconnection of spatially invariant hybrid systems, and for which a new solution concept had to be defined relevant for this NCS context, a bound on the MATI was derived for all the (local) communication networks such that uniform global asymptotic (or exponential) stability or $\mathcal{L}_p$-stability for the overall system is guaranteed. Moreover, these conditions were reformulated using only local information of the subsystem, the interconnection structure, and the adopted scheduling protocol. As a consequence, UGES or $\mathcal{L}_p$-stability for the overall infinite-dimensional system was obtained based only on local properties leading to easy-to-verify conditions, being LMI conditions in the linear case (and $\mathcal{L}_2$-case).

The results presented in this paper also have many natural extensions and applications, including known extensions from NCS-literature such as delays or event-triggered mechanisms, see [16], [29], while, based on the results of [6], [9], it can readily be shown that the composed theorems also hold for different interconnection types such as periodic interconnections and finite interconnections with boundary conditions, as well as interconnections with more spatial variables.

Appendix

Proof of Theorem 1: The proof is based on infinite-dimensional extensions of Lyapunov-based arguments for hybrid systems as stated in [22, Th. 3.18], which also have to be adapted to accommodate the solution concept (which allows

Zeno points). Therefore, the proof will be based on constructing a function $U : \ell_2^{1,m_0} \times \ell_2^{1,m_e} \times \ell^{1,1} \times \ell^{1,1} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its first and second argument, functions $\underline{\alpha}_U, \overline{\alpha}_U \in \mathcal{K}_\infty$ and a positive definite function ϱ such that

$$\underline{\alpha}_U\left(\|\xi_c\|_{\ell_2}\right) \leq U(\xi) \leq \overline{\alpha}_U\left(\|\xi_c\|_{\ell_2}\right) \tag{64a}$$

$$U(\xi^+) - U(\xi) \leq 0 \tag{64b}$$

$$\text{when } \tau(s) \in [\delta, \infty) \text{ for some } s \in \mathbb{Z}$$

$$\langle \nabla U(\xi), F(\xi)\rangle_{\ell_2} \leq -\varrho(\|\xi_c\|_{\ell_2}) \tag{64c}$$

$$\text{when } \tau(s) \in [0, \tau_{\text{mati}}] \text{ for all } s \in \mathbb{Z}$$

with $F(\xi) = (f(x,e), g(x,e), 1, 0)$, $\xi = (\xi_c, \xi_d)$, $\xi_c = (x, e) \in \ell_2^{1,m_0} \times \ell_2^{1,m_+ + m_-}$, and $\xi_d = (\tau, \kappa) \in [0, \tau_{\text{mati}})^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}}$ and where ξ^+ is given by $\xi^+(\tau, s) = \xi(\tau^+, s) = \lim_{t\downarrow\tau, t\notin\mathcal{R}} \xi(t,s)$ for $s \in \mathbb{Z}$ as in Definition 4. These conditions are sufficient to guarantee UGAS of the set $\mathcal{E}$ of (20). If, in addition, there exist $\underline{\alpha}_U^c, \overline{\alpha}_U^c, \varepsilon \in \mathbb{R}_{\geq 0}$ such that $\underline{\alpha}_U^c \|\xi_c\|_{\ell_2}^2 \leq U(x) \leq \overline{\alpha}_U^c \|\xi_c\|_{\ell_2}^2$, and $\varrho(r) \geq \varepsilon^2 r^2$, $r \in \mathbb{R}_{>0}$, then the set $\mathcal{E}$ of (20) is UGES. Of course, this reasoning can only be used if we can also show that each maximal solution is complete.

To show that the conditions of Theorem 1 indeed can be used to obtain a Lyapunov function U such that the conditions in (70) are satisfied, we consider the function $\phi : [0, \tau_{\text{mati}}] \times \mathbb{Z} \to \mathbb{R}_{\geq 0}$, which evolves for every fixed $s \in \mathbb{Z}$ according to

$$\frac{\mathrm{d}}{\mathrm{d}\tau(s)}\phi(\tau(s), s) = -2L\phi(\tau(s), s) - \gamma\left(\phi^2(\tau(s), s) + 1\right)$$

for $\tau(s) \in [0, \tau_{\text{mati}}]$. As shown in [15], when choosing the initial condition to be $\phi(0, s) = \lambda^{-1}$, the function satisfies $\phi(\tau_{\text{mati}}, s) = \lambda$ with τ_{mati} as defined in (27), and $\phi(\tau(s), s) \in [\lambda, \lambda^{-1}]$ for all $\tau(s) \in [0, \tau_{\text{mati}}]$ and every fixed $s \in \mathbb{Z}$. Using this function ϕ, we now define the candidate Lyapunov function as

$$U(\xi) := V(x) + \sum_{s\in\mathbb{Z}} \gamma\phi(\tau(s), s) W^2(\kappa(s), e(s)). \tag{65}$$

Since $\phi(\tau(s), s) \geq \lambda > 0$ for every fixed $s \in \mathbb{Z}$ and all $\tau(s) \in [0, \tau_{\text{mati}}]$, and we have that (23a) holds with $\bar{\kappa} = \kappa(s)$ and $\bar{e} = e(s)$ for every fixed $s \in \mathbb{Z}$, and (25) holds, it can be concluded that there exist functions $\underline{\alpha}_U, \overline{\alpha}_U \in \mathcal{K}$ such that (70a) holds.

To prove that (70b) holds under the conditions proposed in Theorem 1, consider the situation that (only) the subsystem $\mathcal{H}(\hat{s})$ for $\hat{s} \in \mathbb{Z}$ jumps, implying that $\tau^+(\hat{s}) = 0$. This gives

$$\begin{aligned}
U(\xi^+) &= V(x^+) + \sum_{s\in\mathbb{Z}} \gamma\phi(\tau^+(s), s) W^2(\kappa^+(s), e^+(s)) \\
&= V(x) + \sum_{s\in\mathbb{Z}, s\neq\hat{s}} \left(\gamma\phi(\tau(s), s) W^2(\kappa(s), e(s))\right) \\
&\quad + \gamma\phi(0, \hat{s}) W^2\left(\kappa(\hat{s}) + 1, h\left(\kappa(\hat{s}), e(\hat{s})\right)\right) \\
&\leq V(x) + \sum_{s\in\mathbb{Z}, s\neq\hat{s}} \left(\gamma\phi(\tau(s), s) W^2(\kappa(s), e(s))\right) \\
&\quad + \gamma\lambda W(\kappa(\hat{s}), e(\hat{s})) \leq U(\xi)
\end{aligned}$$

where for the latter inequality we used (23b) with $\bar{\kappa} = \kappa(s)$ and $\bar{e} = e(s)$ for every fixed $s \in \mathbb{Z}$, and $\phi(0, \hat{s}) = \lambda^{-1}$. Obviously the same analysis holds when multiple subsystems jump at the same event time, and hence, (70b) holds.

Finally, we will show that (70c) holds. Note that, as W is not differentiable with respect to κ, but as the component in $F(\xi)$ corresponding to κ is zero, $\langle \nabla U(\xi), F(\xi)\rangle$ can still be evaluated with a slight abuse of notation. For all (τ, κ) and almost all (x, e) it holds that

$$\begin{aligned}
&\langle \nabla U(\xi), F(\xi)\rangle_{\ell_2} \\
&= \langle \nabla V(x), f(x,e)\rangle_{\ell_2} + \sum_{s\in\mathbb{Z}} \left[\gamma\dot{\phi}(\tau(s), s) W^2(\kappa(s), e(s))\right] \\
&\quad + \sum_{s\in\mathbb{Z}} \Bigg[2\gamma\phi(\tau(s), s) W(\kappa(s), e(s)) \\
&\qquad \left\langle \frac{\partial W(\kappa(s), e(s))}{\partial e(s)}, g(x,e)(s)\right\rangle\Bigg] \\
&\overset{(24),(26)}{\leq} \sum_{s\in\mathbb{Z}} \big(-\varrho\left(W(\kappa(s), e(s))\right) - H^2(x(s), v(s)) \\
&\quad + \gamma^2 W^2(\kappa(s), e(s))\big) \\
&\quad - \sum_{s\in\mathbb{Z}} \Big(\left(2\gamma L\phi(\tau(s), s) + \gamma^2\left(\phi^2(\tau(s), s) + 1\right)\right) \\
&\qquad W^2(\kappa(s), e(s))\Big) \\
&\quad + \sum_{s\in\mathbb{Z}} \big(W\left(\kappa\left(s\right), e\left(s\right)\right)\left(LW\left(\kappa\left(s\right), e\left(s\right)\right)\right. \\
&\qquad \left. + H\left(x(s), v(s)\right)\right) \\
&\quad \cdot 2\gamma\phi\left(\tau\left(s\right), s\right)\big) - \varrho\left(\|x\|_{\ell_2}\right) \\
&= -\varrho\left(\|x\|_{\ell_2}\right) - \sum_{s\in\mathbb{Z}} \varrho\left(W(\kappa(s), e(s))\right) \\
&\quad - \sum_{s\in\mathbb{Z}} \left(H^2(x(s), v(s)) + \gamma^2\phi^2(\tau(s), s) W^2(\kappa(s), e(s))\right) \\
&\quad - 2\gamma\phi(\tau(s), s) W(\kappa(s), e(s)) H(x(s), v(s)) \\
&\leq -\varrho\left(\|x\|_{\ell_2}\right) - \sum_{s\in\mathbb{Z}} \varrho\left(W(\kappa(s), e(s))\right)
\end{aligned}$$

which gives (70c). Note that we here used that (24) holds for all $s \in \mathbb{Z}$ as a result of Remark 4.

The above reasoning applies to any maximal solution with initial state $\xi_0 \in \mathbb{X}_0$ defined on $[0, T)$. However, to complete the proof, we still have to establish that for any $\xi_0 \in \mathbb{X}_0$ there exists at least one solution on $[0, T)$ with $T > 0$ and moreover that any maximal solution is complete, i.e., $T = \infty$, as this is part of Definition 7. Note that if we prove these properties, then the conditions in (70) are indeed sufficient to establish UGAS or UGES of the set $\mathcal{E}$.

We will start by proving completeness of any maximal solution. A part of the arguments in this proof can also be used to also establish existence of solutions. To show the completeness of $(\xi, \mathcal{R})$, we will proceed by contradiction by assuming that $T < \infty$. In fact, we will show that we can prolong the solution $(\xi, \mathcal{R})$ and get a solution $(\xi', \mathcal{R}')$ on $[0, T')$ with $T' > T$ that satisfies the points 1) and 2) in Definition 5. This would contradict the maximality of the solution $(\xi, \mathcal{R})$ and, thus, T must be infinite.

Observe that for $T < \infty$, each subsystem in the interconnection can jump at most one time in the time period $[T - \delta, T)$ since, as imposed by (8), for each (hybrid) subsystem two consecutive jumps are always at least $\delta > 0$ time units apart. Hence, we know that for each $s \in \mathbb{Z}$ there exists a $t^s \in [T - \delta, T)$ such that $\xi_c(t, s)$ is absolutely continuous on (t^s, T) (this follows from Definition 4 point 3 and the fact that for each subsystem $\mathcal{H}(s)$ no jumps take place in the interval (t^s, T) implying that the solution $\xi_c(s)$ on (t^s, T) consists of the continuous concatenation of absolute continuous pieces that are bounded, see (72) below). In addition, based on the bounds (70b) and (70c), we can also conclude that

$$\underline{\alpha}_U \left(\left\| [\, \xi_c \,\right\|_{\ell_2} \right) \leq U(\xi(t)) \leq U(\xi(0)) < \infty \qquad (66)$$

for all $t \in [0, T) \backslash \mathcal{R}$, implying that there is no finite escape time for the solutions ξ_c. Hence, we have that $\xi_c(t) \in \mathcal{B}$ for all $t \in [0, T)$, where $\mathcal{B}$ is some bounded set in $\ell_2^{1, m_0 + m_e}$.

As a result of these absolutely continuity and boundedness properties of the solutions ξ_c, it follows from a standard result in mathematical analysis [33, Ex. 4.13] that the left-limit $\xi_c(T^-, s) := \lim_{t \uparrow T} \xi_c(t, s)$ exists for each $s \in \mathbb{Z}$. Clearly, also $\tau(T^-, s) := \lim_{t \uparrow T} \tau(t, s)$ and $\kappa(T^-, s) := \lim_{t \uparrow T} \kappa(t, s)$ are well defined for each individual $s \in \mathbb{Z}$, and, hence, we can thus already define the solution ξ for all $t \in [0, T]$ (so on the closed interval). In addition, we have that $\xi_c(T^-) \in \mathcal{B} \subseteq \ell_2^{1, m_0 + m_e}$ as a result of (72). Based on these properties, we can now construct a solution $(\xi', \mathcal{R}')$ that prolongs the considered maximal solution $(\xi, \mathcal{R})$. Consider hereto two possibilities for each subsystem. If $\tau(T^-, s) \geq \delta$ for $s \in \mathbb{Z}$, then, based on Definition 4 point 2, we can define a solution with a reset at time T for this subsystem at s (note that $\xi(T^-, s) \in \mathcal{D}$ in this case). If $\tau(T^-, s) < \delta$, then we define the solution without a reset at time T for the corresponding subsystem at s. Hence, we have $\xi(T^+, s) \in G\left(\xi(T^-, s)\right)$ for $s \in S_r(T)$ and $\xi(T^+, s) = \xi(T^-, s)$ for $s \notin S_r(T)$ where $S_r(T) = \{s \in \mathbb{Z} \mid \tau(T^-, s) \geq \delta\}$. Due to $\tilde{G}$ mapping $\mathbb{X}_0$ to $\mathbb{X}_0$, it follows that $\xi(T^+) \in \mathbb{X}_0$. Note now that we have for all subsystems that $\tau(T^+, s) < \delta$. As a result, for the state $\xi(T^+)$ there exists an $\varepsilon > 0$ and an absolutely continuous $\bar{\xi} : [T, T + \varepsilon) \to \ell_2^{1, m_1} \times \ell^{1, m_2}$ (following from the smoothness condition on F_c) such that $\dot{\bar{\xi}}(t) = F(\bar{\xi}(t), 0)$ and $\bar{\xi}(T) = \xi(T^+)$ for almost all $t \in [T, T + \varepsilon)$. In particular, since $\tau(T^+, s) < \delta$ for all $s \in \mathbb{Z}$, it holds that $\xi(t, s) \in \mathcal{C}$ for all $t \in [T, T + \Delta T)$ and $s \in \mathbb{Z}$, where $\Delta T := \min\left(\varepsilon, \tau_{\text{mati}} - \delta\right) > 0$. Hence, this indicates that $(\xi', \mathcal{R}')$ with $\mathcal{R}' := \mathcal{R} \cup \{T\}$ and

$$\xi'(t) = \begin{cases} \xi(t) & \text{for all } t \in [0, T) \backslash \mathcal{R}' \\ \bar{\xi}(t) & \text{for all } t \in [T, T + \Delta T) \backslash \mathcal{R}' \end{cases}$$

is a solution to $\mathcal{H}$ on $[0, T + \Delta T)$ satisfying points 1 and 2 in Definition 5. This contradicts the maximality of the solution $(\xi, \mathcal{R})$ and thus $T = \infty$.

To establish the existence of a (nontrivial) solution ξ for any $\xi_0 \in \mathbb{X}_0$, the same arguments as used above for prolonging the solution $(\xi, \mathcal{R})$ from the state $\xi(T^-)$ can be used for the initial state $\xi_0 \in \mathbb{X}_0$. This completes the proof. ■

Proof of Theorem 2: Consider again the candidate Lyapunov function as given in (71). Following the same steps as in the proof of Theorem 1, one can conclude that (70a) and (70b) hold, and during flows, i.e., when $\tau(s) \in [0, \tau_{\text{mati}}]$ for all $s \in \mathbb{Z}$, one can obtain that

$$\langle \nabla U(\xi), F(\xi, d) \rangle_{\ell_2} \leq \mu \left(\theta^p \left\| d \right\|_{\ell_2}^p - \left\| q(x, e, d) \right\|_{\ell_2}^p \right) \qquad (67)$$

where $F(\xi, d) = (f(x, e, d), g(x, e, d), 1, 0)$. Combining integrated versions of (73) with (70b) on the interval $[0, \mathcal{T}]$ with $0 \leq \mathcal{T} < T$ and assuming that the considered maximal solution is defined on at least $[0, T)$ yields

$$U\left(\xi(\mathcal{T})\right) - U\left(\xi(0)\right) \leq \mu \int_0^{\mathcal{T}} \left(\theta^p \left\| d \right\|_{\ell_2}^p - \left\| Q(\xi_c, d) \right\|_{\ell_2}^p \right) \mathrm{d}t$$

with $Q(\xi_c, d) = q(x, e, d)$. Hence, recalling (10), we have for all $0 \leq \mathcal{T} < T$ that

$$U(\xi(\mathcal{T})) \leq U\left(\xi(0)\right) + \mu \int_0^{\mathcal{T}} \left(\theta^p \left\| d \right\|_{\ell_2}^p - \left\| z \right\|_{\ell_2}^p \right) \mathrm{d}t. \qquad (68)$$

As a result of this observation, we obtain the bound

$$U\left(\xi(\mathcal{T})\right) \leq U\left(\xi(0)\right) + \mu \theta^p \int_0^{\mathcal{T}} \left\| d \right\|_{\ell_2}^p \mathrm{d}t$$

for all $\mathcal{T} \in [0, T)$.

Using now similar arguments as in the proof of Theorem 1, it can be proven that at least one solution exists and that all maximal solutions are complete solutions, i.e., $T = \infty$. Now using that $U(\xi(\mathcal{T})) \geq 0$ for all $\mathcal{T} \in \mathbb{R}_{\geq 0}$, letting $\mathcal{T} \to \infty$ in (74), and computing the $\mathcal{L}_p$-norm according to Definition 2 yield

$$\begin{aligned} \left\| z \right\|_{\mathcal{L}_p}^p &= \int_0^\infty \left\| z \right\|_{\ell_2}^p \mathrm{d}t \leq \int_0^\infty \theta^p \left\| d \right\|_{\ell_2}^p \mathrm{d}t + \frac{1}{\mu} U\left(\xi(0)\right) \\ &= \theta^p \left\| d \right\|_{\mathcal{L}_p}^p + \frac{1}{\mu} U\left(\xi(0)\right) \\ &\leq \left(\left(\frac{1}{\mu} U\left(\xi\left(0\right)\right) \right)^{\frac{1}{p}} + \theta \left\| d \right\|_{\mathcal{L}_p} \right)^p. \end{aligned} \qquad (69)$$

Since $(U(\xi)/\mu)^{1/p} \leq \beta_U\left(\left\| \xi_c \right\|_{\ell_2}\right)$ for some $\beta_U \in \mathcal{K}_\infty$, (75) leads to (22) thereby completing the proof. ■

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Stefan H. J. Heijmans was born in Weert, The Netherlands, in 1992. He received the M.Sc. degree (*summa cum laude*) in mechanical engineering from the Eindhoven University of Technology (TU/e), Eindhoven, The Netherlands, in August 2015 where he is currently working toward the Ph.D. degree.

His research interests include modeling and control of hybrid systems and analysis of large scale distributed and networked systems.

Dominicus Paulus Borgers (S'14) was born in Eindhoven, The Netherlands, in 1988. He received the M.Sc. degree (*summa cum laude*) in mechanical engineering from TU/e, Eindhoven, The Netherlands, in 2013, where he is currently working toward the Ph.D. degree.

His research interests include hybrid systems, networked control systems, and event-triggered control.

W. P. M. H. Heemels (M'07–SM'10–F'16) received the M.Sc. degree in mathematics and the Ph.D. degree in control theory (both *summa cum laude*) from the Eindhoven University of Technology (TU/e), Eindhoven, The Netherlands, in 1995 and 1999, respectively.

After being an Assistant Professor at the Electrical Engineering Department, TU/e, and a Research Fellow at the Embedded Systems Institute, he is currently a Full Professor in the Control Systems Technology Group, Mechanical Engineering Department, TU/e. He held visiting research positions at the Swiss Federal Institute of Technology, Zurich, Switzerland, in 2001, at Océ, Venlo, The Netherlands, in 2004, and at the University of California at Santa Barbara, USA, in 2008. His current research interests include general control theory, hybrid and cyber-physical systems, networked and event-triggered control, and constrained systems including model predictive control. He received the VICI grant of the Dutch Technology Foundation and the Netherlands Organisation for Scientific Research. In addition, he serves/d on the editorial boards of Automatica, NAHS, TAC, and Annual reviews on Control.