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Solution Concepts and Analysis of Spatially Invariant Hybrid Systems: Exploring Zeno and beyond ⋆

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Abstract: In this paper we consider hybrid systems, consisting of an *infinite* number of interconnected spatially invariant (identical) hybrid subsystems described using the hybrid inclusions framework. It can be shown that such interconnections can be very useful in, e.g., the modeling of interconnected networked subsystems that use packet-based communication for the exchange of information, including autonomously driving platoons of vehicles. Interestingly however, due to the infinite-dimensionality of the overall interconnected hybrid system, establishing proper definitions of solutions becomes a difficult task as standard solution concepts do not apply to the systems under study since Zeno behavior (an infinite number of jumps in a bounded time interval) is inevitable. Therefore, we introduce an alternative and natural solution concept for this class of systems, allowing solutions to be defined beyond Zeno points. In addition, based on this novel solution concept, we derive Lyapunov-based conditions for a specific, but relevant class of infinite-dimensional hybrid systems, as used for modeling, for instance, networked control systems, that guarantee UGAS (or UGES) or $\mathcal{L}_p$-stability of the overall interconnected system.

Keywords: Hybrid dynamical systems, spatially invariant systems, infinite-dimensionality.

1. INTRODUCTION

Hybrid systems theory studies the behavior of dynamical systems that exhibit characteristics of both continuous- and discrete-time dynamical systems. A specific class of hybrid systems that is often considered consists of so-called hybrid inclusions as described in Goebel et al. (2012), as they naturally apply to many dynamical systems, such as certain biological and medical systems, see Aihara and Suzuki (2010), and networked control systems where packet-based communication networks are used for the exchange of information, see Walsh et al. (2002) or Nešić and Teel (2004). The latter ones are often even modeled as an interconnection of such hybrid inclusions based on their large-scale interconnection structure. Such an approach has already been proven to be very effective for the stability and performance analysis, see e.g., Borgers and Heemels (2014) or Dolk et al. (2017).

However, any stability or performance analysis requires the establishment of a proper definition of solutions for the considered interconnections of hybrid systems, which has been proven to be not an easy task, as evidenced by the studies in Sanfelice (2011) and Dashkovskiy and Kosmykov (2013). This becomes even more complicated when the interconnection is composed of an *infinite* number of (identical) hybrid subsystems as was the case in Heijmans et al. (2015) in which it was studied how the stability analysis of the *infinite* interconnections of spatially invariant systems considered in D'Andrea and Dullerud (2003) could be extended from ideal (continuous) communication towards non-ideal packet-based communication. As shown in D'Andrea and Dullerud (2003), Langbort and D'Andrea (2005), and Heijmans et al. (2015), such infinite approximations are very useful for the analysis of large-scale systems. In particular, based on this infinite-dimensional framework, instead of using global monolitical models, *local* conditions can be obtained based only on the information of *one* of the subsystems in the interconnection to analyze stability or performance. In addition, as shown in Langbort and D'Andrea (2005), an infinite approximation may be adequate to analyze interconnections with a large number of subsystems, while its properties are also inherited by periodic or finite interconnections with boundary conditions. An example of an infinite-dimensional system with non-ideal communication is the autonomously driving platoon of vehicles as in Fig. 1, where packet-based communication networks are used for the exchange of velocity, acceleration, and jerk data.

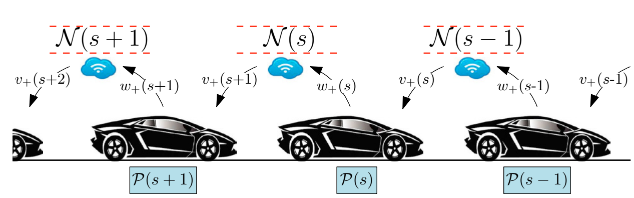

Fig. 1. An infinite string of vehicles interconnected through packet-based communication networks $\mathcal{N}(s)$, $s \in \mathbb{Z}$.

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In these infinite-dimensional "networked" systems, it is often assumed that all communication networks operate independently and asynchronously of each other as each of them has its own set of transmission times at which an update of the networked values occurs. Moreover, the transmission interval (the time elapse between two consecutive transmission times) is in many cases upperbounded by a maximally allowable transmission interval (MATI), see also Walsh et al. (2002) and Nešić and Teel (2004). As a result of this setup, each combination of a subsystem $\mathcal{P}(s)$ and its network $\mathcal{N}(s)$ as in Fig. 1 can be modeled as a hybrid system $\mathcal{H}(s)$, $s \in \mathbb{Z}$, resulting in the infinite interconnection of hybrid subsystems for the overall system, where each update of the networked values corresponds to a jump of a hybrid subsystem $\mathcal{H}(s)$ in this interconnection, see Heijmans et al. (2015) for more details.

Other examples that can (possible) be modeled as infinite interconnection of hybrid systems can been found in, e.g., the medical and biological applications. In particular, one can for instance use the modeling framework to approximate the classical example of the synchronization of an enormous number of fireflies or for the modeling of cellular processes and biological signalling networks, to overcome the various difficulties that the large amount of subsystems introduce, see, e.g. Machado et al. (2011) and Ghosh and Tomlin (2004).

Interestingly however, when we want to define solutions for interconnections of hybrid systems for all times $t \in [0, \infty)$, in general, hybrid solution concepts as in the literature do not apply to many *infinite*-dimensional systems, including the ones described in Heijmans et al. (2015) or as in the setup above. This is because Zeno behavior (an infinite number of jumps in a finite time interval) is inevitable for these infinite interconnections due to the infinite number of subsystems with an upper bound on the time elapse in between two consecutive jumps. Clearly, as many analysis techniques (including Lyapunov) rely on arguments about the system behavior along an execution, we would like to define the solutions beyond such Zeno points, which is not directly possible for many available classical solution concepts, including the ones proposed in Lygeros et al. (2003) or Goebel et al. (2012). As a result, appropriate extensions are needed. In, for instance, Johansson et al. (1999) and Zheng et al. (2006), such extensions of the solutions beyond Zeno points have already been investigated for *finite*-dimensional (hybrid) systems by using regularization techniques. Unfortunately, such techniques require physical knowledge of the systems under study as the solutions depend on the choices of regularizations and are primarily only for the purpose of simulation.

Therefore, inspired by Heemels et al. (2000), in this paper we will introduce a new solution concept in this context, allowing solutions to be defined beyond Zeno points in the sense that solutions can be defined globally on $\mathbb{R}_{\geq 0}$, i.e., for all times $t \in [0, \infty)$. In addition, we will show how this solution concept can be used to analyze UGAS or $\mathcal{L}_p$-stability for the overall system and we will provide Lyapunov-based conditions to guarantee both properties for a specific class of interconnected hybrid systems that is used for modeling, for instance, the above described "networked" systems including the platoon of vehicles of Fig. 1.

2. PRELIMINARIES

The set of non-negative real numbers is denoted by $\mathbb{R}_{\geq 0}$. For vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$, we denote by $(v_1, v_2, \ldots, v_n)$ the vector $[v_1^\top, v_2^\top, \ldots, v_n^\top]^\top$. By $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and the usual inner product are denoted in $\mathbb{R}^n$.

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and, in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each fixed s, the mapping $r \mapsto \beta(r, s)$ belongs to class $\mathcal{K}$ and for each fixed r, the mapping $s \mapsto \beta(r, s)$ is decreasing and $\beta(r, s) \to 0$ when $s \to \infty$. Moreover, the function β is said to be of class exp-$\mathcal{KL}$ if there exist $K, c > 0$, such that $\beta(r, s) = Kr \exp(-cs)$ for all $r, s \in \mathbb{R}_{\geq 0}$. Given a Banach space X, a function $f : X \to X$ is said to be locally Lipschitz continuous if for each $x_0 \in X$ there exists constants $\delta, L > 0$ such that for all $x \in X$ we have that $\|x - x_0\|_X \leq \delta \Rightarrow \|f(x) - f(x_0)\|_X \leq L \|x - x_0\|_X$, where $\|\cdot\|_X$ denotes the norm in X, see also Robinson (2001).

In this paper, the state-space of the considered systems is infinite-dimensional, as we will see below. Therefore, we recall some definitions from D'Andrea and Dullerud (2003). Since the signals are often considered at a fixed time, it is convenient to separate the spatial and the temporal parts of a signal.

Definition 1. The space $\ell^{L,n}$ is the set of functions mapping $\mathbb{Z}^L$ to $\mathbb{R}^n$. The space $\ell_2^{L,n}$ is the set of functions $x \in \ell^{L,n}$ for which

$$\sum_{s_1 \in \mathbb{Z}} \cdots \sum_{s_L \in \mathbb{Z}} x(\mathbf{s})^\top x(\mathbf{s}) < \infty$$

holds equipped with the inner product $\langle \cdot, \cdot \rangle_{\ell_2}$ for $x, y \in \ell_2^{L,n}$ defined as

$$\langle x, y \rangle_{\ell_2} := \sum_{s_1 \in \mathbb{Z}} \cdots \sum_{s_L \in \mathbb{Z}} x(\mathbf{s})^\top y(\mathbf{s}),$$

and the corresponding norm as $\|x\|_{\ell_2} := \sqrt{\langle x, x \rangle_{\ell_2}}$.

When the dimensions L and n are clear from context or not relevant, we sometimes write $\ell_2^{L,n}$ as ℓ_2.

Definition 2. The space $\mathcal{L}_p$ for $1 \leq p < \infty$ is the set of functions mapping $\mathbb{R}_{\geq 0}$ to ℓ_2 with its norm is given by

$$\|\phi\|_{\mathcal{L}_p} := \left(\int_0^\infty \|\phi(t)\|_{\ell_2}^p \mathrm{d}t \right)^{1/p} < \infty.$$

We will consider variables $d \in \mathcal{L}_p$ that are vector-valued functions indexed by $L + 1$ independent variables, i.e., $d = d(t, s_1, \ldots, s_L)$, where $t \in \mathbb{R}_{\geq 0}$ is the (continuous) time and $s_1, s_2, \ldots, s_L \in \mathbb{Z}$ are the spatial variables. The L-tuple $(s_1, s_2, \ldots, s_L)$ is denoted by $\mathbf{s}$. For fixed $t \in \mathbb{R}_{\geq 0}$ and $\mathbf{s} \in \mathbb{Z}^L$, a variable $d(t)$ can be considered as an element of $\ell^{L,n}$ or $\ell_2^{L,n}$ and $d(t, \mathbf{s})$ as an element of $\mathbb{R}^n$, i.e., a real-valued vector. For ease of notation, t is often omitted when considering such variables, however, from the context it will be clear which space is considered. In the case that $L = 1$, which we consider mainly in this paper, we denote s_1 also as s.

3. SYSTEM DESCRIPTION & SOLUTION CONCEPT

In this section, the considered class of systems is motivated and a general modeling framework is introduced. In addition, we will define the notion of solutions for this class of systems under study.

3.1 An infinite interconnection of hybrid systems

As indicated by the studies of D'Andrea and Dullerud (2003), Langbort and D'Andrea (2005), and Heijmans et al. (2015), many large-scale systems can be modeled as an *infinite* interconnection of identical subsystems or "basic building blocks". Examples include flocks of systems as described in Brockett (2010) and vehicle platooning, see, e.g., Ploeg et al. (2014) and the references therein. In addition, each individual subsystem is being more and more often modeled as a hybrid system. An example for this are interconnected systems using packet-based communication networks instead of dedicated point-to-point wired links. These control systems with packet-based communication networks can be easily captured in a hybrid modeling framework as shown in the studies of Nešić and Teel (2004) and Carnevale et al. (2007). If now the overall system consists of an infinite number of these subsystems, we obtain interconnections of an infinite number of (sometimes identical) hybrid systems. As a result, there is an increasing interest in modeling and analyzing techniques concerning (infinite) interconnections of hybrid systems.

In this paper, we are particulary interested in systems that consist of an infinite number of identical hybrid subsystems $\mathcal{H}(s)$, $s \in \mathbb{Z}$, which are interconnected according to a particular structure as indicated in Fig. 2. An example of a system that can be captured by this structure is the platoon of vehicles with packet-based communication networks of Fig. 1, see Heijmans et al. (2015).

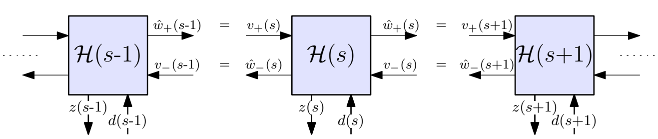

Fig. 2. An infinite interconnection of hybrid dynamical (sub)systems $\mathcal{H}(s)$, $s \in \mathbb{Z}$, where each (sub)system $\mathcal{H}(s)$ has its own external input and output variables $d(s)$ and $z(s)$, respectively. The interconnection for each subsystem is defined by the interconnection variables $v(s) = (v_+(s), v_-(s))$ and $\hat{w}(s) = (\hat{w}_+(s), \hat{w}_-(s))$.

For the interconnection structure of Fig. 2, we consider that each (general) hybrid subsystem $\mathcal{H}(s)$, $s \in \mathbb{Z}$, is described by

$$\mathcal{H}(s) : \begin{cases} \dot{\xi}(s) = F(\xi, d)(s), & \xi(s) \in \mathcal{C} \\ \xi^+(s) \in G(\xi(s)) \quad , & \xi(s) \in \mathcal{D} \end{cases} \tag{1}$$

with $\xi = (\xi_c, \xi_d)$ the state of the overall system where $\xi_c \in \ell_2^{1,m_1}$ comprises the (internal) dynamical states of the system while $\xi_d \in \ell^{1,m_2}$ comprises (auxilary) states like timers and counters, and where $d \in \ell_2^{1,m_d}$ denotes an external (disturbance) input. Note that ξ_d is typically not contained in ℓ_2 and therefore these states are separated from $\xi_c \in \ell_2^{1,m_1}$. In (1) it is assumed that $G : \mathbb{R}^{m_1+m_2} \rightrightarrows \mathbb{R}^{m_1+m_2}$ is a set-valued function with corresponding (diagonal) operator $\tilde{G} : \ell_2^{1,m_1} \times \ell^{1,m_2} \rightrightarrows \ell_2^{1,m_1} \times \ell^{1,m_2}$ defined as $\tilde{G}(\xi)(s) = G(\xi(s))$, $s \in \mathbb{Z}$, $\mathcal{C}$ and $\mathcal{D}$ are closed subsets of $\mathbb{R}^{m_1+m_2}$, and that the function $F := (F_c, F_d)$ with $F_c : \ell_2^{1,m_1+m_d} \to \ell_2^{1,m_1}$ and $F_d : \ell^{1,m_2} \to \ell^{1,m_2}$ is sufficiently smooth such that the dynamical system $\dot{\xi_c} = F_c(\xi_c, d)$ gives rise to solutions ξ_c with values in the space ℓ_2^{1,m_1}. In particular, it is assumed that the function F_c is locally Lipschitz in its first argument and continuous in its second argument such that the existence and uniqueness of the solutions ξ_c is guaranteed for a given internal state $\xi_c \in \ell_2^{1,m_1}$ and disturbance signals $d \in \mathcal{L}_p$, see also Kato (1970) or Robinson (2001). The overall system $\mathcal{H}$ is now composed of the infinite number of subsystems $\mathcal{H}(s)$, all given by (1).

3.2 Solution concept

In order to perform a stability or performance analysis using, for instance, Lyapunov-based arguments, solutions need to be defined globally, i.e., for all times $t \in \mathbb{R}_{\geq 0}$, see also Section 4. Unfortunately however, for the analysis of the overall interconnected system $\mathcal{H}$ as in Fig. 2, standard solution concepts for hybrid systems, see, e.g., Lygeros et al. (2003) or Goebel et al. (2012), are not always applicable as they define solutions only up to Zeno points, but not beyond. For the class of infinite interconnections of hybrid systems considered in this paper, it is highly relevant to have such solution concepts as Zeno behavior is often inevitable due to the infinite number of hybrid subsystems $\mathcal{H}(s)$ as in (1), see also Remark 1 below. As such, inspired by Heemels et al. (2000), in this subsection we introduce a novel, but natural solution concept which allows us to define solutions beyond these Zeno points, i.e., solutions can be defined for all times $t \in \mathbb{R}_{\geq 0}$. Consider hereto the following definitions.

Definition 3. A point $\tau \in \mathcal{R} \subset \mathbb{R}$ is called a right-accumulation point of $\mathcal{R}$ if there exists a sequence $\{\tau_i\}_{i \in \mathbb{N}}$ such that $\tau_i \in \mathcal{R}$ and $\tau_i < \tau$ for all $i \in \mathbb{N}$ and, furthermore, $\lim_{i\to\infty} \tau_i = \tau$. A left-accumulation point is defined similarly by interchanging "<" with ">". A set $\mathcal{R} \subset \mathbb{R}$ is called right-isolated if it contains no left-accumulation points. Hence, we say that $\mathcal{R}$ is right-isolated if for all $\tau \in \mathcal{R}$ it holds that there is an $\varepsilon > 0$ such that $(\tau, \tau + \varepsilon) \cap \mathcal{R} = \varnothing$.

Definition 4. A pair $(\mathcal{R}, \xi)$, where $\mathcal{R}$ is a right-isolated closed subset of $[0, T)$,

$$\xi : [0, T)\backslash\mathcal{R} \to \ell_2^{1,m_1} \times \ell^{1,m_2},$$

is a solution to the overall system $\mathcal{H}$ composed of the identical subsystems $\mathcal{H}(s)$ given by (1), $s \in \mathbb{Z}$, on $[0, T)$ with $T > 0$ or $T = \infty$ for initial state $\xi_0 \in \ell_2^{1,m_1} \times \ell^{1,m_2}$ and external (disturbance) inputs $d \in \mathcal{L}_p$ if the following are satisfied:

(1) $0 \in \mathcal{R}$
(2) For all $\tau \in \mathcal{R}$ and $s \in \mathbb{Z}$, the right-limit $\xi(\tau^+, s) := \lim_{t\downarrow\tau,\, t\notin\mathcal{R}} \xi(t, s)$ exists and for all $\tau \in \mathcal{R}\backslash\{0\}$ and $s \in \mathbb{Z}$ the left-limit $\xi(\tau^-, s) := \lim_{t\uparrow\tau,\, t\notin\mathcal{R}} \xi(t, s)$ exists. Moreover, for all $\tau \in \mathcal{R}$ and $s \in \mathbb{Z}$ it holds that
$$\xi(\tau^+, s) \in G(\xi(\tau^-, s)) \quad \text{or} \quad \xi(\tau^+, s) = \xi(\tau^-, s)$$
when $\xi(\tau^-, s) \in \mathcal{D}$, while for $\xi(\tau^-, s) \notin \mathcal{D}$ it holds that
$$\xi(\tau^+, s) = \xi(\tau^-, s),$$
where $\xi(\tau^-, s) := \xi_0(s)$, $s \in \mathbb{Z}$, when $\tau = 0$.
(3) For all intervals (τ, τ^*) with $\tau \in \mathcal{R}$ and
$$\tau^* := \inf\{\theta > \tau \mid \theta \in \mathcal{R} \cup \{T\}\},$$
it holds that $\xi : (\tau, \tau^*) \to \ell_2^{1,m_1} \times \ell^{1,m_2}$ is absolutely continuous, $\dot{\xi}(t) = F(\xi(t), d(t))$ for almost all $t \in (\tau, \tau^*)$, and that $\xi(t, s) \in \mathcal{C}$ for all $t \in (\tau, \tau^*)$ and $s \in \mathbb{Z}$.

Definition 5. A solution $(\mathcal{R}, \xi)$ to the overall system $\mathcal{H}$ composed out of the identical subsystems $\mathcal{H}(s)$ given by (1), $s \in \mathbb{Z}$, on $[0, T)$ is called maximal if there does not exist a $T' > T$ and a solution $(\mathcal{R}', \xi')$ to the overall system $\mathcal{H}$ on $[0, T')$ for which it holds that

(1) $\mathcal{R}' \cap [0, T) = \mathcal{R}$
(2) $\xi'(t) = \xi(t)$ for all $t \in [0, T) \backslash \mathcal{R}$.

Definition 6. A solution $(\mathcal{R}, \xi)$ to the overall system $\mathcal{H}$ composed out of the identical subsystems $\mathcal{H}(s)$ given by (1), $s \in \mathbb{Z}$, on $[0, T)$ is called complete if $T = \infty$.

Note that complete solutions are always maximal. The set $\mathcal{R}$ contains the jump times, i.e., the times at which there is a jump in one of the subsystems $\mathcal{H}(s)$. Between the successive jump times τ and τ^*, ξ captures the trajectories in the flow phases of the system (as imposed by item 3 of Definition 4 above). Item 2 of Definition 4 connects the flow phases at the jump times and specifies also the initial conditions. In this solution concept, right-accumulation points of event times are included and solutions can indeed be defined beyond Zeno points in the sense that despite the occurrence of right-accumulation points or even an infinite number of jumps at one (continuous) time instant, solutions can still be defined globally on $\mathbb{R}_{\geq 0}$ (i.e., for $T = \infty$), see also Section 4.2.

4. STABILITY AND PERFORMANCE ANALYSIS

With a solution concept now in place, stability (in absence of disturbances) in the sense of UGAS and performance (in presence of disturbances) in the sense of $\mathcal{L}_p$-stability of the overall system can be defined. In this section, these concepts of stability and performance are introduced and Lyapunov-based conditions to guarantee them are given.

4.1 Stability and performance concepts

As a result of the specific structure of the hybrid (sub)systems considered in this paper, we are often only interested in a relevant set of (initial) states specified by $\mathbb{X}_0 \subseteq \ell_2^{1,m_1} \times \ell^{1,m_2}$ for the overall system $\mathcal{H}$ in (1).

With this set of initial states defined, we can analyze uniform global asymptotic stability (UGAS) of the overall system in the case that the external (disturbance) inputs are absent, i.e., $d = 0$.

Definition 7. For the overall system $\mathcal{H}$ with associated set of initial states $\mathbb{X}_0 \subseteq \ell_2^{1,m_1} \times \ell^{1,m_2}$ and composed of the identical subsystems $\mathcal{H}(s)$, $s \in \mathbb{Z}$, given by (1), the set

$$\mathcal{E} = \left\{ \xi = (\xi_c, \xi_d) \in \ell_2^{1,m_1} \times \ell^{1,m_2} \mid \xi_c = 0 \right\} \tag{2}$$

is uniformly globally asymptotic stable (UGAS) if there exists a function $\beta \in \mathcal{KL}$ such that for any initial condition $\xi(0) \in \mathbb{X}_0$, all corresponding maximal solutions $(\mathcal{R}, \xi)$ to $\mathcal{H}$ with $d = 0$ are complete, and for all $t \in [0, \infty) \backslash \mathcal{R}$

$$\|\xi_c(t)\|_{\ell_2} \leq \beta\left(\|\xi_c(0)\|_{\ell_2}, t\right).$$

Moreover, if β is an exp-$\mathcal{KL}$ function, the set $\mathcal{E}$ is uniformly globally exponentially stable (UGES).

In the case of the external inputs being present, i.e., $d \neq 0$, we analyze the performance of the hybrid system $\mathcal{H}$ as being the level of input attenuation with respect to the external output variable

$$z = Q(\xi_c, d) \tag{3}$$

with $Q : \ell_2^{1,m_1+m_d} \to \ell_2^{1,m_z}$ and where we use the $\mathcal{L}_p$-induced gain with $p \in [0, \infty)$ as the performance criterion.

Definition 8. The overall system $\mathcal{H}$ with associated set of initial states $\mathbb{X}_0 \subseteq \ell_2^{1,m_1} \times \ell^{1,m_2}$ and composed of the identical subsystems $\mathcal{H}(s)$, $s \in \mathbb{Z}$, as in (1) with (3), is said to be $\mathcal{L}_p$-stable ($p < \infty$) with an $\mathcal{L}_p$-gain less than or equal to $\theta \geq 0$ from input d to output z, if there exists a function $\beta \in \mathcal{K}$ such that for any exogenous input $d \in \mathcal{L}_p$ and any initial condition $\xi(0) \in \mathbb{X}_0$, all corresponding maximal solutions $(\mathcal{R}, \xi)$ to $\mathcal{H}$ are complete and it holds that

$$\|z\|_{\mathcal{L}_p} \leq \beta\left(\|\xi_c(0)\|_{\ell_2}\right) + \theta \|d\|_{\mathcal{L}_p}. \tag{4}$$

Next, for a specific class of infinite-dimensional interconnected hybrid systems that is, for instance, used for the modeling of interconnected systems that use packet-based communication networks for the exchange of information, we will provide Lyapunov-based conditions that, when satisfied, guarantee UGAS (and sometimes even UGES) of the set $\mathcal{E}$ of (2) and/or $\mathcal{L}_p$-stability with an $\mathcal{L}_p$-gain less than or equal to $\theta \geq 0$.

4.2 Lyapunov conditions for UGAS (or UGES)

Consider again the hybrid modeling framework of (1) for each subsystem $\mathcal{H}(s)$, $s \in \mathbb{Z}$, in the infinite interconnection of Fig. 2 with the "local" state $\xi(s)$. However, we now assume that the function $\tau(t, s) \in \mathbb{R}_{\geq 0}$, which is a timer for each $s \in \mathbb{Z}$ that resets itself to zero after each jump of the subsystem $\mathcal{H}(s)$, i.e.,

$$\begin{cases} \dot{\tau}(t, s) = 1, & \xi(t, s) \in \mathcal{C} \\ \tau(t^+, s) = 0, & \xi(t, s) \in \mathcal{D}, \end{cases} \tag{5}$$

is part of the state $\xi_d(t, s)$ for each subsystem $\mathcal{H}(s)$, $s \in \mathbb{Z}$. Moreover, we also assume that the flow and jump conditions $\xi(t, s) \in \mathcal{C}$ and $\xi(t, s) \in \mathcal{D}$ are now given by

$$\xi(t, s) \in \mathcal{C}(s) \Leftrightarrow \tau(t, s) \in [0, \tau_{mati}^s] \tag{6a}$$

$$\xi(t, s) \in \mathcal{D}(s) \Leftrightarrow \tau(t, s) \in [\delta^s, \infty), \tag{6b}$$

respectively, where $\tau_{mati}^s \in \mathbb{R}_{\geq 0}$ represents a upper bound on the time elapse between two consecutive jumps of the hybrid system $\mathcal{H}(s)$, i.e., the MATI, and $\delta^s > 0$ a lower bound, $s \in \mathbb{Z}$. Here we assume that $\tau_{mati}^s - \delta^s \geq \tau_{min} > 0$ for all $s \in \mathbb{Z}$. This kind of setup with a bounded timer is natural in many applications, including networked control systems, see, e.g., Nešić and Teel (2004), Carnevale et al. (2007), Borgers and Heemels (2014), or Heijmans et al. (2015). Hence, (5)-(6) models that the hybrid subsystem $\mathcal{H}(s)$, $s \in \mathbb{Z}$, has an uncertain duration between two consecutive jump times larger than δ^s, but smaller than τ_{mati}^s. In case of packet-based communication networks this means that the inter-transmission times are between δ^s and τ_{mati}^s, see Nešić and Teel (2004). Note that, while, as a result of (6b) and $\delta^s > 0$, *local* Zeno behavior is prevented, these kind of infinite interconnections of hybrid systems are still typically an example in which an infinite number of jumps occurs in a finite time interval, see also Remark 1 below.

Remark 1. Local Zeno behavior, as mentioned above, refers to the (possible) Zeno behavior of *one single* subsystem in the interconnection. In order to define a solution globally, i.e., for all time $t \in \mathbb{R}_{\geq 0}$, this behavior must be prevented for the same reasons as for the finite dimensional case as in Dashkovskiy and Kosmykov (2013). However, as a result of (6a), *global* Zeno behavior, i.e., Zeno behavior of the overall interconnected system $\mathcal{H}$, cannot be prevented as a result of the infinite number of "jumping" hybrid subsystems. As a result, solution concepts as in Lygeros et al. (2003) or Goebel et al. (2012) cannot be used directly, since they cannot be defined beyond these Zeno points. Therefore, there is the need for a novel solution concept as introduced in Section 3.2 that does allows us to define solutions beyond these Zeno points.

For the systems given by (1) with (5) and (6), as described above, we provide sufficient Lyapunov-based conditions such that UGAS (or UGES) is guaranteed for the overall system. In particular, since the considered systems consist of an infinite number of interconnected hybrid subsystems, the conditions guaranteeing UGAS of the set (2) can be obtained as merely infinite-dimensional extensions of the Lyapunov-based arguments for hybrid systems as stated in Goebel et al. (2012) Theorem 3.18, of course, with proper adaptions to accommodate the new solution concept (which allows Zeno points and trajectories beyond). As such, the analysis is based on constructing a (Lyapunov) function $U : \ell_2^{1,m_1} \times \ell^{1,m_2} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its first and second argument, functions $\underline{\alpha}_U, \overline{\alpha}_U \in \mathcal{K}_\infty$ and a positive definite function ϱ such that

$$\underline{\alpha}_U(\|\xi_c\|_{\ell_2}) \leq U(\xi) \leq \overline{\alpha}_U(\|\xi_c\|_{\ell_2}) \tag{7a}$$

$$U(\xi^+) - U(\xi) \leq 0 \text{ when } \xi(s) \in \mathcal{D} \text{ for some } s \in \mathbb{Z} \tag{7b}$$

$$\langle \nabla U(\xi), F(\xi)\rangle_{\ell_2} \leq -\varrho(\|\xi_c\|_{\ell_2}) \text{ when } \xi(s) \in \mathcal{C}\ \forall\ s \in \mathbb{Z} \tag{7c}$$

with $\xi = (\xi_c, \xi_d)$ and where ξ^+ is given by $\xi^+(\tau, s) = \xi(\tau^+, s) = \lim_{t\downarrow\tau, t\notin\mathcal{R}} \xi(t, s)$ for $s \in \mathbb{Z}$ as in Definition 4. These conditions are sufficient to guarantee UGAS of the set $\mathcal{E}$ of (2). If, in addition, there exist $\underline{\alpha}_U^c, \overline{\alpha}_U^c, \varepsilon \in \mathbb{R}_{\geq 0}$ such that $\underline{\alpha}_U^c \|\xi_c\|_{\ell_2}^2 \leq U(x) \leq \overline{\alpha}_U^c \|\xi_c\|_{\ell_2}^2$, and $\varrho(r) \geq \varepsilon^2 r^2$, $r \in \mathbb{R}_{>0}$, then the set $\mathcal{E}$ of (2) is UGES. Of course, this reasoning can only be used if we can also show the following claim.

Claim 1. If (7) holds, then each maximal solution is complete, i.e., any maximal solution $(\xi, \mathcal{R})$ with initial state $\xi_0 \in \mathbb{X}_0$ should be defined on $[0, T)$ for $T = \infty$.

Note that if we prove this property, then the conditions in (7) are indeed sufficient to establish UGAS or UGES of the set $\mathcal{E}$.

Proof. [Proof of Claim 1] To show completeness of a maximal solution $(\xi, \mathcal{R})$ under the stated conditions, we will proceed by contradiction by assuming that $T < \infty$. In fact, we will show that we can prolong the solution $(\xi, \mathcal{R})$ and get a solution $(\xi', \mathcal{R}')$ on $[0, T')$ with $T' > T$ that satisfies the points 1) and 2) in Definition 5. This would contradict the maximality of the solution $(\xi, \mathcal{R})$ and thus T must be infinite.

Observe that for $T < \infty$, each subsystem in the interconnection can jump at most one time in the time period $[T - \delta^s, T)$. Hence, we know that for each $s \in \mathbb{Z}$ there exists a $t^s \in [T - \delta^s, T)$ such that $\xi_c(t, s)$ is absolutely continuous on (t^s, T) (see Definition 4 point 3). In addition, based on the bounds (7a), (7b), and (7c), we can also conclude that

$$\underline{\alpha}_U\left(\|\xi_c\|_{\ell_2}\right) \leq U(\xi(t)) \leq U(\xi(0)) \leq \infty \tag{8}$$

for all $t \in [0, T)\backslash\mathcal{R}$, implying that there is no finite escape time for the solution ξ_c. Hence, we have that $\xi_c(t) \in \mathcal{B}$ for all $t \in [0, T)$, where $\mathcal{B}$ is some compact set in ℓ_2^{1,m_1}.

As a result of the absolute continuity and boundedness of the solutions ξ_c, it follows from a standard result in mathematical analysis (Rudin, 1976, ex. 4.13) that $\xi_c(T^-, s) := \lim_{t\uparrow T} \xi_c(t, s)$ exists for each $s \in \mathbb{Z}$ and we thus have that $\xi_c(T^-) \in \mathcal{B} \subseteq \ell_2^{1,m_1}$ as a result of (8).

Based on these properties we can now construct a solution $(\xi', \mathcal{R}')$ that prolongs the considered maximal solution $(\xi, \mathcal{R})$. Consider hereto two possibilities for each subsystem. If $\tau(T^-, s) \geq \delta^s$ for $s \in \mathbb{Z}$, then we can define a solution with a jump at time T for this subsystem at s (note that $\xi(T^-, s) \in \mathcal{D}$ in this case, see (6)). If $\tau(T^-, s) < \delta^s$, then we define the solution without a reset at time T for the corresponding subsystem at s. Hence, we have $\xi(T^+, s) \in G(\xi(T^-, s))$ for $s \in S_r(T)$ and $\xi(T^+, s) = \xi(T^-, s)$ for $s \notin S_r(T)$ where $S_r(T) = \{s \in \mathbb{Z} \mid \tau(T^-, s) \geq \delta^s\}$. Due to (5), it follows that $\tau(T^+, s) \in [0, \delta^s)$.

For the state $\xi(T^+)$ there now exist a constant $\varepsilon > 0$ and an absolutely continuous $\bar{\xi} : [T, T + \varepsilon) \to \ell_2^{1,m_1} \times \ell^{1,m_2}$ such that $\bar{\xi}(T) = \xi(T^+)$ and $\dot{\bar{\xi}}(t) = F(\bar{\xi}(t), 0)$ for almost all $t \in [T, T + \varepsilon)$. Since, $\tau(T^+, s) < \delta^s$ for all $s \in \mathbb{Z}$, it holds that $\xi(t, s) \in \mathcal{C}$ for all $t \in [T, T + \Delta T)$ and $s \in \mathbb{Z}$, where $\Delta T := \inf_{s\in\mathbb{Z}} (\varepsilon, \tau^s_{mati} - \tau(T^+, s)) > 0$. Hence, this indicates that $(\xi', \mathcal{R}')$ with $\mathcal{R}' := \mathcal{R} \cup \{T\}$ and

$$\xi'(t) = \begin{cases} \xi(t) & \text{for all } t \in [0, T)\backslash\mathcal{R}' \\ \bar{\xi}(t) & \text{for all } t \in [T, T + \Delta T)\backslash\mathcal{R}' \end{cases}$$

is a solution to $\mathcal{H}$ on $[0, T + \Delta T)$ satisfying points 1) and 2) in Definition 5. This contradicts the maximality of the solution $(\xi, \mathcal{R})$ and thus $T = \infty$, implying that indeed every maximal solution is a complete solution. □

4.3 Lyapunov conditions for $\mathcal{L}_p$-stability

In a similar fashion as for the case with disturbances, we can also compose Lyapunov-based conditions that guarantee $\mathcal{L}_p$-stability of the overall system according to Definition 8 when disturbances are present. Consider hereto again the candidate Lyapunov function $U : \ell_2^{1,m_1} \times \ell^{1,m_2} \to \mathbb{R}_{\geq 0}$ such that the conditions (7a) and (7b) hold. However, during flows, i.e., when $\xi(s) \in \mathcal{C}$ for all $s \in \mathbb{Z}$, we now require that it should hold that

$$\langle \nabla U(\xi), F(\xi, d)\rangle_{\ell_2} \leq \mu\left(\theta^p \|d\|_{\ell_2}^p - \|q(x, e, d)\|_{\ell_2}^p\right) \tag{9}$$

for some constants $\mu > 0$ and $\theta \geq 0$. To show that the conditions (7a), (7b), and (9) are indeed sufficient to guarantee $\mathcal{L}_p$-stability of the overall system, consider the following. Combining integrated versions of (9) with (7b) on the interval $[0, \mathcal{T}]$ with $0 \leq \mathcal{T} < T$ and assuming that the considered maximal solution is defined on at least $[0, T)$ yields

$$U(\xi(\mathcal{T})) - U(\xi(0)) \leq \mu \int_0^{\mathcal{T}} \left(\theta^p \|d\|_{\ell_2}^p - \|Q(\xi_c, d)\|_{\ell_2}^p\right) \mathrm{d}t.$$

Hence, recalling (3), we have for all $0 \leq \mathcal{T} < T$ that

$$U(\xi(\mathcal{T})) \leq U(\xi(0)) + \mu \int_0^{\mathcal{T}} \left(\theta^p \|d\|_{\ell_2}^p - \|z\|_{\ell_2}^p\right) \mathrm{d}t. \tag{10}$$

As a result of this observation, we obtain the bound

$$U(\xi(\mathcal{T})) \leq U(\xi(0)) + \mu\theta^p \int_0^{\mathcal{T}} \|d\|_{\ell_2}^p \,\mathrm{d}t$$

for all $\mathcal{T} \in [0, T)$. Using now similar arguments as for the case with disturbances, it can be proven that all maximal solutions are complete solutions, i.e., $T = \infty$. Now using that $U(\xi(\mathcal{T})) \geq 0$ for all $\mathcal{T} \in \mathbb{R}_{\geq 0}$, letting $\mathcal{T} \to \infty$ in (10), and computing the $\mathcal{L}_p$-norm according to Definition 2 yield

$$\|z\|_{\mathcal{L}_p}^p = \int_0^\infty \|z\|_{\ell_2}^p \mathrm{d}t \leq \int_0^\infty \theta^p \|d\|_{\ell_2}^p \mathrm{d}t + \frac{1}{\mu} U(\xi(0))$$
$$= \theta^p \|d\|_{\mathcal{L}_p}^p + \frac{1}{\mu} U(\xi(0)) \leq \left(\left(\frac{1}{\mu} U(\xi(0)) \right)^{\frac{1}{p}} + \theta \|d\|_{\mathcal{L}_p} \right)^p . \quad (11)$$

Since $(U(\xi)/\mu)^{1/p} \leq \beta_U(\|\xi_c\|_{\ell_2})$ for some $\beta_U \in \mathcal{K}_\infty$, (11) leads to (4) we indeed have proven that the existence of a Lyapunov function U satisfying the conditions (7a), (7b), and (9) is sufficient to guarantee $\mathcal{L}_p$-stability of the overall system composed out of the hybrid subsystems $\mathcal{H}(s)$, $s \in \mathbb{Z}$, of (1) with (5) and (6).

5. CONCLUDING REMARKS

In this paper we considered a class of infinite-dimensional hybrid systems, or more precisely, interconnections consisting of an infinite number of spatially invariant hybrid (sub)systems. These kind of interconnections have been proven to be very useful for the modeling of large-scale systems which use packet-based communication networks for the exchange of information, see, e.g., Heijmans et al. (2015). However, this modeling setup requires the establishment of a proper definition of solutions since classical solution concepts typically do not define solutions beyond Zeno points, which is relevant for the class of systems studied here. Therefore, we provided a novel solution concept for the considered infinite interconnections of hybrid systems, which can incorporate Zeno points and allows us to define solutions globally, i.e., for all time $t \in \mathbb{R}_{\geq 0}$. In addition, we have provided sufficient Lyapunov-based conditions for a subclass of hybrid systems relevant for modeling, for instance, networked control systems based on the new definition of solutions, such that UGAS or $\mathcal{L}_p$-stability is guaranteed for the overall system.

This novel solution concept and sufficient (Lyapunov-based) conditions to guarantee global stability properties (UGAS or $\mathcal{L}_p$-stability) clear the path for a more in depth analysis of the topic of infinite interconnections of hybrid systems. In particular, one can, for instance, exploit the spatial invariance of the hybrid subsystems in the interconnection to construct Lyapunov functions satisfying the condition (7a), (7b), and (7c) or (9) based only on the *local* dynamics of one of the subsystems in the interconnection structure. Some first results in this direction are presented in Heijmans et al. (2015), expanding ideas from D'Andrea and Dullerud (2003).

REFERENCES

Aihara, K. and Suzuki, H. (2010). Theory of hybrid dynamical systems and its applications to biological and medical systems. *Mathematical, Physical and Engineering Sciences*, 368(1930), 4893–4914.

Borgers, D.P. and Heemels, W.P.M.H. (2014). Stability analysis of large-scale networked control systems with local networks: A hybrid small-gain approach. In *Hybrid Systems: Computation and Control*, 103–112.

Brockett, R.W. (2010). On the control of a flock by a leader. *Proc. Steklov Inst. of Math.*, 268(1), 49–57.

Carnevale, D., Teel, A.R., and Nešić, D. (2007). A Lyapunov proof of an improved maximum allowable transfer interval for networked control systems. *IEEE Trans. Automat. Contr.*, 52(5), 892–897.

D'Andrea, R. and Dullerud, G.E. (2003). Distributed control design for spatially interconnected systems. *IEEE Trans. Automat. Contr.*, 48(9), 1478–1495.

Dashkovskiy, S.N. and Kosmykov, M. (2013). Input-to-state stability of interconnected hybrid systems. *Automatica*, 49(4), 1068–1074.

Dolk, V.S., Borgers, D.P., and Heemels, W.P.M.H. (2017). Output-based and decentralized dynamic event-triggered control with guaranteed $\mathcal{L}_p$-gain performance and Zeno-freeness. *IEEE Trans. Automat. Contr.*

Ghosh, R. and Tomlin, C. (2004). Symbolic reachable set computation of piecewise affine hybrid automata and its application to biological modelling: Delta-notch protein signalling. *Systems Biology*, 1(1), 170–183.

Goebel, R., Sanfelice, R.G., and Teel, A.R. (2012). *Hybrid dynamical systems.* Princeton University Press, U.K.

Heemels, W.P.M.H., Schumacher, J.M., and Weiland, S. (2000). Linear complementarity systems. *SIAM Applied Mathematics*, 60(4), 1234–1269.

Heijmans, S.H.J., Borgers, D.P., and Heemels, W.P.M.H. (2015). Stability analysis of spatially invariant interconnected systems with networked communication. *IFAC-PapersOnLine*, 48(22), 221–226.

Johansson, K.H., Egerstedt, M., Lygeros, J., and Sastry, S.S. (1999). On the regularization of Zeno hybrid automata. *Sys. Contr. Lett.*, 38(3), 141–150.

Kato, T. (1970). Evolution equations in Banach spaces. *Nonlinear Functional Analysis: Proceedings*, 18, 138.

Langbort, C. and D'Andrea, R. (2005). Distributed control of spatially reversible interconnected systems with boundary conditions. *SIAM Contr. Opt.*, 44(1), 1–28.

Lygeros, J., Johansson, K.H., Simić, S.N., Zhang, J., and Sastry, S.S. (2003). Dynamical properties of hybrid automata. *IEEE Trans. Automat. Contr.*, 48(1), 2–17.

Machado, D., Costa, R.S., Rocha, M., Ferreira, E.C., Tidor, B., and Rocha, I. (2011). Modeling formalisms in systems biology. *AMB express*, 1(1), 1–14.

Nešić, D. and Teel, A.R. (2004). Input-output stability properties of networked control systems. *IEEE Trans. Automat. Contr.*, 49(10), 1650–1667.

Ploeg, J., van de Wouw, N., and Nijmeijer, H. (2014). $\mathcal{L}_p$-string stability of cascaded systems: Application to vehicle platooning. *IEEE Trans. Contr. Sys. Tech.*, 22(2), 786–793.

Robinson, J.C. (2001). *Infinite-dimensional dynamical systems: An introduction to dissipative parabolic PDEs and the theory of global attractors*, volume 28. Cambridge University Press.

Rudin, W. (1976). *Principles of mathematical analysis.* McGraw-Hill, New York.

Sanfelice, R.G. (2011). Interconnections of hybrid systems: Some challenges and recent results. *Nonlinear Sys. Appli.*, 2(1-2), 111–121.

Walsh, G.C., Ye, H., and Bushnell, L.G. (2002). Stability analysis of networked control systems. *IEEE Trans. Contr. Sys. Tech.*, 10(3), 438–446.

Zheng, H., Lee, E.A., and Ames, A.D. (2006). Beyond Zeno: Get on with it! In *Hybrid Systems: Computation and Control*, 568–582. Springer.